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Discrete generalized Wirtinger's inequalities

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Abstract. Let n, k be fixed natural numbers $1 \le k \le n$. We study the following generalized weighted discrete inequalities of Wirtinger type:

$$\alpha_{\pm}^{(i)} \sum_{j=0}^{n} p_j |x_j|^2 \le \sum_{j} {}^{(i)} r_j |x_j \pm x_{j+k}|^2 \le \beta_{\pm}^{(i)} \sum_{j=0}^{n} p_j |x_j|^2$$

where $x_0, x_1, \dots, x_n$ are arbitrary complex numbers, $p_0, p_1, \dots, p_n$ and $r_{-k}, \dots, r_0$, $r_1, \dots, r_n, \dots, r_{n+k}$ are given positive weights, $\alpha_{\pm}^{(i)}$, $\beta_{\pm}^{(i)}$ are constants and either the $+$ or the $-$ sign has to be taken. $i = 1, 2, 3, 4$ indicates the type of the summation, for example

$$\sum_{j} {}^{(2)} r_j |x_j \pm x_{j+k}|^2 = \sum_{j=0}^{n} r_j |x_j \pm x_{j+k}|^2 \quad \text{with } x_{n+1} = \dots = x_{n+k} = 0.$$

Our aim is to find the best constants $\alpha_{\pm}^{(i)}$, $\beta_{\pm}^{(i)}$.

The weighted versions with positive sign, shift $k = 1$ and $i = 2, 3$ were studied by G. V. Milovanović and I. Ž. Milovanović [9], the unweighted versions were studied by the author [6].

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1. Introduction

A version of Wirtinger's inequality (see e.g. [5], p. 184–187) states that if $x, x' \in L^2[0,\pi]$, $x(0) = x(\pi) = 0$, then

$$\int_0^\pi x^2(t)\,dt < \int_0^\pi x'^2(t)\,dt$$

unless $x(t) = A\sin t$.

Fan, Taussky and Todd [4] studied discrete analogs of Wirtinger's inequality. They found the best constants β in the inequalities

$$\sum_j (x_j - x_{j+1})^2 \geq \beta \sum_{j=0}^n x_j^2 \tag{1}$$

where $x_0, x_1, \dots x_n$ are arbitrary real numbers and the summation on the left hand side goes from 0 to $n-1$, or from 0 to n with $x_{n+1} = 0$, or from -1 to n with $x_{-1} = x_n = 0$. They also considered inequalities similar to (1) where on the left side the first differences $x_j - x_{j+1}$ are replaced by second differences $x_j - 2x_{j+1} + x_{j+2}$.

Redheffer [11] gave elementary proofs for some of the inequalities (1).

G. V. Milovanović and I. Ž. Milovanović [9] considered the weighted versions of (1) and their reverse inequalities by determining the best constants α, β in

$$\alpha \sum_{j=0}^n p_j x_j^2 \leq \sum_j r_j (x_j - x_{j+1})^2 \leq \beta \sum_{j=0}^n p_j x_j^2$$

where $x_0, x_1, \dots x_n$ are arbitrary real numbers $p_0, p_1, \dots p_n$; $r_{-1}, r_0, r_1, \dots r_n$ are given sequences of positive numbers and the sum in the middle term varies according to the boundary conditions for the sequence (x_j).

The author [6] determined the best constants $\alpha_\pm^{(i)}$, $\beta_\pm^{(i)}$ in

$$\alpha_\pm^{(i)} \sum_{j=0}^n |x_j|^2 \leq \sum_j{}^{(i)} |x_j \pm x_{j+k}|^2 \leq \beta_\pm^{(i)} \sum_{j=0}^n |x_j|^2 \tag{2}$$

where n, k are fixed natural numbers with $1 \leq k \leq n$, $x_0, x_1, \dots, x_n$ are arbitrary complex numbers and the summation $\sum_j{}^{(i)}$ involves four possibilities $(i = 1, \dots 4)$. Some related results can be found in [7].

In the above papers the best constants were the smallest and largest eigenvalues of suitable multi diagonal matrices. Concerning this see Rutherford [12],

[13]. Alzer [1] gave short elementary proofs for some of the Fan–Taussky–Todd inequalities and their converses.

Lunter [8] used discrete Fourier transform technique to prove some inequalities of Fan–Taussky–Todd involving second differences and also obtained some generalizations.

The aim of this paper is to extend the results concerning (2) to the weighted case.

2. Weighted Wirtinger's inequality with special weights

Let n, k be fixed natural numbers $1 \le k \le n$. We study the following generalized weighted discrete inequalities of Wirtinger type:

$$\alpha_{\pm}^{(i)} \sum_{j=0}^{n} p_j |x_j|^2 \le \sum_j {}^{(i)} r_j |x_j \pm x_{j+k}|^2 \le \beta_{\pm}^{(i)} \sum_{j=0}^{n} p_j |x_j|^2 \tag{3}$$

where $x_0, x_1, \dots, x_n$ are arbitrary complex numbers, $p_0, p_1, \dots, p_n$ and $r_{-k}, \dots, r_0$, $r_1, \dots, r_n, \dots, r_{n+k}$ are given positive weights, $\alpha_{\pm}^{(i)}$, $\beta_{\pm}^{(i)}$ are constants and either the $+$ or the $-$ sign has to be taken.

The upper index $i = 1, 2, 3, 4$ of the summation sign indicates the type of the summation

$$\sum_j {}^{(1)} r_j |x_j \pm x_{j+k}|^2 = \sum_{j=0}^{n-k} r_j |x_j \pm x_{j+k}|^2,$$

$$\sum_j {}^{(2)} r_j |x_j \pm x_{j+k}|^2 = \sum_{j=0}^{n} r_j |x_j \pm x_{j+k}|^2 \qquad \text{with } x_{n+1} = \dots = x_{n+k} = 0,$$

$$\sum_j {}^{(3)} r_j |x_j \pm x_{j+k}|^2 = \sum_{j=-k}^{n} r_j |x_j \pm x_{j+k}|^2 \quad \text{with } x_{-k} = \dots = x_{-1} = 0 = x_{n+1} = \dots = x_{n+k},$$

$$\sum_j {}^{(4)} r_j |x_j \pm x_{j+k}|^2 = \sum_{j=-k}^{n-k} r_j |x_j \pm x_{j+k}|^2 \quad \text{with } x_{-k} = \dots = x_{-1} = 0.$$

It is easy to see that apart from the notation of the variables the cases $i = 2$ and $i = 4$ are the same. Thus we shall consider only the cases $i = 1, 2, 3$. Our aim is to find the best constants $\alpha_{\pm}^{(i)}$, $\beta_{\pm}^{(i)}$. They depend on n, k and the weights. In this section we consider the dependence of the best constants on n, k, thus, if needed, we use the notation $\alpha_{\pm}^{(i)}(n,k)$, $\beta_{\pm}^{(i)}(n,k)$.

For instance consider the unweighted inequality with $n = 10$, $k = 3$, $i = 3$ implying that $x_{-3} = x_{-2} = x_{-1} = 0 = x_{11} = x_{12} = x_{13}$ and $-$ sign

$$\alpha_-^{(3)}(10,3)\sum_{j=0}^{10}|x_j|^2 \le |x_0|^2 + |x_1|^2 + |x_2|^2 + |x_0 - x_3|^2 + |x_1 - x_4|^2 + |x_2 - x_5|^2$$
$$+ |x_3 - x_6|^2 + |x_4 - x_7|^2 + |x_5 - x_8|^2 + |x_6 - x_9|^2 + |x_7 - x_{10}|^2$$
$$+ |x_8|^2 + |x_9|^2 + |x_{10}|^2 \le \beta_-^{(3)}(10,3)\sum_{j=0}^{10}|x_j|^2$$

With division $10 + 1 = 3q + r$, $0 \le r < 3$ we get $q = 3$, $r = 2$. Let

$$\begin{aligned} &x_0 = u_0,\ x_3 = u_1,\ x_6 = u_2,\ x_9 = u_3 \\ &x_1 = u_4,\ x_4 = u_5,\ x_7 = u_6,\ x_{10} = u_7 \\ &x_2 = u_8,\ x_5 = u_9,\ x_8 = u_{10}. \end{aligned} \tag{4}$$

With these new variables our inequality can be written as

$$\begin{aligned} &\alpha_-^{(3)}(10,3)(|u_0|^2 + |u_1|^2 + |u_2|^2 + |u_3|^2 + |u_4|^2 + |u_5|^2 + |u_6|^2 + |u_7|^2 \\ &\quad + |u_8|^2 + |u_9|^2 + |u_{10}|^2) \\ &\le |u_0|^2 + |u_0 - u_1|^2 + |u_1 - u_2|^2 + |u_2 - u_3|^2 + |u_3|^2 + |u_4|^2 + |u_4 - u_5|^2 \\ &\quad + |u_5 - u_6|^2 + |u_6 - u_7|^2 + |u_7|^2 + |u_8|^2 + |u_8 - u_9|^2 + |u_9 - u_{10}|^2 + |u_{10}|^2 \\ &\le \beta_-^{(3)}(10,3)(|u_0|^2 + |u_1|^2 + |u_2|^2 + |u_3|^2 + |u_4|^2 + |u_5|^2 + |u_6|^2 + |u_7|^2 \\ &\quad + |u_8|^2 + |u_9|^2 + |u_{10}|^2). \end{aligned}$$

We can see that with the new variables inequality (3) is decomposed into the sum of three inequalities of the same type $i = 3$ with 4, 4, 3 variables and shift 1. In these inequalities the best constants are $\alpha_-^{(3)}(3,1)$, $\alpha_-^{(3)}(3,1)$, $\alpha_-^{(3)}(2,1)$ on the left side and the corresponding beta on the right side. This implies that

$$\alpha_-^{(3)}(10,3) = \min\{\alpha_-^{(3)}(3,1), \alpha_-^{(3)}(2,1)\} = \alpha_-^{(3)}(3,1) = \alpha_-^{(3)}\left(\left[\frac{10}{3}\right], 1\right)$$
$$\beta_-^{(3)}(10,3) = \max\{\beta_-^{(3)}(3,1), \beta_-^{(3)}(2,1)\} = \beta_-^{(3)}(3,1) = \beta_-^{(3)}\left(\left[\frac{10}{3}\right], 1\right)$$

as clearly $\alpha_\pm^{(i)}(n,k)$ is decreasing $\beta_\pm^{(i)}(n,k)$ is increasing in n.

This method works (essentially with the same proof as above) in the unweighted case. Therefore, if $p_i = r_i = 1$ for all possible indices then we have for $i = 1, 2, 3$

$$\alpha_{\pm}^{(i)}(n,k) = \alpha_{\pm}^{(i)}\left(\left[\frac{n}{k}\right],1\right), \qquad \beta_{\pm}^{(i)}(n,k) = \beta_{\pm}^{(i)}\left(\left[\frac{n}{k}\right],1\right).$$

In the weighted case this method works only if $\frac{n+1}{k}$ is an integer and the weights p_i, r_i satisfy certain proportionality conditions.

Theorem 1. *Let n, k be fixed natural numbers with $1 \le k \le n$, let $\frac{n+1}{k} = q$ be an integer, further p_i $(i = 0, \dots, q)$, r_i, $(i = -1, 0, \dots, q)$ be arbitrary given fixed positive weights satisfying the conditions*

$$\begin{aligned} p_{j+ik} &:= c(j)p_i && \text{if } j = 0, \dots, k-1;\ i = 0, \dots, q-1 \\ r_{j+ik} &:= c(j)r_i && \text{if } j = 0, \dots, k-1;\ i = -1, 0, \dots, q-1 \end{aligned} \tag{5}$$

where $c(j)$ $(j = 0, \dots, k-1)$ are arbitrary positive numbers (i.e. the weights p_{j+ik}, r_{j+ik} for $j = 0, \dots, k-1$ are proportional with the given weights p_i, r_i, the numbers $c(j)$ being the proportionality factors). Then for the best constants in the inequality

$$\alpha_{\pm}^{(i)}(n,k)\sum_{j=0}^{n} p_j|x_j|^2 \le \sum_j{}^{(i)} r_j|x_j \pm x_{j+k}|^2 \le \beta_{\pm}^{(i)}(n,k)\sum_{j=0}^{n} p_j|x_j|^2 \tag{6}$$

we have

$$\alpha_{\pm}^{(i)}(n,k) = \alpha_{\pm}^{(i)}\left(\left[\frac{n}{k}\right],1\right), \qquad \beta_{\pm}^{(i)}(n,k) = \beta_{\pm}^{(i)}\left(\left[\frac{n}{k}\right],1\right) \tag{7}$$

where $\alpha_{\pm}^{(i)}\left(\left[\frac{n}{k}\right],1\right)$, $\beta_{\pm}^{(i)}\left(\left[\frac{n}{k}\right],1\right)$ are the best constants in the inequality

$$\alpha_{\pm}^{(i)}\left(\left[\frac{n}{k}\right],1\right)\sum_{j=0}^{[\frac{n}{k}]} p_j|x_j|^2 \le \sum_j{}^{(i)} r_j|x_j \pm x_{j+k}|^2 \le \beta_{\pm}^{(i)}\left(\left[\frac{n}{k}\right],1\right)\sum_{j=0}^{[\frac{n}{k}]} p_j|x_j|^2.$$

PROOF. Introducing new variables u_s by

$$u_{j(q+1)+i} := x_{j+ik} \quad \text{if} \quad \begin{array}{l} j = 0, \dots, r-1;\ i = 0, \dots, q \\ j = r, \dots, k-1;\ i = 0, \dots, q-1. \end{array}$$

(6) is decomposed into the sum of k inequalities of $q = \left[\frac{n}{k}\right]$ variables with shift 1. Due to (5) the jth inequality can be obtained as the first one multiplied by $c(j)$. Therefore (7) holds. □

3. The general case and its reduction to eigenvalue problem

Let $y_j = \sqrt{p_j}x_j$ and

$$\langle \mathbf{y}, \mathbf{z} \rangle = \sum_{j=0}^{n} y_j \overline{z_j}$$

be the inner product in the complex unitary $n+1-$space for vectors $\mathbf{y} = (y_0, \dots, y_n)^T$, $\mathbf{z} = (z_0, \dots, z_n)^T$. Then the middle term of our inequality (3) changes to

$$\sum_j{}^{(i)} \left(\frac{r_j}{p_j}|y_j|^2 \pm \frac{r_j}{\sqrt{p_j p_{j+k}}}\, y_j \bar{y}_{j+k} \pm \frac{r_j}{\sqrt{p_j p_{j+k}}}\, \bar{y}_j y_{j+k} + \frac{r_j}{p_{j+k}}|y_{j+k}|^2 \right)$$

Writing this sum as an inner product $\langle M^{(i)}\mathbf{y}, \mathbf{y}\rangle$ with suitable $(n+1)\times(n+1)$ three-diagonal matrices $M^{(i)}$ $(i = 1, 2, 3)$ inequality (3) yields

$$\alpha_{\pm}^{(i)}\|\mathbf{y}\|^2 \le \left\langle M^{(i)}\mathbf{y}, \mathbf{y} \right\rangle \le \beta_{\pm}^{(i)}\|\mathbf{y}\|^2. \tag{8}$$

Assuming $n+1-2k \ge 0$ the matrix $M^{(i)}$ is of the form

$$\left(\begin{array}{ccc|ccc|ccc}
d_0^{(i)} & & & t_0 & & & & & \\
& \ddots & & & \ddots & & & & \\
& & d_{k-1}^{(i)} & & & & & & \\ \hline
t_0 & & & d_k^{(i)} & & & & & \\
& t_1 & & & d_{k+1}^{(i)} & & & & \\
& & \ddots & & \ddots & & \ddots & & \\
& & & \ddots & & d_{n-k+1}^{(i)} & & t_{n-k+1} & \\
& & & & & d_{n-k}^{(i)} & & & t_{n-k} \\ \hline
& & & & & & d_{n-k+1}^{(i)} & & \\
& & & & \ddots & & & \ddots & \\
& & & & & t_{n-k} & & & d_n^{(i)}
\end{array}\right)$$

where the elements

$$t_j = \pm\frac{r_j}{\sqrt{p_j p_{j+k}}} \quad (j = 0, \dots, n-k),$$

in the side diagonals do not depend on i. The entries $m_{jl}^{(i)}$ $(j,l=0,1,\dots,n)$ of $M^{(i)}$ are

$$m_{jl}^{(i)} = \{ \begin{cases} t_l & \text{if } \ j=l+k, \ l=0,\dots,n-k \\ 0 & \text{if } \ 0<|j-l|\neq k \\ t_j & \text{if } \ l=j+k, \ j=0,\dots,n-k \\ d_j^{(i)} & \text{if } \ l=j, \ j=0,\dots,n \end{cases}$$

where the elements in the main diagonal are given by

$$d_j^{(1)} = \begin{cases} \dfrac{r_j}{p_j} & \text{if } \ j=0,\dots,k-1, \\ \dfrac{r_j+r_{j-k}}{p_j} & \text{if } \ j=k,\dots,n-k, \\ \dfrac{r_{j-k}}{p_j} & \text{if } \ j=n-k+1,\dots,n, \end{cases} \qquad \text{if } n+1-k\geq 0,$$

$$d_j^{(1)} = \begin{cases} \dfrac{r_j}{p_j} & \text{if } \ j=0,\dots,n-k, \\ 0 & \text{if } \ j=n-k+1,\dots,k-1, \\ \dfrac{r_{j-k}}{p_j} & \text{if } \ j=k,\dots,n, \end{cases} \qquad \text{if } n+1-k<0,$$

$$d_j^{(2)} = \begin{cases} \dfrac{r_j}{p_j} & \text{if } \ j=0,\dots,k-1, \\ \dfrac{r_j+r_{j-k}}{p_j} & \text{if } \ j=k,\dots,n, \end{cases}$$

$$d_j^{(3)} = \begin{cases} \dfrac{r_j+r_{j-k}}{p_j} & \text{if } \ j=0,\dots,n. \end{cases}$$

Let now A be an Hermitean matrix of order $n+1$ with eigenvalues $\lambda_0\leq\lambda_1\leq\dots\lambda_n$ and let $\mathbf{z_0},\mathbf{z_1},\dots,\mathbf{z_n}$ be the corresponding eigenvectors.

It is known (see e.g. [2]) that then the inequality

$$\lambda_0\|\mathbf{y}\|^2\leq\langle A\mathbf{y},\mathbf{y}\rangle\leq\lambda_n\|\mathbf{y}\|^2 \tag{9}$$

holds for every vector $\mathbf{y}$ in the complex unitary $n+1$-space. Equality on the left side of (9) occurs if and only if $\mathbf{y}=\mathbf{0}$ or $\mathbf{y}$ is an eigenvector corresponding to λ_0 (if $\lambda_0<\lambda_1$, then $\mathbf{y}$ is a scalar multiple of $\mathbf{z_0}$). Similarly equality occurs on the right hand side of (9) if and only if $\mathbf{y}=\mathbf{0}$ or $\mathbf{y}$ is an eigenvector corresponding to λ_n.

Thus the best constants $\alpha_\pm^{(i)}$, $\beta_\pm^{(i)}$ in (3) are the minimal and maximal eigenvalues of $M^{(i)}$.

4. Eigenvalues and eigenvectors of three-diagonal matrices

The aim of this section is to find the eigenvalues and eigenvectors of three-diagonal matrices of the form

$$M(\mathbf{u},\mathbf{t}) = \left(\begin{array}{ccc|ccc|cc}
u_0 & & & t_0 & & & & \\
 & \ddots & & & \ddots & & & \\
 & & u_{k-1} & & & & & \\
\hline
t_0 & & & u_k & & & & \\
 & t_1 & & & u_{k+1} & & \ddots & \\
 & & \ddots & & & \ddots & & \ddots \\
 & & & & & u_{n-k} & & t_{n-k} \\
\hline
 & & & & & & u_{n-k+1} & \\
 & & & & \ddots & & & \ddots \\
 & & & & & t_{n-k} & & u_n
\end{array}\right).$$

where $\mathbf{t} = (t_0, \dots, t_{n-k})$, $\mathbf{u} = (u_0, \dots, u_n)$ and we assumed that $n + 1 - 2k \geq 0$.

Let $n + 1 = kq + r$ $(0 \leq r < k)$ and rearrange both the rows and columns of the matrix $M(\mathbf{u} - \lambda\mathbf{e}, \mathbf{t})$ where $\mathbf{e} = (1, \dots, 1)$ in the order of indices

$$\begin{aligned} &0, k, 2k, \dots, qk;\ 1, k+1, 2k+1, \dots, qk+1;\ \dots; \\ &\quad r-1, k+r-1, 2k+r-1, \dots, qk+r-1 \end{aligned} \tag{10}$$

$$\begin{aligned} &r, k+r, 2k+r, \dots, (q-1)k+r; r+1, k+r+1, 2k+r+1, \dots, (q-1)k+r+1; \dots; \\ &\quad k-1, 2k-1, 3k-1, \dots, (q-1)k+k-1. \end{aligned} \tag{11}$$

(10) contains r groups of $q+1$ indices while (11) has $k-r$ groups of q indices. If $r = 0$, the group (10) is empty and all indices are contained in (11). We used the same rearrangement to introduce new variables in the proof of Theorem 1. The rearrangement (10), (11) is not new, it has been applied by the author in [7] and much earlier by Egerváry and Szász [3].

It is easy to see that the rearranged matrix is of the form

$$\begin{pmatrix}
A_0(\lambda) & & & & & \\
 & \ddots & & & & \\
 & & A_{r-1}(\lambda) & & & \\
 & & & A_r(\lambda) & & \\
 & & & & \ddots & \\
 & & & & & A_{k-1}(\lambda)
\end{pmatrix} \tag{12}$$

where for $0 \le s \le r-1$ the matrices $A_s(\lambda)$ are $(q+1)\times(q+1)$ three-diagonal matrices of the form

$$\begin{pmatrix} u_s-\lambda & t_s & 0 & 0 & \dots & 0 & 0 & 0 \\ t_s & u_{k+s}-\lambda & t_{k+s} & 0 & \dots & 0 & 0 & 0 \\ 0 & t_{k+s} & u_{2k+s}-\lambda & t_{2k+s} & \dots & 0 & 0 & 0 \\ 0 & 0 & t_{2k+s} & u_{3k+s}-\lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & t_{(q-2)k+s} & u_{(q-1)k+s}-\lambda & t_{(q-1)k+s} \\ 0 & 0 & 0 & 0 & \dots & 0 & t_{(q-1)k+s} & u_{qk+s}-\lambda \end{pmatrix}$$

while for $r \le s \le k-1$ $A_s(\lambda)$ are $q\times q$ three-diagonal matrices of similar form (in the above formula q has to be replaced by $q-1$).

From (12) it follows that the determinant $D(\lambda,\mathbf{u},\mathbf{t})$ of the matrix $M(\mathbf{u}-\lambda\mathbf{e},\mathbf{t})$ can be obtained as

$$D(\lambda,\mathbf{u},\mathbf{t}) = \prod_{s=0}^{k-1} \det A_s(\lambda).$$

For $s = 0,\dots,r-1$; $j = 0,\dots,q$ and $s = r,\dots,k-1$; $j = 0,\dots,q-1$ define the sequence of polynomials $P_{s,j}$ by

$$\begin{cases} P_{s,0}(\lambda) = 1, \quad P_{s,1}(\lambda) = u_s - \lambda, \\ P_{s,j+1}(\lambda) = (u_{jk+s}-\lambda)P_{s,j}(\lambda) - t^2_{(j-1)k+s}P_{s,j-1}(\lambda) \quad (j = 1,2,\dots,q). \end{cases} \tag{13}$$

We claim that

$$\begin{cases} \det A_s(\lambda) = P_{s,q+1}(\lambda) & \text{if } 0 \le s \le r-1, \\ \det A_s(\lambda) = P_{s,q}(\lambda) & \text{if } r \le s \le k-1. \end{cases} \tag{14}$$

Denote namely the left upper $j\times j$ $(1 \le j \le q+1)$ subdeterminant of the matrix $A_s(\lambda)$ by $P_{s,j}(\lambda)$. Then $P_{s,1}(\lambda) = u_s - \lambda$. Expanding $P_{s,2}(\lambda)$ by the last column we can see that the recursive formula in the second line of (13) holds for $j = 1$, if we set $P_{s,0}(\lambda) = 1$.

Expanding the left upper $j\times j$ subdeterminant by its last column we can easily prove the recursive formula in the second line of (13) by induction. Thus (14) is justified.

Theorem 2. *The eigenvalues of the matrix $M(\mathbf{u},\mathbf{t})$ are the zeros of the polynomials*

$$P_{0,q+1}, P_{1,q+1}, \dots, P_{r-1,q+1}; \quad P_{r,q}, P_{r+1,q}, \dots, P_{k-1,q}.$$

Proof. Our statement follows from (14). □

To formulate the next result we introduce the notations

$$\begin{aligned} I_0 &= \{(0,q+1),(1,q+1),\dots,(r-1,q+1)\}\,,\\ I_1 &= \{(r,q),(r+1,q),\dots,(k-1,q)\}\,,\\ I &= I_0 \cup I_1. \end{aligned}$$

It is clear that I is exactly the set of indices of the polynomials in Theorem 2.

Theorem 3. *Suppose that $t_0 t_1 \cdots t_{n-k} \neq 0$ and the eigenvalue λ of $M(\mathbf{u},\mathbf{t})$ is such that*

$$\begin{aligned} P_{s,j}(\lambda) &= 0 \quad \text{for } (s,j) \in J(\lambda),\\ P_{s,j}(\lambda) &\neq 0 \quad \text{for } (s,j) \in I - J(\lambda) \end{aligned}$$

for some non-empty subset $J(\lambda)$ of I. Then the components of the eigenvectors $\mathbf{y} = (y_0,\dots,y_n)^T$ corresponding to λ are given by

$$y_{jk+s} = \begin{cases} P_{s,j}(\lambda)C_s \text{ for } j=0,\dots,q & \text{if } (s,q+1) \in J(\lambda)\cap I_0,\\ 0 \text{ for } j=0,\dots,q & \text{if } (s,q+1) \in (I-J(\lambda))\cap I_0,\\ P_{s,j}(\lambda)C_s \text{ for } j=0,\dots,q-1 & \text{if } (s,q) \in J(\lambda)\cap I_1,\\ 0 \text{ for } j=0,\dots,q-1 & \text{if } (s,q) \in (I-J(\lambda))\cap I_1, \end{cases} \tag{15}$$

where C_s are arbitrary constants.

Proof. The eigenvectors of $M(\mathbf{u},\mathbf{t})$ corresponding to λ are the solutions $\mathbf{y} = (y_0,\dots,y_n)^T$ of the equation

$$M(\mathbf{u}-\lambda\mathbf{e},\mathbf{t})\mathbf{y} = \mathbf{0}. \tag{16}$$

Rearranging the system (16) in the order of (10), (11) we can see that it decomposes to the systems

$$A_s(\lambda)\mathbf{y_s} = \mathbf{0} \quad (0 \le s \le k-1) \tag{17}$$

where

$$\begin{aligned} \mathbf{y_s} &= (y_s, y_{k+s},\dots,y_{qk+s})^T && \text{if } 0 \le s \le r-1,\\ \mathbf{y_s} &= (y_s, y_{k+s},\dots,y_{(q-1)k+s})^T && \text{if } r \le s \le k-1. \end{aligned}$$

The detailed form of (17) for $0 \le s \le r-1$ is

$$\begin{cases} (u_s-\lambda)y_s + t_s y_{k+s} = 0 \\ t_s y_s + (u_{k+s}-\lambda)y_{k+s} + t_{k+s}y_{2k+s} = 0 \\ \vdots \\ t_{(q-2)k+s}y_{(q-2)k+s} + (u_{(q-1)k+s}-\lambda)y_{(q-1)k+s} + t_{(q-1)k+s}y_{qk+s} = 0 \\ t_{(q-1)k+s}y_{(q-1)k+s} + (u_{qk+s}-\lambda)y_{qk+s} = 0. \end{cases} \tag{18}$$

From the first and second equation of (18) we get that

$$\begin{aligned} y_{k+s} &= -\frac{1}{t_s}[u_s-\lambda]y_s = -\frac{1}{t_s}P_{s,1}(\lambda)y_s, \\ y_{2k+s} &= -\frac{1}{t_{k+s}}\left[(u_{k+s}-\lambda)y_{k+s} + t_s y_s\right] \\ &= -\frac{1}{t_{k+s}}\left[(u_{k+s}-\lambda)\left(-\frac{1}{t_s}\right)P_{s,1}(\lambda)y_s + t_s P_{s,0}(\lambda)y_s\right] \\ &= \left(-\frac{1}{t_s}\right)\left(-\frac{1}{t_{k+s}}\right)\left[(u_{k+s}-\lambda)P_{s,1}(\lambda) - t_s^2 P_{s,0}(\lambda)\right]y_s \\ &= \left(-\frac{1}{t_s}\right)\left(-\frac{1}{t_{k+s}}\right)P_{s,2}(\lambda)y_s. \end{aligned}$$

By induction

$$y_{jk+s} = \left(-\frac{1}{t_s}\right)\left(-\frac{1}{t_{k+s}}\right)\cdots\left(-\frac{1}{t_{(j-1)k+s}}\right)P_{s,j}(\lambda)y_s \qquad (j=1,\dots,q;\ s=0,\dots,r-1). \tag{19}$$

Substituting $y_{(q-1)k+s}$, y_{qk+s} from (19) into the last equation of (18) and multiplying the equation obtained by $(-1)^q t_s t_{k+s}\dots t_{(q-1)k+s}$ we get

$$\left[-t_{(q-1)k+s}^2 P_{s,q-1}(\lambda) + (u_{qk+s}-\lambda)P_{s,q}(\lambda)\right]y_s = 0.$$

By (13) the expression in the bracket is exactly $P_{s,q+1}(\lambda)$ hence

$$P_{s,q+1}(\lambda)y_s = 0.$$

If $(s,q+1) \in J(\lambda)\cap I_0$ then $P_{s,q+1}(\lambda)=0$ hence $y_s = C_s^*$ is arbitrary and by (19)

$$y_{jk+s} = \left(-\frac{1}{t_s}\right)\left(-\frac{1}{t_{k+s}}\right)\cdots\left(-\frac{1}{t_{(j-1)k+s}}\right)P_{s,j}(\lambda)y_s = P_{s,j}(\lambda)C_s \quad (j=0,\dots,q)$$

where C_s are new arbitrary constants. If $(s, q+1) \in (I - J(\lambda)) \cap I_0$ then $P_{s,q+1}(\lambda) \neq 0$ hence $y_s = 0$ and

$$y_{jk+s} = \left(-\frac{1}{t_s}\right)\left(-\frac{1}{t_{k+s}}\right)\cdots\left(-\frac{1}{t_{(j-1)k+s}}\right)P_{s,j}(\lambda)y_s = 0 \quad (j = 0, \dots, q).$$

We can prove the second part of (15) in a similar way. □

5. The main result

Our main result is the consequence of Theorem 2 and Theorem 3. Corresponding to the entries of our matrix $M^{(i)}$ we have to modify the definitions of the polynomials $P_{s,j}$. Let the numbers t_j, $d_j^{(i)}$ defined at the end of Section 3.

Replacing in the definition (13) $P_{s,j}$ by $P_{s,j}^{(i)}$ and u_s by $d_s^{(i)}$ we get

$$\begin{cases} P_{s,0}^{(i)}(\lambda) = 1, \quad P_{s,1}^{(i)}(\lambda) = v_s^{(i)} - \lambda, \\ P_{s,j+1}^{(i)}(\lambda) = (v_{jk+s}^{(i)} - \lambda)P_{s,j}^{(i)}(\lambda) - t_{(j-1)k+s}^2 P_{s,j-1}^{(i)}(\lambda) \quad (j = 1, \dots, q). \end{cases} \tag{20}$$

Theorem 4. *Let n, k be fixed natural numbers, $1 \leq k \leq n$; $n+1 = kq + r$ $(0 \leq r < k)$, $p_0, \dots, p_n$ and $r_{-k}, \dots, r_{-1}, r_0, r_1, \dots, r_{n+k}$ be given positive numbers. The inequalities*

$$\alpha_{\pm}^{(i)} \sum_{j=0}^{n} p_j |x_j|^2 \leq \sum_j {}^{(i)} r_j |x_j \pm x_{j+k}|^2 \leq \beta_{\pm}^{(i)} \sum_{j=0}^{n} p_j |x_j|^2 \tag{21}$$

for $i = 1, 2, 3$ with $+$ or $-$ signs hold for every complex $(n+1)$-vector $\mathbf{x} = (x_0, \dots, x_n)^T$. For the best constants $\alpha_{\pm}^{(i)}, \beta_{\pm}^{(i)}$ we have

$$\alpha_+^{(i)} = \alpha_-^{(i)} = \lambda_{\min}$$

$$\beta_+^{(i)} = \beta_-^{(i)} = \lambda_{\max}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and the largest zeros of the polynomials

$$P_{0,q+1}^{(i)}, P_{1,q+1}^{(i)}, \dots, P_{r-1,q+1}^{(i)}; \quad P_{r,q}^{(i)}, P_{r+1,q}^{(i)}, \dots, P_{k-1,q}^{(i)}. \tag{22}$$

(defined by (20)). Equality holds on the left hand side of (21) if and only if

$$\mathbf{x} = \langle \mathbf{y}(\lambda_{\min}), \mathbf{p} \rangle,$$

equality holds on the right hand side of (21) *if and only if*

$$\mathbf{x} = \langle \mathbf{y}(\lambda_{max}), \mathbf{p} \rangle\,,$$

where $\mathbf{p} = \left(\frac{1}{\sqrt{p_1}}, \dots, \frac{1}{\sqrt{p_n}}\right)$ *and the components of the vectors* $\mathbf{y}(\lambda_{\min})$ *and* $\mathbf{y}(\lambda_{max})$ *can be obtained from* (15) *replacing* $P_{s,j}$ *by* $P^{(i)}_{s,j}$ *and* λ *by* $\lambda_{\min}$ *and* λ_{max} *respectively.*

It is remarkable that *the best constants do not depend on the sign* $\pm$ in the middle of our inequality. This is true since the $\pm$ signs appear only in the components of $\mathbf{t}$ but these components are always squared in the definition of $P^{(i)}_{s,j}$.

6. Examples

Using (7) for the unweighted case and some results of Fan–Taussky–Todd [4] we can get a simpler proof of theorem 4 of the author [6].

Corollary 5. *Let* n, k *be fixed natural numbers,* $1 \le k \le n$. *The inequalities*

$$\alpha^{(i)}_{\pm}(n,k) \sum_{j=0}^{n} |x_j|^2 \le \sum_j{}^{(i)} |x_j \pm x_{j+k}|^2 \le \beta^{(i)}_{\pm}(n,k) \sum_{j=0}^{n} |x_j|^2$$

for $i = 1,2,3$ *and with* $+$ *or* $-$ *signs hold for every* $x_j \in \mathbb{C}$ $(j = 0,\dots,n)$. *The best constants are given by*

$$\alpha^{(1)}_{\pm}(n,k) = 0, \qquad \beta^{(1)}_{\pm}(n,k) = 4\cos^2 \frac{\pi}{2\left([\frac{n}{k}]+1\right)},$$

$$\alpha^{(2)}_{\pm}(n,k) = 4\sin^2 \frac{\pi}{2\left(2[\frac{n}{k}]+3\right)}, \qquad \beta^{(2)}_{\pm}(n,k) = 4\cos^2 \frac{\pi}{2[\frac{n}{k}]+3},$$

$$\alpha^{(3)}_{\pm}(n,k) = 4\sin^2 \frac{\pi}{2\left([\frac{n}{k}]+2\right)}, \qquad \beta^{(3)}_{\pm}(n,k) = 4\cos^2 \frac{\pi}{2\left([\frac{n}{k}]+2\right)}.$$

Corollary 6. *Let* n, k *be fixed natural numbers,* $1 \le k \le n$ *and* $\frac{n+1}{k} = q$ *be an integer. Let* $r_i = i+1$ $(i = 0,\dots,q)$ *and suppose that*

$$r_{j+ik} = c(j) r_i \quad \text{if} \;\; j = 0,\dots,k-1; \;\; i = 0,\dots,q-1,$$

where $c(j)$ $(j = 0,\dots,k-1)$ *are arbitrary positive numbers. Then the best constants* $\alpha^{(i)}_{\pm}(n,k)$, $\beta^{(i)}_{\pm}(n,k)$ *in the inequality*

$$\alpha^{(i)}_{\pm}(n,k) \sum_{j=0}^{n} |x_j|^2 \le \sum_j{}^{(i)} r_j |x_j \pm x_{j+k}|^2 \le \beta^{(i)}_{\pm}(n,k) \sum_{j=0}^{n} |x_j|^2 \tag{23}$$

are the minimal and maximal zeros of the the polynomials

$$L_{[\frac{n}{k}]+1}(\lambda)+\frac{[\frac{n}{k}]-1}{[\frac{n}{k}]+1}L_{[\frac{n}{k}]}(\lambda) \quad \text{if } i=1,$$

$$L_{[\frac{n}{k}]+1}(\lambda) \quad \text{if } i=2,$$

where L_n is the Laguerre polynomial of degree n defined by

$$L_n(x)=\frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right)=\sum_{j=0}^{n}(-1)^j\binom{n}{n-j}\frac{x^j}{j!}.$$

Proof. Consider first the case $i=2$, $k=1$ with $p_j=1$, $r_j=j+1$, $j=(0,\dots,n)$. The matrix $A_0(\lambda)$ in (12) (with $t_j=j+1$, $u_j=v_j^{(2)}=2j+1$ $j=(0,\dots,n)$) has the form

$$\begin{pmatrix} 1-\lambda & \pm1 & 0 & \dots & 0 & 0 & 0\\ \pm1 & 3-\lambda & \pm2 & \dots & 0 & 0 & 0\\ 0 & \pm2 & 5-\lambda & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \pm(2n-3)-\lambda & \pm(n-1) & 0\\ 0 & 0 & 0 & \dots & \pm(n-1) & 2n-1-\lambda & \pm n\\ 0 & 0 & 0 & \dots & 0 & \pm n & 2n+1-\lambda. \end{pmatrix} \tag{24}$$

□

Lemma 7. *For the principal minors $P_{0,j}^{(2)}(\lambda)$ of order j of the matrix* (24) *we have*

$$P_{0,j}^{(2)}(\lambda)=j!L_j(\lambda) \quad (j=1,\dots,n+1). \tag{25}$$

Proof. If we set $P_{0,0}^{(2)}(\lambda)=1$, we see that $P_{0,j}^{(2)}(\lambda)$ satisfies (13), i.e.

$$\begin{cases} P_{0,0}^{(2)}(\lambda)=1, \quad P_{0,1}^{(2)}(\lambda)=1-\lambda,\\ P_{0,j+1}^{(2)}(\lambda)=(2j+1-\lambda)P_{0,j}^{(2)}(\lambda)-j^2P_{0,j-1}^{(2)}(\lambda) \quad (j=1,2,\dots,n). \end{cases}$$

Comparing this with the recurrence relation satisfied by the Laguerre polynomials L_j (see e.g. [14] p. 101, (5.1.10))

$$\begin{cases} L_0(\lambda)=1, \quad L_1(\lambda)=1-\lambda,\\ (j+1)L_{j+1}(\lambda)=(2j+1-\lambda)L_j(\lambda)-jL_{j-1}(\lambda) \quad (j=1,2,\dots,n), \end{cases}$$

we see that multiplying the last equation by $j!$ the recurrence equations for $P_{0,j}^{(2)}$ and $j!L_j$ are identical, hence (25) holds.

In the case $i=1$, $k=1$ with $p_j=1$, $r_j=j+1$, $(j=0,\dots,n)$ the matrix $A_0(\lambda)$ in (12) (with $t_j=j+1$, $u_j=v_j^{(1)}=2j+1$ $(j=0,\dots,n-1)$, $u_n=v_n^{(1)}=n$) is the same as the matrix (24) except that the last element of the main diagonal changes from $2n+1-\lambda$ to $n-\lambda$. Thus for the principal minors $P_{0,j}^{(1)}(\lambda)$ of order j of the modified matrix we have

$$P_{0,j}^{(1)}(\lambda)=j!L_j(\lambda)\quad(j=1,\dots,n)$$
$$P_{0,n+1}^{(1)}(\lambda)=(n-\lambda)P_{0,n}^{(1)}(\lambda)-n^2P_{0,n-1}^{(1)}(\lambda).$$

From the last equation applying the recursion formula $nL_{n-1}=(2n+1-\lambda)L_n-(n+1)L_{n+1}$ we get

$$\begin{aligned}P_{0,n+1}^{(1)}(\lambda)&=(n-\lambda)n!L_n(\lambda)-n^2(n-1)!L_{n-1}(\lambda)\\&=(n-\lambda)n!L_n(\lambda)-n!\left[(2n+1-\lambda)L_n(\lambda)-(n+1)L_{n+1}(\lambda)\right]\\&=(n+1)!\left[L_{n+1}(\lambda)+\frac{n-1}{n+1}L_n(\lambda)\right]\end{aligned}$$

Applying this, Lemma 1 and Theorem 1 we get the best constants in (23) if $i=1,2$, completing the proof of Corollary 2. □

For $i=3$ with $r_i=i+1$ $(i=-1,0,\dots,q)$ the positivity requirement of the weights r_i is not satisfied. However for $i=3$ formally we get exactly the same inequality as for $i=2$, thus the best constants are the same as for $i=2$.

We remark that results related to Corollary 2 were found by G. V. Milovanović and I. Ž. Milovanović [9], (Corollaries 1 and 3 corresponding to our cases $k=1$ with negative sign, $i=1$ and $i=2$ respectively).

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