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Killing vector fields on compact Finsler manifolds

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Abstract. In this paper, we obtain the Weitzenböck type formula for Killing vector fields on a compact Finsler manifold. By using the "Bochner technique", we prove that Killing vector fields are parallel or vanish identically under certain curvature condition and other extra condition. In particular, we discuss Killing vector fields on some compact special Finsler manifolds. Moreover, we prove that the number of the independent Killing vector fields in a Minkowskian space is equal to the dimension of a Minkowskian space.

1. Introduction

Bochner [6]–[9] initiated a method, the well-known "Bochner technique", which used the Laplace operator and the general maximum principle of Hopf to deal with the relation between vector or tensor fields and the curvature of manifolds, and got the global properties of manifolds. From then on, the Bochner technique became a very useful method in geometrical study. Such as, both in Riemannian and Kählerian manifolds, the Bochner technique has been discussed in details in [10], [32], [19]. The Bochner technique is to integrate the Laplacian of the pointwise square norm of a harmonic form over a compact Riemannian manifolds, yielding thereby two terms. One is the global square norm of the covariant derivatives of the harmonic form. The other involves the curvature tensor. Under the suitable assumption of the curvature tensor, it can be obtained that the harmonic form must be zero or parallel. In the papers of Bochner [6]–[9], he

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obtained some vanishing theorems for harmonic forms and Killing vector fields. In the generalized space, the Killing equations were obtained by HOKARI [16], KNEBELMAN [18] and SOÓS [30]. Recently, under the initiation of S. S. Chern, the global differential geometry of real and complex Finsler manifolds has gained a great development [13]–[26]. S. S. CHERN has pointed out that "complex Finsler geometry is very important in the research of complex analysis in several complex variables, since on every complex manifold with or without boundary there exist a Carathéodory metric and a Kobayashi metric, and under proper condition they are $C^{(2)}$ metrics, and the most important fact is that naturally they are Finsler metrics, ..., to extend harmonic integral to the case of Finslerian will be a new research region of differential geometry, and we expect the prospects are boundless" [13], [14]. T. D. ZHONG and C. P. ZHONG [33] and J. X. XIAO, T. D. ZHONG and C. H. QIU [23] discussed the Bochner technique in real Finsler manifolds and strongly Kähler–Finsler manifolds, respectively. J. L. LI, C. H. QIU and T. D. ZHONG [21] researched an extension of Hodge theorem to the natural projection of complex horizontal Laplacian on complex Finsler manifolds. In addition, there appeared some papers about using complex connections of Finsler geometry to research the theory of integral representation of functions in several complex variables on complex Finsler manifolds [24]. In this paper, we try to discuss the Killing vector fields and we can obtain the Weitzenböck type formula for Killing vector fields on a compact Finsler manifold. By using the "Bochner technique", we prove that Killing vector fields are parallel or vanish identically under certain curvature condition and other extra condition. In particular, we discuss Killing vector fields on some compact special Finsler manifolds.

2. Preliminaries

Let M be a smooth manifold of dimension n and $\pi : TM \to M$ be the tangent bundle of M. We denote by $\tilde{M}$ the complement of zero section $o(M)$ in TM and $\tilde{\pi} : \tilde{M} \to M$ is the slit tangent bundle of M. Our geometrical objects are sections of the pulled-back bundle $\tilde{\pi}^*TM$ or its dual $\tilde{\pi}^*T^*M$, or their tensor products, where $\tilde{\pi}^*TM := \{(v, w) \in \tilde{M} \times TM \mid \tilde{\pi}(v) = \pi(w)\}$ is the sub-bundle of the bundle $\tilde{M} \times TM$.

Let $\{\pi^{-1}(U), (x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^n)\}$ be the local coordinates on TM induced by the covering of the system of local coordinate neighborhoods $\{U, u = (u^1, \ldots, u^n)\}$ on M, where

$$x^i := u^i \circ \pi, y^i(v) := v(u^i)(v \in \pi^{-1}(U)).$$

In the above local coordinates, if $p \in M$ and $v \in T_pM$, then

$$v = y^i(v)\left(\frac{\partial}{\partial u^i}\right)_p \in T_pM; \quad \left(\frac{\partial}{\partial x^i}\right)_v, \left(\frac{\partial}{\partial y^i}\right)_v \in T_vTM.$$

And if we denote by $\mathfrak{X}(M)$ the space of smooth vector fields on M, then

$$\frac{\partial}{\partial u^i} \in \mathfrak{X}(U); \quad \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} \in \mathfrak{X}(\pi^{-1}(U)).$$

Every vector field X on M induces a Finsler vector $\widehat{X} = \tilde{\pi}^* X$ on $\tilde{M}$ such that

$$\widehat{X}(v) := (v, X(\pi(v)) \quad \text{for all } v \in \tilde{M}.$$

In general, we denote by $\widehat{T} := \tilde{\pi}^* T$ the pull-back tensor field of any tensor field T on M. Local coordinates $\{U, u = (u^1, \ldots, u^n)\}$ on M produce the basis sections $\{\frac{\partial}{\partial u^i}\}$ and $\{du^i\}$, respectively, for TM and T^*M. So $\widehat{\frac{\partial}{\partial u^i}}$ is the local section of the pulled-back bundle $\tilde{\pi}^* TM$, and we denote by $\widehat{du^i} := \tilde{\pi}^*(du^i)$ the local section of $\tilde{\pi}^* T^*M$. If f is a function on M, we consider its vertical lift $f^{\mathsf{v}} := f \circ \pi$ on TM.

A function $F : TM \to \mathbb{R}$ is called a (positive definite) Finsler metric if the following conditions are satisfied [22]:

(i) F is continuous on TM and smooth on $\tilde{M}$;

(ii) F is positive-homogeneous of degree 1;

(iii) $F(v) > 0$ for all $v \neq 0$;

(iv) The fundamental tensor g, defined locally by its components

$$g_{ij} := \frac{1}{2}\frac{\partial^2 E}{\partial y^i \partial y^j}, E = F^2,$$

is positive definite.

Remark 2.1. In fact, it suffices to assume that g is fibrewise non-degenerate (see, e.g. [17], [31]).

A manifold M endowed with a Finsler metric is called a Finsler manifold. Now, we give some notations and preliminary knowledge on Finsler manifolds.

The Cartan tensor is given by

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4}\frac{\partial^3 E}{\partial y^i \partial y^j \partial y^k},$$

which is homogeneous of degree -1 and symmetric in all three of its indices. And we obtain

$$C_{ijk}y^i = C_{ijk}y^j = C_{ijk}y^k = 0. \tag{1}$$

The formal Christoffel symbols γ^i_{jk} are given by

$$\gamma^i_{jk} = \frac{1}{2}g^{ih}\left(\frac{\partial g_{hj}}{\partial x^k} + \frac{\partial g_{kh}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h}\right),$$

where (g^{ij}) is the inverse matrix of (g_{ij}). In local coordinates, the equations of the geodesics can be written in the form

$$\frac{d^2x^i}{d^2s} + 2G^i\left(x, \frac{dx}{ds}\right) = 0,$$

where

$$G^i = \frac{1}{2}\gamma^i_{jk}y^jy^k$$

is homogeneous of degree two in y^i's. The successive derivatives of G^i with respect to u are denoted by

$$G^i_j = \frac{\partial G^i}{\partial y^j}$$

and

$$G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}, \tag{2}$$

where G^i_j is the Christoffel symbol of the non-linear connection $\tilde{D}$ associated to the Cartan connection [1] and G^i_{jk} is the Berwald curvature tensor of the Berwald derivative [29], [4]. And for any $\xi \in \mathcal{X}(M)$, we have

$$\tilde{D}\xi = \xi^k{}_{;h}dx^h \otimes \frac{\partial}{\partial x^k} = \left[\frac{\partial \xi^k}{\partial u^h} + G^k_h \circ \xi\right] dx^h \otimes \frac{\partial}{\partial x^k},$$

where

$$G^k_h \circ \xi : U \longrightarrow \tilde{\pi}^{-1}(U) \longrightarrow \mathbb{R}, p \longmapsto G^k_h(\xi(p)).$$

Let $T = T^{i_1\cdots i_p}_{j_1\cdots j_q}\widehat{\frac{\partial}{\partial u^{i_1}}}\otimes\cdots\otimes\widehat{\frac{\partial}{\partial u^{i_p}}}\otimes\widehat{du^{j_1}}\otimes\cdots\otimes\widehat{du^{j_q}}$ be an arbitrary smooth local section of $\underbrace{\pi^*TM \otimes \cdots \otimes \pi^*TM}_{p} \otimes \underbrace{\pi^*T^*M \otimes \cdots \otimes \pi^*T^*M}_{q}$. Then the horizontal covariant derivative $T^{i_1\cdots i_p}_{j_1\cdots j_q\,|k}$ in the sense of Cartan [11] is given by

$$T^{i_1\cdots i_p}_{j_1\cdots j_q\,|k} = \frac{\partial T^{i_1\cdots i_p}_{j_1\cdots j_q}}{\partial x^k} - \frac{\partial T^{i_1\cdots i_p}_{j_1\cdots j_q}}{\partial y^l}G^l_k + T^{i_1\cdots i_{\alpha-1}li_{\alpha+1}\cdots i_p}_{j_1\cdots j_q}\Gamma^{i_\alpha}_{lk} - T^{i_1\cdots i_p}_{j_1\cdots j_{\beta-1}lj_{\beta+1}\cdots j_q}\Gamma^l_{j_\beta k},$$

where the subscript $|k$ denotes the horizontal covariant derivative with respect to the Cartan connection, and the Cartan connection coefficients Γ^i_{kj} are given by

$$\Gamma^i_{kj} = g^{ih}\Gamma_{khj}, \tag{3}$$

and

$$\Gamma_{kij} = \gamma_{kij} - C_{jih}\frac{\partial G^h}{\partial y^k} - C_{kih}\frac{\partial G^h}{\partial y^j} + C_{kjh}\frac{\partial G^h}{\partial y^i}, \tag{4}$$

so we have

$$G^i_j = \Gamma^i_{kj} y^k. \tag{5}$$

The curvature tensor R^i_{jhk} induced by the horizontal part of Cartan connection is defined by

$$R^i_{jhk} = \left(\frac{\partial \Gamma^i_{jh}}{\partial x^k} - \frac{\partial \Gamma^i_{jh}}{\partial y^l}\frac{\partial G^l}{\partial y^k}\right) - \left(\frac{\partial \Gamma^i_{jk}}{\partial x^h} - \frac{\partial \Gamma^i_{jk}}{\partial y^l}\frac{\partial G^l}{\partial y^h}\right) + \Gamma^i_{mk}\Gamma^m_{jh} - \Gamma^i_{mh}\Gamma^m_{jk}. \tag{6}$$

And the third curvature tensor $\widetilde{R}^i_{jhk}$ induced by the Cartan connection is the form

$$\widetilde{R}^i_{jhk} = R^i_{jhk} + C^i_{jm}R^m_{rhk}y^r, \tag{7}$$

where

$$C^i_{jk} = g^{ih}C_{jhk}.$$

From (1) and (7), we have

$$\widetilde{R}^i_{jhk}y^j = R^i_{jhk}y^j.$$

The horizontal covariant derivative $T^{i_1\cdots i_p}_{j_1\cdots j_q\,(k)}$ in the sense of Berwald [4] is given by

$$T^{i_1\cdots i_p}_{j_1\cdots j_q\,(k)} = \frac{\partial T^{i_1\cdots i_p}_{j_1\cdots j_q}}{\partial x^k} - \frac{\partial T^{i_1\cdots i_p}_{j_1\cdots j_q}}{\partial y^l}G^l_k + T^{i_1\cdots i_{\alpha-1} l i_{\alpha+1}\cdots i_p}_{j_1\cdots j_q}G^{i_\alpha}_{lk} - T^{i_1\cdots i_p}_{j_1\cdots j_{\beta-1} l j_{\beta+1}\cdots j_q}G^l_{j_\beta k},$$

where the subscript (k) denotes the horizontal covariant derivative with respect to the Berwald connection, and the Berwald connection coefficient G^i_{jk} is given by (2). The hh-curvature tensor induced by the Berwald connection is given by

$$H^i_{jhk} = \left(\frac{\partial G^i_{jh}}{\partial x^k} - G^i_{jhl}\frac{\partial G^l}{\partial y^k}\right) - \left(\frac{\partial G^i_{jk}}{\partial x^h} - G^i_{jkl}\frac{\partial G^l}{\partial y^h}\right) + G^i_{mk}G^m_{jh} - G^i_{mh}G^m_{jk},$$

where

$$G^i_{jhl} = \frac{\partial G^i_{jh}}{\partial y^l}.$$

Let

$$A^i_{jk} = FC^i_{jk},$$

and

$$l^i = \frac{y^i}{F}.$$

Note that the relation between the two connections is [25]

$$G^k_{ij} = \Gamma^k_{ij} + C^k_{ij|h} y^h = \Gamma^k_{ij} + A^k_{ij|h} l^h, \tag{8}$$

where the subscript $|h$ denotes the horizontal covariant derivative with respect to the Cartan connection.

Let

$$A_{ijk} = FC_{ijk},$$

we have

$$g_{ij(k)} = -2A_{ijk|h} l^h, \tag{9}$$

and

$$2A_{ijk|h} y^i = 0. \tag{10}$$

So

$$g_{ij(k)} y^i = 0.$$

The Riemann curvature tensor R^j_k naturally arises from the geodesic variation of geodesics [25], which is defined by

$$R^i_k := R^i_{jhk} y^j y^h.$$

From (3), (4) and (6), we have

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^h \partial y^k} y^h + 2G^i_{kl} G^l - \frac{\partial G^i}{\partial y^l} \frac{\partial G^l}{\partial y^k}.$$

Now we set

$$R_{ijhk} = g_{rj} R^r_{ihk}, \widetilde{R}_{ijhk} = g_{rj} \widetilde{R}^r_{ihk}, H_{ijhk} = g_{rj} H^r_{ihk},$$

and

$$R_{ij} = R^k_{ijk}, \widetilde{R}_{ij} = \widetilde{R}^k_{ijk}, H_{ij} = H^k_{ijk}.$$

The Ricci curvature is defined by

$$\mathrm{Ric} := R^k_k,$$

and it has the following properties [25]

$$\mathrm{Ric} = R_{ij} y^i y^j = \widetilde{R}_{ij} y^i y^j = H_{ij} y^i y^j.$$

3. Weitzenböck type formula for the Killing vector fields on compact Finsler manifolds

For any real-valued smooth function $f \in C^\infty(TM)$, it is easy to find that the local form $g^{ij} f_{(i)(j)}$ is well defined on the whole tangent bundle $\tilde{M}$. Now, let $M_X = \{p \in M \mid X(p) \neq 0\}$ for any $X \in \mathfrak{X}(M)$. Therefore, for any $f \in C^\infty(M_X) \cap C^0(M)$, there is an operator defined locally by

$$\Delta_X f := \begin{cases} g^{ij} f^{\mathbf{v}}_{(i)(j)} \circ X & \text{on } M_X, \\ 0 & \text{on } M \backslash M_X, \end{cases}$$

The above operator is well defined on the whole manifold M, locally, we have

$$\triangle_X f = g^{ij} \left[\left(\frac{\partial^2 f}{\partial u^i \partial u^j} \right)^{\mathbf{v}} - G^h_{ij} \left(\frac{\partial f}{\partial u^h} \right)^{\mathbf{v}} \right] \circ X \quad \text{on } U \cap M_X. \tag{11}$$

Here $g^{ij} \circ X$ and $G^h_{ij} \circ X$ are smooth functions on $U \cap M_X$, so $\triangle_X f$ is an elliptic operator with smooth coefficients on $U \cap M_X$.

Theorem 3.1. *Let M be a compact Finsler manifold and $\xi \in \mathfrak{X}(M)$. If ξ satisfies*

$$\triangle_\xi |\xi|^2 \geq 0$$

on M_ξ, then $|\xi|^2 = const$ and

$$\triangle_\xi |\xi|^2 = 0$$

everywhere on M, where $|\xi|^2$ is the square of the length of the vector field ξ given by

$$|\xi|^2 = F^2(\xi) = (g_{ij} \circ \xi) \xi^i \xi^j.$$

Proof. Let m be the maximum value of $|\xi|^2$ on M and $V = \{p \in M \mid |\xi|^2(p) = m\}$. Since M is compact, we have $V \neq \varnothing$. It is only to prove that $V = M$ when m is not equal to zero.

Since $f = |\xi|^2$ is a continuous function on M, V is the closed subset of M. For any point $p \in V$, that is $|\xi|^2(p) = m \neq 0$, there is an open local coordinate neighborhood U of p such that $U \subseteq M_\xi$. So $\triangle_\xi f$ is an elliptic operator on U, and f has the maximum value at p in U, then using the maximum principle of Hopf([10]), we have $|\xi|^2 = |\xi|^2(p) = m$ in U, that is $U \subset V$, so V is an open subset of M. We obtain that V is a closed and open subset of M, then $V = M$. □

Let $f = |\xi|^2 \in C^\infty(M_\xi)$, we compute (11) on $U \cap M_\xi$. Locally, we use the symbol $(\xi^k)^{\mathbf{v}}$ to denote the components of the tensor field $\widehat{\xi} = \tilde{\pi}^*\xi$. And we set

$$\gamma_{ij} = g_{ij} \circ \xi = g_{ij}(\xi); \quad \gamma^{ij} = g^{ij} \circ \xi = g^{ij}(\xi) \quad \text{on } U \cap M_\xi.$$

So

$$\begin{aligned}\triangle_\xi |\xi|^2 &= g^{ij}(\gamma_{kl}\xi^k\xi^l)^{\mathbf{v}}_{(i)(j)} \circ \xi \\ &= 2\gamma^{ij}\gamma_{kl}((\xi^k)^{\mathbf{v}}_{(i)} \circ \xi)((\xi^l)^{\mathbf{v}}_{(j)} \circ \xi) + 2\gamma^{ij}\gamma_{kl}((\xi^k)^{\mathbf{v}}_{(i)(j)} \circ \xi)\xi^l \\ &\quad + \gamma^{ij}([\gamma_{kl\,(i)}\xi^k\xi^l])^{\mathbf{v}}_{(j)} \circ \xi + 2\gamma^{ij}((\gamma_{kl})^{\mathbf{v}}_{(j)} \circ \xi)((\xi^k)^{\mathbf{v}}_{(i)} \circ \xi)\xi^l. \end{aligned} \tag{12}$$

Because

$$(\gamma_{kl})^{\mathbf{v}}_{(j)} = \frac{\partial g_{kl}}{\partial x^j} \circ \xi + \left(\frac{\partial g_{kl}}{\partial y^m} \circ \xi\right) \frac{\partial (\xi^m)^{\mathbf{v}}}{\partial x^j} - \gamma_{ml}G^m_{kj} - \gamma_{km}G^m_{lj}, \tag{13}$$

and

$$g_{kl\,(j)} = \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{kl}}{\partial y^m}\frac{\partial G^m}{\partial y^j} - g_{ml}G^m_{kj} - g_{km}G^m_{lj}, \tag{14}$$

from (13), (14) and (9), we have

$$(\gamma_{kl})^{\mathbf{v}}_{(j)} \circ \xi = 2(C_{klm} \circ \xi)\left(\frac{\partial \xi^m}{\partial u^j} + \frac{\partial G^m}{\partial y^j} \circ \xi\right) - 2(A_{klj|h} \circ \xi)(l^h \circ \xi). \tag{15}$$

From (5), (8) and (10), we have

$$\xi^m{}_{;j} = \frac{\partial \xi^m}{\partial u^j} + \frac{\partial G^m}{\partial y^j} \circ \xi = (\xi^m)^{\mathbf{v}}_{(j)} \circ \xi. \tag{16}$$

From (15) and (16), it follows that

$$(\gamma_{kl})^{\mathbf{v}}_{(j)} \circ \xi = 2(C_{klm} \circ \xi)\xi^m{}_{;j} - 2(A_{klj|h} \circ \xi)(l^h \circ \xi),$$

so

$$((\gamma_{kl})^{\mathbf{v}}_{(j)} \circ \xi)\xi^l = 0. \tag{17}$$

By the similar calculation of (15), we can obtain

$$\begin{aligned}&[((\gamma_{kl})^{\mathbf{v}}_{(i)} \circ \xi)(\xi^k)^{\mathbf{v}}(\xi^l)^{\mathbf{v}}]_{(j)} \circ \xi - ((\gamma_{kl})^{\mathbf{v}}_{(i)}(\xi^k)^{\mathbf{v}}(\xi^l)^{\mathbf{v}})_{(j)} \circ \xi \\ &\qquad = \left[\frac{\partial((\gamma_{kl})^{\mathbf{v}}_{(i)}(\xi^k)^{\mathbf{v}}(\xi^l)^{\mathbf{v}})}{\partial y^m} \circ \xi\right]\xi^m{}_{;j}. \end{aligned} \tag{18}$$

From (17) and (18), we have

$$\gamma^{ij}([\gamma_{kl\,(i)}\xi^k\xi^l])^{\mathbf{v}}_{(j)} \circ \xi = 2\gamma^{ij}\gamma_{ml}(G^m_{kih} \circ \xi)\xi^h{}_{;j}\xi^k\xi^l.$$

Since G^m_{hi} is homogeneous of degree 0, we have $(G^m_{kih} \circ \xi)\xi^k = \big(\frac{\partial G^m_{hi}}{\partial y^k} y^k\big) \circ \xi = 0$.

The above calculation gives the formula

$$\frac{1}{2}\triangle_\xi |\xi|^2 = g^{ij}(\xi)g_{kl}(\xi)\xi^k{}_{;i}\xi^l{}_{;j} + g^{ij}(\xi)g_{kl}(\xi)(\xi^k)^{\mathbf{v}}_{(i)(j)}(\xi)\xi^l \quad \text{on } M_\xi.$$

In the following, we discuss Killing vector fields on Finsler manifold (M, F). A vector field ξ on M is called a Killing vector field on (M, F) if its one-parameter group $(\varphi_t)_{t\in\mathbb{R}}$ consists of Finslerian isometries, i.e.,

$$F \circ (\varphi_t)_* = F \quad \text{for all } t \in \mathbb{R},$$

where $(\varphi_t)_* : TM \longrightarrow TM$ is the derivative of a smooth mapping $M \longrightarrow M$. Since M is compact, the vector field ξ is complete. It can be easily seen that ξ is a Killing vector field on (M, F) if and only if $\xi^c F = 0$, where ξ^c is a complete lift of ξ.

Let ξ be a Killing vector field on an n-dimensional Finsler manifold M, it must satisfy the Killing equations [25]

$$g_{ih}(\xi^h)^{\mathbf{v}}_{(j)} + g_{jh}(\xi^h)^{\mathbf{v}}_{(i)} + g_{ij\,(k)}(\xi^k)^{\mathbf{v}} + 2C_{ijh}(\xi^h)^{\mathbf{v}}_{(r)}y^r = 0,$$

or

$$g_{ih}(\xi^h)^{\mathbf{v}}_{|j} + g_{jh}(\xi^h)^{\mathbf{v}}_{|i} + 2C_{ijh}(\xi^h)^{\mathbf{v}}_{|r}y^r = 0.$$

However, we recall that a motion carries a geodesic into a geodesic. We have the equations [18]

$$(\xi^i)^{\mathbf{v}}{}_{(j)(k)} + H^i_{jkh}\xi^h + G^i_{jkh}(\xi^h)^{\mathbf{v}}_{(r)}y^r = 0.$$

Thus the greatest number of linearly independent motions which may be admitted by a Finsler manifold M of dimensional n is $\frac{1}{2}n(n+1)$ [18].

On the subset $U \cap M_\xi$, we have

$$g^{ij}(\xi)g_{kl}(\xi)(\xi^k)^{\mathbf{v}}_{(i)(j)}(\xi)\xi^l = -g^{ij}(\xi)H_{iljh}(\xi)\xi^h\xi^l + \phi_\xi(\xi),$$

where

$$\phi_\xi = -g^{ij}g_{kl}G^k_{ijh}(\xi^h)^{\mathbf{v}}_{(r)}y^r y^l \quad \text{on } \tilde{M},$$

and

$$\phi_\xi(\xi) = -(g^{ij} \circ \xi)(g_{kl} \circ \xi)(G^k_{ijh} \circ \xi)((\xi^h)^{\mathbf{v}}_{(r)} \circ \xi)\xi^r\xi^l. \tag{19}$$

Note that ([25])

$$H_{iljk}y^i = -H_{lijk}y^i = R_{iljk}y^i = \tilde{R}_{iljk}y^i,$$

and

$$H_{iljk} = -H_{ilkj},$$

we have

$$g^{ij}(\xi)H_{iljh}(\xi)\xi^h\xi^l = (H^j_{lhj}y^l y^h)\circ\xi = (H_{lh}y^l y^h)\circ\xi = \mathrm{Ric}\circ\xi = \mathrm{Ric}(\xi) \quad \text{on } M_\xi.$$

Therefore, we have the following Weitzenböck type formula.

Theorem 3.2. *Let M be a compact Finsler manifold. Then for any Killing vector field ξ on M, we have the Weitzenböck type formula*

$$\frac{1}{2}\triangle_\xi|\xi|^2 = g^{ij}(\xi)g_{kl}(\xi)\xi^k{}_{;i}\xi^l{}_{;j} - \mathrm{Ric}(\xi) + \phi_\xi(\xi) \quad \text{on } M_\xi,$$

where $\phi_\xi(\xi)$ is defined by (19).

On the subset M_ξ of M, we know that

$$g^{ij}(\xi)g_{kl}(\xi)\xi^k{}_{;i}\xi^l{}_{;j}$$

is a positive definite form in $\xi^l{}_{;j}$. Therefore, if ξ satisfies

$$\mathrm{Ric}(\xi) \le 0 \quad \text{on } M_\xi,$$

and

$$\phi_\xi(\xi) \ge 0 \quad \text{on } M_\xi,$$

then we have

$$\frac{1}{2}\triangle_\xi|\xi|^2 = g^{ij}(\xi)g_{kl}(\xi)\xi^k{}_{;i}\xi^l{}_{;j} - \mathrm{Ric}(\xi) + \phi_\xi(\xi) \ge 0 \quad \text{on } M_\xi.$$

Consequently, from Theorem 3.1, we get

$$g^{ij}(\xi)g_{kl}(\xi)\xi^k{}_{;i}\xi^l{}_{;j} - \mathrm{Ric}(\xi) + \phi_\xi(\xi) = 0,$$

or

$$\xi^l{}_{;j} = 0, \qquad \mathrm{Ric}(\xi) = 0.$$

Thus, if $\mathrm{Ric} < 0$ on $\tilde{M}$, then $\xi = 0$. So we have

Theorem 3.3. *Let M be a compact Finsler manifold, if $\mathrm{Ric} \le 0$ on $\tilde{M}$ and $\phi_\xi(\xi) \ge 0$ on M_ξ for any smooth vector field ξ, then every Killing vector field ξ must be parallel, i.e., $\tilde{D}\xi = 0$, and then $\mathrm{Ric}(\xi) = 0$.*

Thus if $\mathrm{Ric} < 0$ on $\tilde{M}$ and $\phi_\xi(\xi) \ge 0$ on M_ξ for any smooth vector field ξ, then there exists no Killing vector field other than zero vector.

Theorem 3.4. *Let M be a compact Finsler manifold, if* Ric *is negative on $\tilde{M}$, then there is no Killing vector field ξ satisfied*

$$\xi^l{}_{;j} = 0$$

other than zero vector.

Proof. Note that if a vector satisfies $\xi^l{}_{;j} = 0$, then $\phi_\xi(\xi) = 0$. From Theorem 3.3, we can conclude Theorem 3.4. □

Remark 3.1. If the Ricci tensor defined by

$$\mathrm{Ric}_{ij} := \frac{\partial^2 \left[\frac{1}{2}\,\mathrm{Ric}\right]}{\partial y^i \partial y^j}$$

is negative definite, then $\mathrm{Ric} < 0$ on $\tilde{M}$. And if M is a Riemannian manifold, then $\phi_\xi \equiv 0$, so Theorem 3.4 contains the similar theorem of Bochner [10].

Remark 3.2. In Theorem 3.3, the Killing vector fields ξ must be parallel under certain extra condition, that is, $\xi^l{}_{;j} = 0$. These equations are quasi-linear equations, so the greatest number of linearly independent motions which may be admitted by an n-dimensional Finsler manifold M under certain extra condition is n.

4. Killing vector fields on some compact special Finsler manifolds

F is said to be a Berwald metric if Γ^i_{jk} is independent of y, or G^i_{jk} is independent of y, or $A^k_{ij|h} l^h = 0$ [26], [25]. In a Berwald manifold, we have $(\xi^m{}_{;j})^{\mathbf{v}} = (\xi^m)^{\mathbf{v}}_{(j)}$.

Theorem 4.1. *Let M be a compact Berwald manifold, if* $\mathrm{Ric} \le 0$ *on $\tilde{M}$, then for any Killing vector field ξ, we have*

$$(\xi^l)^{\mathbf{v}}_{(j)} = 0,$$

and $\mathrm{Ric}(\xi) = 0$.

Thus if $\mathrm{Ric} < 0$ *on $\tilde{M}$, then there exists no Killing vector field other than zero vector.*

Proof. If M is a Berwald manifold, note that G^i_{jk} is not depend on the fiber y, that is, $G^i_{jkl} = 0$, so $\phi_\xi \equiv 0$ on $\tilde{M}$ and $(\xi^m{}_{;j})^{\mathbf{v}} = (\xi^m)^{\mathbf{v}}_{(j)}$. From Theorem 3.3, it follows that Theorem 4.1 is valid. □

Let F be a Finsler metric on an n-dimensional manifold M. F is called an Einstein metric with Einstein scalar σ if

$$\mathrm{Ric} = (n-1)\sigma F^2,$$

where σ is a scalar function on M. In particular, F is said to be Ricci constant (resp. Ricci flat) if $\sigma = \text{const.}$ (resp. $\sigma = 0$).

Theorem 4.2. *Let M be a compact Finsler manifold with a Finsler metric F. Support that F is a Einstein metric with non-positive Einstein scalar σ and $\phi_\xi(\xi) \geq 0$ on M_ξ for any vector field ξ .*

(1) *If F is not Ricci flat, then there exists no Killing vector field other than zero vector.*

(2) *If F is Ricci flat, then every Killing vector field ξ must be parallel, i.e., $\tilde{D}\xi = 0$.*

It was proved recently in [15] that a connected Berwald–Einstein manifold is either Riemannian or Ricci flat, so we have the following Corollary 4.3.

Corollary 4.3. *Suppose that M is a connected compact non-Riemannian Berwald–Einstein manifold. Then for any Killing vector field ξ, we have*

$$(\xi^l)^{\mathrm{v}}_{(j)} = 0.$$

Corollary 4.4. *Let M be a compact Finsler manifold, if the tensors H^i_{jhk} and G^i_{jhk} vanish identically, then for the Killing vector field ξ*

$$(\xi^l)^{\mathrm{v}}_{(j)} = 0.$$

Remark 4.1. The Finsler manifold is Minkowskian if the tensors H^i_{jhk} and G^i_{jhk} vanish identically([5]). Moreover, the tensors $R^i_{jhk} = 0$ iff $H^i_{jhk} = 0$, and also $C^i_{hk|r} = 0$ iff $G^i_{jhk} = 0$.

Take an arbitrary plane $P \subset T_xM$ and $y \in P$, the flag curvature $K(x,y,P)$ is defined by

$$K(x,y,P) = \frac{R_{ijhk}y^i y^h \eta^j \eta^k}{[g_{ih}g_{jk} - g_{ij}g_{hk}]y^i y^h \eta^j \eta^k},$$

where η is an arbitrary vector in P such that $P = \mathrm{span}\{y, \eta\}$.

In a Finsler manifold M, F is said to be of scalar flag curvature if $K(x,y,P) = K$ is independent of P, F is said to be of constant flag curvature If $K(x,y,P) = \sigma$, where σ is constant.

If M is the Finsler manifold of scalar flag curvature, we have

$$R^i_j = KF^2(\delta^i_j - F^{-2} g_{jk} y^k y^i),$$

and then

$$\mathrm{Ric} = (n-1)KF^2,$$

so we have

Theorem 4.5. *Let M be a compact Finsler manifold of scalar flag curvature. If $K < 0$ on $\tilde{M}$, then there is no Killing vector field ξ satisfied*

$$\xi^l{}_{;j} = 0$$

other than zero vector. Furthermore, if $K < 0$ on $\tilde{M}$ and $\phi_\xi(\xi) \geq 0$ on M_ξ for any vector field ξ, then there exists no Killing vector field other than zero vector.

If $K = 0$, then F is Ricci flat, it has been discussed in Theorem 4.2.

In the Finsler manifold with constant flag curvature $K = \sigma$, we have

$$H_{ik} = (n-1)\sigma g_{ik}, \tag{20}$$

and if $\sigma < 0$, then (H_{ik}) is negative definite, so

Theorem 4.6. *Let M be a compact Finsler manifold with negative constant flag curvature, if $\phi_\xi(\xi) \geq 0$ on M_ξ for any vector field ξ, then there exists no Killing vector field other than zero vector.*

If the compact manifold with constant flag curvature $\sigma = 0$ and $\phi_\xi(\xi) \geq 0$ on M_ξ for any vector field ξ, then for every Killing vector field ξ must be parallel, i.e., $\tilde{D}\xi = 0$.

And if a compact manifold with constant flag curvature $\sigma < 0$, then there is no Killing vector field ξ satisfied

$$\xi^l{}_{;j} = 0$$

other than zero vector.

Theorem 4.7. *Let M be a compact Berwald Finsler manifold with constant flag curvature $K = \sigma$.*

(1) *If $\sigma \neq 0$, then M is Riemannian. So if $\sigma < 0$, there is no Killing vector field other than zero vector.*
(2) *If $\sigma = 0$, then for any Killing vector field ξ, we have*

$$(\xi^l)^{\mathbf{v}}_{(j)} = 0.$$

PROOF. If M is a Berwald manifold, then H_{ij} is independent of u. If the constant flag curvature $K = \sigma \neq 0$, then it follows from (20) that M is Riemannian. The other part of the theorem is the direct corollary of Theorem 4.1. □

In Finsler geometry, (α, β)-metrics are important classes of Finsler metrics. Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and a 1-form $\beta = b_i y^i$ on an n-dimensional manifold M. An (α, β) metric F is defined by

$$F = \alpha\phi(s), \qquad s = \frac{\beta}{\alpha},$$

where $\phi(s)$ is a C^∞ positive function on $(-b_0, b_0)$. ϕ satisfies

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \leqslant b = \|\beta\|_\alpha < b_0).$$

When $n \geqslant 0$ and under certain extra condition, B. LI and Z. SHEN [20] have proved that if α is projectively flat and β is parallel with respect to α, then $F = \alpha\phi(s)$ is a projectively flat Berwald metric with constant flag curvature $K = \sigma$. From Theorem 4.7, if the metric F is non-Riemannian, then $\sigma = 0$. So

Theorem 4.8. *Let M be a compact manifold with a non-Riemannian (α, β)-metric $F = \alpha\phi(s)$, and α is projectively flat and β is parallel with respect to α, then for the Killing vector field ξ respect to F, we have*

$$(\xi^l)^{\mathbf{v}}_{(j)} = 0.$$

Remark 4.2. For a Riemannian metric, Beltrami Theorem tells us a Riemannian metric is projectively flat iff it is of constant sectional curvature. But the same theorem is not true for a Finsler metric. Therefore, in the projectively flat manifold [12], the conditions of Theorem 3.3 are difficult to simplify, but in Riemannian manifold it is easy [10].

There exist some (α, β)-metrics. The Randers metric is an (α, β)-metric, where $\phi(s) = 1 + s$, i.e., $F = \alpha + \beta$. The Kropina metric is an (α, β)-metric, where $\phi(s) = \frac{1}{s}$, i.e., $F = \frac{\alpha^2}{\beta}$. The Matsumoto metric is an (α, β)-metric, where $\phi(s) = \frac{1}{1-s}$, i.e., $F = \frac{\alpha^2}{\alpha - \beta}$.

Lemma 4.9 (see e.g. [3]). *Suppose that M is a connected compact boundaryless Einstein Randers manifold with Ricci constant σ.*

(1) *If $\sigma < 0$, then M is Riemannian.*

(2) *If $\sigma = 0$, then M is Berwald.*

So we have the following corollaries.

Corollary 4.10. *If M is a connected compact boundaryless non-Riemannian Einstein Randers manifold with Ricci constant σ, then $\sigma = 0$, and for any Killing vector field ξ, we have*

$$(\xi^l)^{\mathbf{v}}_{(j)} = 0.$$

In [27], X. L. Zhang and Y. B. Shen have proved that for a non-Riemannian Kropina metric $F = \frac{\alpha^2}{\beta}$, if F is Ricci flat, then F is Berwald. So

Corollary 4.11. *Suppose that M is a compact non-Riemannian Kropina manifold with Ricci flat metric F, then for any Killing vector field ξ, we have*

$$(\xi^l)^{\mathbf{v}}_{(j)} = 0.$$

Corollary 4.12. *Suppose that M is a compact manifold with a conformal flat Einstein Randers metric (or Kropina metric) F, then for any Killing vector field ξ, we have*

$$(\xi^l)^{\mathbf{v}}_{(j)} = 0.$$

Proof. In [27], every conformal flat Einstein Randers metric (or Kropina metric) F must be Minkowskian. Thus it follows from Corollary 4.4 that Corollary 4.12 is valid. □

Corollary 4.13. *Let $F = \frac{\alpha^2}{\alpha - \beta}$ be a non-Riemannian Einstein Matsumoto metric on an n-dimensional compact manifold M, $n \geq 3$.*

(1) *Suppose that the length of β with respect to α is constant, then for any Killing vector field ξ, we have*

$$(\xi^l)^{\mathbf{v}}_{(j)} = 0.$$

(2) *Suppose that S-curvature vanishes, then for any Killing vector field ξ, we have*

$$(\xi^l)^{\mathbf{v}}_{(j)} = 0.$$

Proof. In [28], X. L. Zhang and Y. B. Shen have proved that α is Ricci flat and β is parallel with respect to α in the two cases. So F is a Ricci flat Berwald metric. From Corollary 4.3, we can get Corollary 4.13. □

Remark 4.3. In the above discussion, we obtain that for any Killing vector field ξ on some special Finsler spaces M such as compact Ricci flat Berwald space and Minkowskian space, we have $(\xi^l)^{\mathbf{v}}_{(j)} = 0$. So the greatest number of linearly independent motions in the n-dimensional Minkowskian space is n.

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