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An explicit André–Oort type result for $\mathbb{P}^1(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C})$ based on logarithmic forms

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Abstract. Using linear forms in logarithms we prove an explicit result of André–Oort type for $\mathbb{P}^1(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C})$. In this variation the special points of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C})$ are of the form (α, λ), with α a singular modulus and λ a root of unity. The qualitative version of our result states that if $\mathcal{C}$ is a closed algebraic curve in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C})$, defined over a number field, not containing a horizontal or vertical line, then $\mathcal{C}$ contains only finitely many special points. The proof is based on linear forms in logarithms. This differs completely from the method used by the author recently in the proof of the same kind of statement, where class field theory was applied.

1. Introduction and result

Kühne in [11] and also Wüstholz in [17], and independently Bilu, Masser and Zannier in [4] have studied the André–Oort conjecture in the case of the Shimura variety $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, where $\mathbb{P}^1(\mathbb{C})$ is the modular curve $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^*$. They obtain the first nontrivial, unconditional, effective results in the area. In [14] the author investigates the variant $\mathbb{P}^1(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C})$, where the special points are of the form $(j(\tau), \lambda)$, with τ an imaginary quadratic number and $\lambda \in \mathbb{C}$ a root of unity. For further background on the André–Oort problem, see [1], [15] and [17]. Note that the non-effective version of our Theorem 1.1 is a very special case of Theorem 1.1 of Pila's paper.

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André in [2] has considered a related question for an elliptic pencil over $\mathbb{P}^1$, however his results are not effective. Wüstholz in [17] deals with the more general situation for a product of finite number of such pencils over an arbitrary curve, and his result is effective. Habegger has also worked on related problems (see e.g., [9]).

In this article we attack the same problem as in [14], using a different method. We prove a weaker version of the main explicit result of [14], therefore we also reprove the main non-effective result. The better bounds in [14] are achieved using more sophisticated class field theory. We use less class field theory here, and instead employ linear forms in logarithms. Even though the bounds are worse here, the methods presented could still be useful for other similar problems. Note that [11] and [4] use elliptic logarithms, while we only use lower bounds for linear forms of ordinary logarithms.

Let $\mathcal{H}$ denote the complex upper half-plane. We call $(\alpha, \lambda) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C})$ a special point, if $\alpha = j(\tau)$ for some imaginary quadratic $\tau \in \mathcal{H}$ and $\lambda \in \mathbb{C}$ is a root of unity. We work with the same assumptions as in Theorem 2.2 of [14]. So K is a number field of degree d over $\mathbb{Q}$ with a fixed embedding into $\mathbb{C}$, and $F \in K[X, Y]$ is a nonconstant polynomial with $\delta_1 = \deg_X F$ and $\delta_2 = \deg_Y F$. We assume that zero set of $F(X, Y) = 0$ contains no vertical or horizontal line, i.e. $F(X, Y)$ does not have a nonconstant divisor $f \in K[X]$ or $g \in K[Y]$. Then clearly $\delta_1, \delta_2 > 0$. Let $h(F)$ denote the height of the polynomial F (so $h(F)$ is the absolute logarithmic Weil height of the point defined by the nonzero coefficients of F in projective space, see the definition in Section 2). Let (α, λ) be a special point of $\mathcal{C}$, where $\alpha = j(\tau)$ for some $\tau \in \mathcal{H}$. Let

$$C = 2^{36} d^3 \delta_2^3 (\log(4d\delta_2))^2 \max\left(dh(F) + (d-1)(\delta_1 + \delta_2)\log 2, 1\right).$$

Let Δ denote the discriminant of the endomorphism ring of the complex elliptic curve $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, and let N be the smallest positive integer such that $\lambda^N = 1$.

Theorem 1.1. *In the above situation*

$$|\Delta| < \left(\frac{1}{d\delta_2} C \log C\right)^2 \tag{1}$$

and

$$N < C(\log C)^2 \log\log C \tag{2}$$

This result implies Theorem 2.1 of [14], because there are only finitely many special points (α, λ) with Δ and N bounded. Similarly to [14], we can reduce the proof to the case when $K = \mathbb{Q}$ and $\mathbb{Z} + \mathbb{Z}\tau$ is an order.

In the following section we collect some preliminary definitions and statements, and also prove some auxiliary results. In Section 3 we prove Theorem 1.1.

2. Preliminaries

For the reader's convenience we recall some definitions. The (absolute logarithmic Weil) height of a point $P = (a_0 : \ldots : a_n) \in \mathbb{P}^n(\overline{\mathbb{Q}})$ is defined by

$$h(P) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log(\max_i |a_i|_v),$$

where K is any number field containing all a_i, M_K is the set of places of K, and for any place v, $|\cdot|_v$ is the absolute value on K extending a standard absolute value of $\mathbb{Q}$. Similarly, the (absolute logarithmic Weil) height of a polynomial $F \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ with nonzero coefficients c_i is defined by

$$h(F) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log(\max_i |c_i|_v),$$

where K is a number field containing the coefficients of F. We use the notation $H(F) = e^{h(F)}$. If $F \in \mathbb{Z}[X_1, \ldots, X_n]$, and the gcd of the coefficients of F is 1, then $H(F)$ is equal to the maximum of the euclidean absolute values of the coefficients of F.

If K is a number field and $\alpha \in K$, then the (absolute logarithmic Weil) height of α is

$$h(\alpha) = h((\alpha : 1)) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \max(1, |\alpha|_v).$$

We use the notation $H(\alpha) = e^{h(\alpha)}$.

Let $f = a_d X^d + \cdots + a_0 = a_d(X - \alpha_1) \cdots (X - \alpha_d) \in \mathbb{C}[X]$, where $a_d \neq 0$. The Mahler measure of f (see e.g. [5]) is defined by

$$M(f) = \exp\left(\int_0^1 \log |f(e^{2\pi i t})| dt\right) = |a_d| \prod_{j=1}^{d} \max(1, |\alpha_j|).$$

If $\alpha \in \overline{\mathbb{Q}}$ and $f \in \mathbb{Z}[X]$ is the minimal polynomial of α, then $h(\alpha) = \frac{\log M(f)}{\deg f}$ (see Proposition 1.6.6 in [5]).

If $\mathcal{O}$ is an order in an imaginary quadratic number field L, then the class number $h(\mathcal{O})$ denotes the number of equivalence classes of proper fractional ideals of $\mathcal{O}$ (see e.g. [7]). Since $\mathcal{O}$ is an order in L, we can write it in the form $\mathbb{Z} + \mathbb{Z}\tau_0$ for some τ_0 in $L \cap \mathcal{H}$. Then the discriminant of the order $\mathcal{O}$ is $D(\mathcal{O}) = -4(\operatorname{Im} \tau_0)^2$ (see e.g. §7, Ch. 2 in [7]). This is a negative integer congruent to 0 or 1 modulo 4.

If $D < 0$ is an integer congruent to 0 or 1 modulo 4, then the class number $h(D)$ denotes the number of proper equivalence classes of primitive quadratic forms of discriminant D (see [7] or [10]). Theorem 7.7 in [7] says that if $\mathcal{O}$ is an order with discriminant D in an imaginary quadratic number field, then $h(\mathcal{O}) = h(D)$.

The following theorem is the main result of [3]. Note that the notation of [3] for heights differs from ours: we use absolute heights, which do not depend on the containing number field (because of the factor $\frac{1}{[K:\mathbb{Q}]}$ in the definition), whereas [3] uses relative heights (see [3, §2]), which do. The statement of the following theorem is adjusted to our notation.

Theorem 2.1 (Baker, Wüstholz)**.** *Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C} \setminus \{0, 1\}$ be algebraic numbers, with fixed determinations of logarithms $\log \alpha_1, \ldots, \log \alpha_n$. The degree of the field extension $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)/\mathbb{Q}$ is denoted by d. Let $L(z_1, \ldots, z_n) = b_1 z_1 + \cdots + b_n z_n$ be a linear form, where $b_1, \ldots, b_n$ are integers such that at least one b_i is nonzero. We use the notation $h'(\alpha_i) = \max\left(h(\alpha_i), \frac{1}{d}|\log \alpha_i|, \frac{1}{d}\right)$ (which depends on the choice of $\log \alpha_i$) and $h'(L) = \max\left(h(L), \frac{1}{d}\right)$, where $h(L)$ is the absolute logarithmic Weil height of $(b_1 : \ldots : b_n)$ in $\mathbb{P}^{n-1}_{\mathbb{Q}}$. If $\Lambda = L(\log \alpha_1, \ldots, \log \alpha_n) \neq 0$, then*

$$\log |\Lambda| > -C(n, d) h'(\alpha_1) \cdots h'(\alpha_n) h'(L),$$

where

$$C(n, d) = 18(n + 1)! n^{n+1} (32d)^{n+2} \log(2nd).$$

The following lemma gives us a lower bound on the distance between an algebraic number and a root of unity. Note that throughout this paper, if N is a positive integer, then by an N^{th} root of unity we mean any complex number λ such that $\lambda^N = 1$ (so e.g., 1 is an N^{th} root of unity for every positive integer N).

Lemma 2.1. *Let $\lambda \in \mathbb{C}$ be an N^{th} root of unity, where N is a positive integer. If $\gamma \in \mathbb{C}$ is algebraic of degree d over $\mathbb{Q}$, and $\lambda \neq \gamma$, then*

$$\log |\lambda - \gamma| > -cd^3 \log(4d) \max\left(h(\gamma), \frac{4}{d}\right) \log \max(N, 2)$$

with $c = 2^{25} 3^3 \pi + 1$.

Proof. The right hand side is the same for $N = 1$ and for $N = 2$, so we may assume that $N \geq 2$. Suppose first that $d = 1$. Then $\gamma \in \mathbb{Q}$, so there are coprime integers a, b such that $b > 0$ and $\gamma = \frac{a}{b}$. If $\lambda \in \{1, -1\}$, then $|\lambda - \gamma| = \left|\frac{a \pm b}{b}\right| \geq \frac{1}{b} \geq \frac{1}{H(\gamma)}$, hence $\log |\lambda - \gamma| \geq -h(\gamma)$. Now let $\lambda \notin \{1, -1\}$,

then $N \geq 3$, and $|\lambda - \gamma| \geq |\operatorname{Im}(\lambda)| \geq \sin\left(\frac{2\pi}{N}\right)$. The sine function is concave in the interval $[0, \pi]$, so $\sin x \geq \frac{\sin(2\pi/3)}{2\pi/3} x = \frac{3\sqrt{3}}{4\pi} x$ for every $x \in \left[0, \frac{2\pi}{3}\right]$. Thus $|\lambda - \gamma| \geq \frac{3\sqrt{3}}{4\pi} \cdot \frac{2\pi}{N} = \frac{3\sqrt{3}}{2N} \geq \frac{1}{N}$, therefore $\log|\lambda - \gamma| \geq -\log N$. So the statement is true for $d = 1$.

From now on we assume that $N, d \geq 2$. If $|\lambda - \gamma| \geq \frac{1}{4}$, then $\log|\lambda - \gamma| \geq -\log 4$, which is clearly greater than the bound needed. So we may assume that $|\lambda - \gamma| < \frac{1}{4}$. Let us define the logarithm function in the open unit disc around 1 such that $\log(1) = 0$, and let $s = \log\left(\frac{\gamma}{\lambda}\right)$. Here s is well defined, because $|\frac{\gamma}{\lambda} - 1| = |\gamma - \lambda| < \frac{1}{2}$. It is a basic fact from analysis that if $z \in \mathbb{C}$ and $|z| < \frac{1}{2}$, then $|\log(1+z)| \leq 2|z|$. Using this for $z = \frac{\gamma}{\lambda} - 1$, we get that $|s| \leq 2|\frac{\gamma}{\lambda} - 1| = 2|\lambda - \gamma|$. In particular $|s| < \frac{1}{2}$.

We can find an integer u such that $|u| \leq \frac{N}{2}$ and $\lambda = e^{\frac{u}{N} 2\pi i}$. Note that $e^{s + \frac{u}{N} 2\pi i} = \gamma$ and $e^{\pi i} = -1$, so we may choose the logarithms $\log\gamma$ and $\log(-1)$ to be $s + \frac{u}{N} 2\pi i$ and πi. Define the linear form $L(z_1, z_2) = Nz_1 - 2uz_2$. We will apply the Baker–Wüstholz estimate (Theorem 2.1) for $\Lambda = L(\log\gamma, \log(-1))$. Here $\Lambda = N\log\gamma - 2u\pi i = N\left(s + \frac{u}{N} 2\pi i\right) - 2u\pi i = Ns \neq 0$, because otherwise $\log\gamma = \frac{u}{N} 2\pi i$, so $\gamma = e^{\frac{u}{N} 2\pi i} = \lambda$, contradicting $\lambda \neq \gamma$. Note that $-1, \gamma \notin \{0, 1\}$, because $d \geq 2$. Theorem 2.1 tells us that

$$\log|\Lambda| > -C(2,d) h'(\gamma) h'(-1) h'(L),$$

where

$$\begin{aligned}
C(2,d) &= 18 \cdot 6 \cdot 8 (32d)^4 \log(4d), \\
h'(\gamma) &= \max\left(h(\gamma), \frac{1}{d}|\log\gamma|, \frac{1}{d}\right), \\
h'(-1) &= \max\left(h(-1), \frac{1}{d}|\log(-1)|, \frac{1}{d}\right) = \frac{\pi}{d}, \\
h'(L) &= \max\left(h\left(\frac{2u}{N}\right), \frac{1}{d}\right) \leq \max\left(\log N, \frac{1}{d}\right) \leq \max\left(\log N, \frac{1}{2}\right) = \log N.
\end{aligned}$$

Note that $|\log\gamma| = |s + \frac{2u}{N}\pi i| \leq |s| + \pi \leq \pi + \frac{1}{2} < 4$, so $h'(\gamma) \leq \max\left(h(\gamma), \frac{4}{d}\right)$. Collecting these inequalities together, we get

$$\log|\Lambda| > -c' d^3 \log(4d) \max\left(h(\gamma), \frac{4}{d}\right) \log N,$$

where $c' = 18 \cdot 6 \cdot 8 \cdot 32^4 \cdot \pi = 2^{25} 3^3 \pi = c - 1$. From $|s| \leq 2|\lambda - \gamma|$ and $\Lambda = Ns$

we obtain that $|\lambda - \gamma| \geq \frac{|\Lambda|}{2N}$. So

$$\log|\lambda-\gamma| \geq \log|\Lambda| - \log(2N) \geq \log|\Lambda| - 2\log N$$
$$> -(c'+1)d^3\log(4d)\max\left(h(\gamma), \frac{4}{d}\right)\log N = -cd^3\log(4d)\max\left(h(\gamma), \frac{4}{d}\right)\log N.$$

□

Remark 2.1. We have used the Baker–Wüstholz bound in Lemma 2.1 for a linear form in two logarithms. There are other similar bounds, see for example [8], [12] and [13]. Note that many of the more easily applicable results in these papers assume multiplicative independence, which is not true in our case. Moreover, except for a possibly better constant, we were unable to get a significantly better bound using the results of these papers. Hence we used the Baker–Wüstholz bound, which we found the easiest to apply.

In the following lemma, we get a bound for the value of a polynomial at a root of unity.

Lemma 2.2. *Let N and d be positive integers. Let $\lambda \in \mathbb{C}$ be an N^{th} root of unity, and let $g \in \mathbb{Z}[X]$ be a polynomial of degree at most δ such that $g(\lambda) \neq 0$. Then*

$$\log|g(\lambda)| > -2^{35}\delta^2(\log(4\delta))^2\max(h(g),1)\log\max(N,2).$$

Proof. We will prove the inequality

$$\log|g(\lambda)| > -c_2\delta^2\log(4\delta)\log(2M(g))\log\max(N,2), \tag{3}$$

where $c_0 = 2^{25}3^3\pi + 1$, $c_1 = \frac{4}{\log 2}c_0$ and $c_2 = c_1 + 6$.

We argue by induction on δ. The right hand side of (3) is the same for $N = 1$ and for $N = 2$, so we may assume that $N \geq 2$. If $\deg g = 0$, then $g(\lambda) \in \mathbb{Z}\setminus\{0\}$, so $\log|g(\lambda)| \geq 0$. Now let $\deg g \geq 1$. The right hand side of (3) is a monotone decreasing function of δ, so we may assume that $\deg g = \delta$, and that (3) is true for smaller values of δ. We may also assume that the gcd of the coefficients of g is 1, because multiplying g by a positive integer increases the left hand side and decreases the right hand side of (3). Suppose g is reducible, then $g = g_1g_2$ for some polynomials $g_1, g_2 \in \mathbb{Z}[X]$ of positive degrees d_1 and d_2. Note that $\delta = d_1 + d_2$ and $M(g) = M(g_1)M(g_2) \geq M(g_1), M(g_2)$. Using the induction hypothesis for g_1 and g_2, we get

$$\begin{aligned}\log|g(\lambda)| &= \log|g_1(\lambda)| + \log|g_2(\lambda)| \\ &\geq -c_2\log(2M(g))(\log N)\left(d_1^2\log(4d_1) + d_2^2\log(4d_2)\right) \\ &\geq -c_2\log(2M(g))(\log N)\delta^2\log(4\delta),\end{aligned}$$

because

$$\delta^2 \log(4\delta) = (d_1^2 + d_2^2 + 2d_1 d_2) \log(4\delta) \geq d_1^2 \log(4d_1) + d_2^2 \log(4d_2).$$

Finally, let g be irreducible. Let $g = a(X-\gamma_1)\cdots(X-\gamma_\delta)$, where $a \in \mathbb{Z}\setminus\{0\}$ and $\gamma_1, \ldots, \gamma_\delta \in \mathbb{C}$. Choose a $k \in \{1, \ldots, n\}$ such that $|\lambda - \gamma_k|$ is minimal. During the proof of Theorem A.3 in [6] it is shown that if $P \in \mathbb{Z}[X] \setminus \{0\}$ is separable polynomial of degree n, and $\alpha, \beta \in \mathbb{C}$ are such that $P(\alpha) = P(\beta) = 0$ and $\alpha \neq \beta$, then

$$|\alpha - \beta|^2 \frac{n^3}{3} \max(1, |\alpha|, |\beta|)^{-2} n^{n-1} M(P)^{2n-2} > 1.$$

So in fact

$$|\alpha - \beta| > \sqrt{3} n^{-(n+2)/2} \max(1, |\alpha|, |\beta|) M(P)^{-(n-1)} \geq \sqrt{3} n^{-(n+2)/2} M(P)^{-(n-1)}.$$

Applying this result for $P = g$ and $\alpha = \gamma_k$, $\beta = \gamma_i$ with $i \neq k$, we get that $|\gamma_k - \gamma_i| > R$ with $R = \sqrt{3}\delta^{-(\delta+2)/2} M(g)^{-(\delta-1)}$. Then

$$R < |\gamma_k - \gamma_i| \leq |\gamma_k - \lambda| + |\lambda - \gamma_i| \leq 2|\lambda - \gamma_i|,$$

so

$$|\lambda - \gamma_i| > \frac{R}{2} \tag{4}$$

for every $i \neq k$. Applying Lemma 2.1 and using $\log M(g) = \delta h(\gamma_k) \geq 0$, we obtain

$$\begin{aligned} \log|\lambda - \gamma_k| &> -c_0 \delta^2 \log(4\delta) \max(\log M(g), 4) \log N \\ &\geq -c_1 \delta^2 \log(4\delta) \log(2M(g)) \log N. \end{aligned} \tag{5}$$

Let $A = \delta^2 \log(4\delta) \log(2M(g)) \log N$. The bounds (4) and (5) together imply that

$$\begin{aligned} \log|g(\gamma)| &> (\delta - 1) \log\left(\frac{R}{2}\right) - c_1 A \\ &= -(\delta - 1) \log\left(\frac{2}{\sqrt{3}}\right) - \frac{(\delta-1)(\delta+2)}{2} \log\delta - (\delta-1)^2 \log M(g) - c_1 A. \end{aligned}$$

It is easy to check that the terms

$$(\delta - 1) \log\left(\frac{2}{\sqrt{3}}\right), \quad \frac{(\delta-1)(\delta+2)}{2} \log\delta, \quad (\delta-1)^2 \log M(g)$$

are all smaller than $2A$, so $\log|g(\gamma)| > -(c_1 + 6)A = -c_2 A$. This finishes the proof of (3).

To prove the statement of the lemma, first note that we may assume that the gcd of the coefficients of g is 1. Then every coefficient of g has euclidean absolute value at most $H(g)$, so $M(g) \le \sqrt{\delta+1}H(g)$ (see Lemma 1.6.7 in [5]), hence

$$\begin{aligned}\log|g(\lambda)| &> -c_2\delta^2\log(4\delta)\log(2\sqrt{\delta+1}H(g))\log\max(N,2)\\ &\ge -2c_2\delta^2\log(4\delta)^2\max(h(g),1)\log\max(N,2),\end{aligned}$$

because

$$\log 2+\frac{\log(\delta+1)}{2}+h(g)\le 2\log(4\delta)\max(h(g),1).$$

The statement of the lemma follows from $2c_2 < 2^{35}$. □

If λ is a primitive N^{th} root of unity, then the degree of λ over $\mathbb{Q}$ is $\varphi(N)$, where φ denotes Euler's totient function. During the proof of Theorem 1.1 we will need a lower bound for the degree of λ. A more or less trivial bound would be $\varphi(N) \ge c(\varepsilon)N^{1-\varepsilon}$ for every positive ε. We can actually do better.

Proposition 2.1. *If $N > 30$ is an integer, then $\varphi(N) > \frac{N}{3\log\log N}$.*

Proof. The statement can be easily verified case by case for $31 \le N \le 66$, so we may assume that $N \ge 67$. Theorem 15 in [16] implies that

$$\varphi(N) > \frac{N}{e^\gamma\log\log N+\frac{2.51}{\log\log N}}$$

for $N \ge 3$, with γ denoting the Euler constant. Now $\log\log N \ge \log\log 67 > \sqrt{\frac{2.51}{3-e^\gamma}}$, hence $(3-e^\gamma)\log\log N > \frac{2.51}{\log\log N}$, therefore

$$\varphi(N) > \frac{N}{e^\gamma\log\log N+\frac{2.51}{\log\log N}} > \frac{N}{3\log\log N}.$$ □

We will use the following upper bound for the class number $h(D)$.

Proposition 2.2. *If $D < 0$ is an integer congruent to 0 or 1 modulo 4, then*

$$h(D) < \frac{1}{\pi}\sqrt{|D|}(2+\log|D|).$$

Proof. The class number formula (see Theorem 10.1, Ch. 12 in [10]) says that

$$h(D) = \frac{w\sqrt{|D|}}{2\pi}L_D(1),$$

where w is equal to 6, 4 and 2 for $D = -3$, $D = -4$ and $D < -4$ respectively, and $L_D(1) = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{D}{n}\right)$. If $D = -3$ or -4, then $h(D) = 1$, thus the statement of the proposition is true. So we may assume that $D < -4$. Then $w = 2$, so $h(D) = \frac{\sqrt{|D|}}{\pi} L_D(1)$. Theorem 14.3, Ch. 12 in [10] says that $0 < L_D(1) < 2 + \log|D|$, which gives $h(D) < \frac{\sqrt{|D|}}{\pi}(2 + \log|D|)$. □

We will use the following auxiliary lemma in the proof of Theorem 1.1.

Lemma 2.3. *Let $p > 0$, $q > e$ and $A > 10^4$ be real numbers such that*

$$p \le A \log q \quad \text{and} \quad \frac{q}{\log\log q} \le p \log p.$$

Then $p < 2A \log A$ and $q < 3A(\log A)^2 \log\log A$.

Proof. Note that $0 < \frac{q}{\log\log q} \le p \log p$, so $p > 1$. Since $p \log p$ is an increasing function of $p \in (1, \infty)$, we may assume that $p = A \log q$. Then $\frac{q}{\log\log q} \le A(\log q)\log(A \log q)$, so $q \le G(q)$, where

$$G(x) = A(\log x)(\log\log x)\log(A \log x).$$

We claim that $G(x)/x$ is strictly decreasing function of x in the interval (e^4, ∞). Indeed, if $x > e^4$, then $\log(Ax) \ge \log\log x > 1$, so

$$G'(x) = \frac{A}{x}((1 + \log\log x)(1 + \log(A \log x)) - 1) < \frac{4A}{x}(\log\log x)\log(A \log x),$$

hence

$$(G(x)/x)' = \frac{1}{x^2}(G'(x)x - G(x)) < \frac{A}{x^2}(4 - \log x)(\log\log x)\log(A \log x) < 0.$$

We claim that $G(Q) < Q$, where $Q = 3A(\log A)^2 \log\log A$. This will prove the upper bound on q, because $G(Q)/Q < 1 \le G(q)/q$ implies $q < Q$, since $G(x)/x$ is a decreasing function in (e^4, ∞), and $Q > e^4$. So we need to show

$$A(\log Q)(\log\log Q)\log(A \log Q) < 3A(\log A)^2 \log\log A,$$

or equivalently

$$\frac{\log Q}{\log A} \cdot \frac{\log(A \log Q)}{\log A} \cdot \frac{\log\log Q}{\log\log A} < 3.$$

One can easily check that $3 < A^{1/8}$, $\log A < A^{1/4}$ and $\log\log A < A^{1/8}$ for $A > 10^4$. Thus $3(\log A)^2(\log\log A) < A^{3/4}$, hence $Q < A^{7/4}$ and $\frac{\log Q}{\log A} < \frac{7}{4}$. Then

$$\log Q < \frac{7}{4} \log A < A^{1/3},$$

because $\frac{7}{4} < A^{1/12}$ and $\log A < A^{1/4}$ for $A > 10^4$. So $A \log Q < A^{4/3}$ and $\frac{\log(A \log Q)}{\log A} < \frac{4}{3}$. We have

$$\log \log Q < \log \left(\frac{7}{4} \log A \right) < \log \left((\log A)^{9/7} \right),$$

because $\frac{7}{4} < (\log A)^{2/7}$ for $A > 10^4$. So $\frac{\log \log Q}{\log \log A} < \frac{9}{7}$. This proves $q < Q$, because $\frac{7}{4} \cdot \frac{4}{3} \cdot \frac{9}{7} = 3$.

Finally, we have seen that $q < Q < A^{7/4} < A^2$, so $p = A \log q < 2A \log A$. □

3. Proof of Theorem 1.1

One can show in the same way as in the proof of Theorem 2.2 in [14], that it is enough to prove the statement in the case when $K = \mathbb{Q}$ and $\mathbb{Z} + \mathbb{Z}\tau$ is an order. So let $K = \mathbb{Q}$ and $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\tau$ an order. Then $d = 1$ and

$$C = 2^{36} \delta_2^3 (\log(4\delta_2))^2 \max(h(F), 1).$$

The polynomial $g(Y) = F(\alpha, Y) \in \mathbb{Q}(\alpha)[Y]$ is nonzero (recall that $\alpha = j(\tau)$), because the zero set of F contains no vertical line. Moreover $g(\lambda) = 0$, so $[\mathbb{Q}(\alpha, \lambda) : \mathbb{Q}(\alpha)] \le \deg g \le \delta_2$. This gives

$$\varphi(N) = [\mathbb{Q}(\lambda) : \mathbb{Q}] \le [\mathbb{Q}(\alpha, \lambda) : \mathbb{Q}] = [\mathbb{Q}(\alpha, \lambda) : \mathbb{Q}(\alpha)] \cdot [\mathbb{Q}(\alpha) : \mathbb{Q}] \le \delta_2 \cdot h(\mathcal{O}).$$

The discriminant of the order $\mathcal{O}$ is $\Delta = -4(\operatorname{Im} \tau)^2$. We know that $\Delta < 0$ and Δ is congruent to 0 or 1 modulo 4. Moreover $h(\mathcal{O}) = h(\Delta)$. Using Proposition 2.2, we obtain that

$$\varphi(N) \le \delta_2 h(\Delta) \le \frac{\delta_2}{\pi} \sqrt{|\Delta|} (2 + \log |\Delta|). \tag{6}$$

Suppose $|\Delta| < 25$. Then (1) is true. If $N \le 30$, then (2) is also true. If $N > 30$, then using Proposition 2.1, we obtain

$$\sqrt{N} < \frac{N}{3 \log \log N} < \varphi(N) < \frac{\delta_2}{\pi} 5(2 + \log 25) < 9\delta_2,$$

which implies $N < 81\delta_2^2$. This proves (2) if $|\Delta| < 25$. From now on we assume that $|\Delta| \ge 25$.

Since $|\Delta| \ge 25$, we have $\operatorname{Im} \tau = \frac{\sqrt{|\Delta|}}{2} \ge \frac{5}{2} > \frac{1}{2\pi} \log 6912$, and from [14, Proposition 3.1], we deduce as in [14] that $\frac{|\alpha|}{e^{2\pi \operatorname{Im} \tau}} \in \left[\frac{1}{2}, 2\right]$. Taking logarithms

leads to $\log|\alpha| \geq 2\pi \operatorname{Im}\tau - \log 2 > 6 \operatorname{Im}\tau$, because $\operatorname{Im}\tau \geq \frac{5}{2} > \frac{\log 2}{2\pi - 6}$. Substituting $\operatorname{Im}\tau = \frac{\sqrt{|\Delta|}}{2}$ we obtain

$$3\sqrt{|\Delta|} < \log|\alpha|. \tag{7}$$

We multiply F by a nonzero rational number, so that F will have integer coefficients with gcd equal to 1. Then the maximum of the euclidean absolute values of the coefficients of F is $H(F) = e^{h(F)}$. Let $F = \sum_{i=0}^{\delta_1} g_i(Y)X^i$, where $g_i(Y) \in \mathbb{Z}[Y]$. Here each g_i has degree at most δ_2. Since $F(X, \lambda) \in \mathbb{C}[X]$ is a nonzero polynomial, $g_i(\lambda) \neq 0$ for some i. Let m be the maximal such i. It is proved in [14, p. 160] that

$$|g_m(\lambda)| < \frac{(\delta_2 + 1)H(F)}{|\alpha| - 1}.$$

Since $|\Delta| \geq 25$, inequality (7) implies that $\log|\alpha| > 15$, hence $|\alpha| > e^{15} > 2$. Thus

$$\frac{(\delta_2 + 1)H(F)}{|\alpha| - 1} \leq \frac{4\delta_2 H(F)}{|\alpha|},$$

therefore

$$\log|g_m(\lambda)| < \log(4\delta_2) + h(F) - \log|\alpha|. \tag{8}$$

On the other hand, we can apply Lemma 2.2 for g_m and λ. Since $\deg g_m \leq \delta_2$, and the coefficients of g_m have euclidean absolute values at most $H(F) = e^{h(F)}$, we have $h(g_m) \leq h(F)$, and Lemma 2.2 says

$$\log|g_m(\lambda)| > -2^{35}\delta_2^2(\log(4\delta_2))^2 \max(h(F), 1) \log\max(N, 2). \tag{9}$$

The inequalities (7), (8) and (9) together imply

$$\begin{aligned} 3\sqrt{|\Delta|} &< 2^{35}\delta_2^2(\log(4\delta_2))^2 \max(h(F), 1) \log\max(N, 2) + \log(4\delta_2) + h(F) \\ &< (2^{35} + 2)\delta_2^2(\log(4\delta_2))^2 \max(h(F), 1) \log\max(N, 2). \end{aligned} \tag{10}$$

If $N \leq 30$, then we get

$$\sqrt{|\Delta|} < 2^{36}\delta_2^2(\log(4\delta_2))^2 \max(h(F), 1) = \frac{1}{\delta_2} C < \frac{1}{\delta_2} C \log C,$$

hence both (1) and (2) are true. From now on we assume that $N > 30$.

Applying (6) and Proposition 2.1, we get

$$\frac{N}{3 \log\log N} < \frac{\delta_2}{\pi}\sqrt{|\Delta|}(2 + \log|\Delta|). \tag{11}$$

Let $p = \frac{12}{\pi}\delta_2\sqrt{|\Delta|}$ and $q = 2N$, then (10) and (11) imply

$$p < A\log q \quad \text{and} \quad \frac{q}{\log\log q} < p\log p$$

with $A = \frac{4}{\pi}(2^{35}+2)\delta_2^3(\log(4\delta_2))^2\max(h(F),1)$. Applying Lemma 2.3, we obtain

$$p < 2A\log A \quad \text{and} \quad q < 3A(\log A)^2\log\log A.$$

Then $\sqrt{|\Delta|} < \frac{\pi}{6\delta_2}A\log A$ and $N < \frac{3}{2}A(\log A)^2\log\log A$. The inequalities (1) and (2) follow from these, because $\frac{\pi}{6}A < A < \frac{3}{2}A < C$.

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