Publ. Math. Debrecen
88/1-2 (2016), 59–78
DOI: 10.5486/PMD.2016.7257

Bounds for Diophantine quintuples II

By MIHAI CIPU (Bucharest), ALAN FILIPIN (Zagreb)
and YASUTSUGU FUJITA (Narashino)

Dedicated to Professor Kálmán Győry on the occasion of the 75th anniversary

Abstract. A set of positive integers $a_1, a_2, \dots, a_m$ with the property that $a_i a_j + 1$ is a perfect square for all distinct indices i and j between 1 and m is called Diophantine. In this paper, we show that if $\{a, b, c, d, e\}$ is a Diophantine quintuple with $a < b < c < d < e$ and $g = \gcd(a, b)$, then $b > 3ag$; moreover, if $c > a + b + 2\sqrt{ab+1}$ then $b > \max\{24\,ag, 2\,a^{3/2}g^2\}$. Similar results are given assuming that either ab is odd or $c = a + b + 2\sqrt{ab+1}$.

1. Introduction

A set of positive integers $a_1, a_2, \dots, a_m$ with the property that $a_i a_j + 1$ is a perfect square for all distinct indices i and j between 1 and m is called *Diophantine*. When $m = 2$ (3, 4, 5 or 6), we shall speak of Diophantine pair (triple, quadruple, quintuple or sextuple, respectively). Unless stated otherwise, the elements of a Diophantine set are enumerated in increasing order.

Euler knew that any Diophantine pair $\{a, b\}$ can be extended to a Diophantine triple $\{a, b, c_+\}$ with $c_+ := a + b + 2\sqrt{ab+1}$. In [1], Arkin, Hoggatt,

---

Mathematics Subject Classification: 11D09, 11B37, 11J86, 11J68.
Key words and phrases: Diophantine m-tuples, Pell equations, linear forms in logarithms, hypergeometric method.
The second author was supported by Croatian Science Foundation under the project no. 6422.
The third author was partially supported by Japan Society for the Promotion of Science KAKENHI Grant Number 25400025.

and Strauss show that any Diophantine triple $\{a, b, c\}$ can be prolongated to a Diophantine quadruple by

$$d_+ := a + b + c + 2abc + 2\sqrt{(ab+1)(bc+1)(ac+1)}.$$

Such a Diophantine quadruple $\{a, b, c, d_+\}$ is called *regular*. It can be shown that d_+ is the smallest integer greater than $\max\{a, b, c\}$ having this extension property. Moreover, one always has

$$4abc < d_+ < 4c(ab+1).$$

As no example of Diophantine quintuple is known, a natural supposition emerged.

Conjecture A. *There exists no Diophantine quintuple.*

A stronger assumption was formulated by Arkin, Hoggatt, and Strauss in [1] and independently by Gibbs in [18].

Conjecture B. *Any Diophantine triple $\{a, b, c\}$ can be uniquely prolongated to a Diophantine quadruple $\{a, b, c, d\}$ with $d > \max\{a, b, c\}$.*

By a result of Dujella [9], it is known that no Diophantine sextuple exists and that there are only finitely many Diophantine quintuples. The latest upper bounds on their number are found in [11] and [5].[1]

Presently a lot of properties that a hypothetical Diophantine quintuple must have are known. Baker and Davenport ([2]) made a breakthrough by showing that the Diophantine triple $\{1, 3, 8\}$ cannot be extended to a Diophantine quintuple. The same property has been established for the triple $\{k-1, k+1, 4k\}$ with integer $k \geq 2$ by Dujella ([7]), and for the pair $\{1, 3\}$ by Dujella and Pethő ([10]). A further generalization ([15]) asserts that the Diophantine pair $\{k-1, k+1\}$ $(k \geq 2)$ cannot be extended to a Diophantine quintuple. The above result of Dujella in [7] has other generalizations ([19], [20]), which state that the Diophantine triple $\{k, A^2k + 2A, (A+1)^2k + 2(A+1)\}$ for positive integers A, k with either $A \leq 10$ or $A \geq 52330$ cannot be extended to a Diophantine quintuple. In fact, all the Diophantine quadruples containing the pairs and triples mentioned above are known to be regular (see [4] besides the references cited above), in other words, they support Conjecture B. From [16] it is also known

[1]After submitting this work, the authors became aware of the paper T. Trudgian, Bounds on the number of Diophantine quintuples, *J. Number Theory* **157** (2015), 233–249, containing sharper estimates on the number of Diophantine quintuples than those given in [11] and [5].

that any Diophantine quintuple contains a unique regular Diophantine quadruple, which is obtained by removing its largest element.

Most of the papers dealing with the above conjectures grew out of efforts to diminish the gap existing between the largest elements of a putative Diophantine quintuple. Quite recently, the approach based on a thorough study of the smaller elements has been successfully employed by Filipin, Fujita, and Togbé. In [13] they published upper bounds for the minimal c in terms of b, where $\{a, b\}$ $(a < b)$ is a fixed Diophantine pair, such that $\{a, b, c\}$ is a Diophantine triple extendible to an irregular Diophantine quadruple. The first lower bound greater than 1 for the ratio b/a of the two smallest elements of a Diophantine quintuple was recently obtained in [6], where the next results have been proved.

Theorem A. *There exists no Diophantine quintuple* $\{a, b, c, d, e\}$ *with* $a < b < c < d < e$ *and* $b \leq 3\,a$.

The gap is even larger when the third element is not the smallest possible one.

Theorem B. *There exists no Diophantine quintuple* $\{a, b, c, d, e\}$ *with* $a < b < c < d < e$, $c > a + b + 2\sqrt{ab+1}$ *and* $b \leq \max\{21\,a,\ 2\,a^{3/2}\}$.

The present paper contains refinements of these results in several directions. A first line of thought takes into account the greatest common divisor of the two smallest entries in a hypothetical Diophantine quintuple.

Theorem 1.1. *Let* $\{a, b\}$ *be a Diophantine pair with* $a < b$ *and put* $g = \gcd(a, b)$. *Then, there exists no Diophantine quintuple* $\{a, b, c, d, e\}$ *with* $a < b < c < d < e$ *satisfying any of the following:*

(1) $b \leq 3ag$.

(2) $c > a + b + 2\sqrt{ab+1}$ *and* $b \leq \max\{24\,ag, 2\,a^{3/2}g^2\}$.

A further idea worth pursuing is to restrict attention to Diophantine sets whose elements satisfy additional hypotheses. In this vein we have the following results.

Theorem 1.2. *Let* $\{a, b\}$ *be a Diophantine pair with* $a < b$ *and* $a \equiv b \equiv 1 \pmod 2$. *Then, there exists no Diophantine quintuple* $\{a, b, c, d, e\}$ *with* $a < b < c < d < e$ *satisfying any of the following:*

(1) $b \leq 40a/9$.

(2) $c > a + b + 2\sqrt{ab+1}$ *and* $b \leq \max\{42\,a, 4\,a^{3/2}\}$.

Theorem 1.3. *Let $\{a, b, c\}$ be a Diophantine triple with $c = a+b+2\sqrt{ab+1}$. Then, there exists no Diophantine quintuple $\{a, b, c, d, e\}$ with $a < b < c < d < e$ satisfying any of the following:*

(1) $b \geq a^3$.

(2) $\gcd(b, c) > 1$.

(3) $a \equiv b \equiv 0 \pmod 2$.

Comparison of Theorem B to part (2) of Theorem 1.1 reveals a noticeable improvement. This is due to several factors. First of all, we prove new variants for Rickert's theorem on the simultaneous approximation of quadratic irrationals, see Section 2. A further source of advancement is a systematic use of computer packages PARI/GP ([23]) and MATHEMATICA ([25]), which is visible not only in the proofs of our main results but also in a slight improvement of our knowledge on the range of validity of Conjecture B (see Lemma 3.4 in Section 3). As a third explanation for ameliorations one can suggest the way we mix various ingredients already available in literature.

2. Versions of Rickert's theorem

Any Diophantine set gives rise to a system of generalized Pell equations solvable in positive integers. Tight bounds on the solutions of the relevant equations can be obtained by using Padé approximations to hypergeometric functions. Since the results of the kind available in the literature do not take into account all of our hypotheses, we provide here versions needed in subsequent sections.

Theorem 2.1. *Let A, B be integers with $0 < A < B$ and N a multiple of AB. Put $A' = \max\{B - A, A\}$ and $g = \gcd(A, B)$. Assume that $B/g \geq 5$ and $N \geq 3.804A'B^2(B - A)^2/g^4$. Then the numbers $\theta_1 = \sqrt{1 + B/N}$ and $\theta_2 = \sqrt{1 + A/N}$ satisfy*

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > \left(\frac{1.435 \cdot 10^{28} A'BN}{Ag^2}\right)^{-1} q^{-\lambda}$$

for all integers p_1, p_2, q with $q > 0$, where

$$\lambda = 1 + \frac{\log(10A^{-1}A'BNg^{-2})}{\log(2.629A^{-1}B^{-1}(B - A)^{-2}N^2g^2)} < 2. \tag{2.1}$$

Theorem 2.2. *Let* A, B *be odd integers with* $0 < A < B$ *and* N *a multiple of* AB. *Put* $A' = \max\{B - A, A\}$. *Assume that* $N \geq 0.927A'B^2(B - A)^2$ *and* $B > 2000$. *Then the numbers* $\theta_2 = \sqrt{1 + B/N}$ *and* $\theta_2 = \sqrt{1 + A/N}$ *satisfy*

$$\max\left\{\left|\theta_2 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > \left(\frac{7.065 \cdot 10^{27} A'BN}{A}\right)^{-1} q^{-\lambda}$$

for all integers p_1, p_2, q *with* $q > 0$, *where*

$$\lambda = 1 + \frac{\log(5A^{-1}A'BN)}{\log(5.394A^{-1}B^{-1}(B - A)^{-2}N^2)} < 2. \tag{2.2}$$

Both theorems can be obtained by specialization from a very general result recalled for reader's convenience.

Lemma 2.3 ([3, Lemma 3.1])**.** *Let* $\theta_1, \dots, \theta_m$ *be arbitrary real numbers and* $\theta_0 = 1$. *Assume that there exist positive real numbers* l, p, L *and* P *with* $L > 1$ *such that for each positive integer* k, *we can find integers* p_{ijk} $(0 \leq i, j \leq m)$ *with nonzero determinant,*

$$|p_{ijk}| \leq pP^k \quad (0 \leq i, j \leq m)$$

and

$$\left|\sum_{j=0}^{m} p_{ijk}\theta_j\right| \leq lL^{-k} \quad (0 \leq i \leq m).$$

Then

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \dots, \left|\theta_m - \frac{p_m}{q}\right|\right\} > cq^{-\lambda}$$

holds for all integers $p_1, \dots, p_m, q$ *with* $q > 0$, *where*

$$\lambda = 1 + \frac{\log P}{\log L} \quad \text{and} \quad c^{-1} = 2mpP\left(\max\{1, 2l\}\right)^{\lambda - 1}.$$

Proof of Theorem 2.1. We apply Lemma 2.3 with $m = 2$ and θ_1, θ_2 as in Theorem 2.1. For $0 \leq i, j \leq 2$ and arbitrary integers a_i, let $p_{ij}(x)$ be the polynomial defined by

$$p_{ij}(x) = \sum_{ij} \binom{k + \frac{1}{2}}{h_j}(1 + a_j x)^{k - h_j} x^{h_j} \prod_{l \neq j} \binom{-k_{il}}{h_l}(a_j - a_l)^{-k_{il} - h_l},$$

where $k_{il} = k + \delta_{il}$ with δ_{il} the Kronecker delta, $\sum_{ij}$ denotes the sum over all non-negative integers h_0, h_1, h_2 satisfying $h_0 + h_1 + h_2 = k_{ij} - 1$, and $\prod_{l \neq j}$

denotes the product from $l = 0$ to $l = 2$ omitting $l = j$ (which is the expression (3.7) in [24] with $\nu = 1/2$). Then, we have

$$p_{ij}(1/N) = \sum_{ij} \binom{k+\frac{1}{2}}{h_j} C_{ij}^{-1} \prod_{l \neq j} \binom{-k_{il}}{h_l},$$

where

$$C_{ij} = \frac{N^k}{(N+a_j)^{k-h_j}} \prod_{l \neq j} (a_j - a_l)^{k_{il}+h_l}.$$

Now we take $a_0 = 0$, $a_1 = A = A_0 g$, $a_2 = B = B_0 g$, and $N = ABN_0$ for some positive integer N_0. If $j = 0$, then

$$|C_{i0}| = \frac{A^{k_{i1}+h_0+h_1-k} B^{k_{i2}+h_0+h_2-k} N^k}{N_0^{k-h_0}}.$$

By $k_{il} + h_j + h_l - k \leq k$ we have $A^k B^k N^k C_{i0}^{-1} = A_0^k B_0^k N^k g^{2k} C_{i0}^{-1} \in \mathbb{Z}$ for all i. If $j = 1$, then

$$\begin{aligned} |C_{i1}| &= \frac{A^{k_{i0}+h_0+h_1-k}(B-A)^{k_{i2}+h_2} N^k}{(BN_0+1)^{k-h_1}} \\ &= \frac{A_0^{k_{i0}+h_0+h_1-k}(B_0-A_0)^{k_{i2}+h_2} N^k g^{k_{i0}+k_{i2}+h_0+h_1+h_2-k}}{(BN_0+1)^{k-h_1}}. \end{aligned}$$

Since $k_{il} + h_l \leq k_{il} + k_{ij} - 1 \leq 2k$ and $k_{i0} + k_{i2} + h_0 + h_1 + h_2 - k = 2k$, we have $A_0^k (B_0 - A_0)^{2k} N^k g^{2k} C_{i1}^{-1} \in \mathbb{Z}$ for all i. If $j = 2$, then in a similar way to the above we have $B_0^k (B_0 - A_0)^{2k} N^k g^{2k} C_{i2}^{-1} \in \mathbb{Z}$ for all i. Hence, we obtain

$$\left\{A_0 B_0 (B_0 - A_0)^2 N g^2\right\}^k C_{ij}^{-1} = \left\{AB(B-A)^2 N g^{-2}\right\}^k C_{ij}^{-1} \in \mathbb{Z}$$

for all i, j. As seen in the proof of Lemma 4.3 in [24], we have

$$2^{h_j + h'_j} \binom{k+\frac{1}{2}}{h_j} \in \mathbb{Z}, \tag{2.3}$$

where $h'_j = \max\{0, h_j - 1\}$, for all j. It follows from the proof of Theorem 2.5 in [12] that

$$p_{ijk} := 2^{-1} \left\{4AB(B-A)^2 N g^{-2}\right\}^k \Pi_2(k)^{-1} p_{ij}(1/N) \in \mathbb{Z},$$

where $\Pi_2(k)$ is an integer satisfying $\Pi_2(k) > 1.6^k/(4.09 \cdot 10^{13})$ (see equation (2.2) in [6]). Hence, the proof of Theorem 21 in [16], together with the assumptions $N \geq 3.804 A' B^2 (B-A)^2/g^4$ and $B/g \geq 5$, enables us to get

$$|p_{ijk}| < pP^k, \quad \left|\sum_{j=0}^{2} p_{ijk}\theta_j\right| < lL^{-k},$$

where

$$p = \frac{4.09 \cdot 10^{13}}{2}\left(1 + \frac{A'}{2N}\right)^{1/2} < 2.051 \cdot 10^{13}, \quad P < \frac{10A'BN}{Ag^2},$$
$$l = \frac{4.09 \cdot 10^{13}}{2} \cdot \frac{27}{64}\left(1 - \frac{B}{N}\right)^{-1} < 8.743 \cdot 10^{12},$$
$$L = \frac{1.6g^2}{4AB(B-A)^2N} \cdot \frac{27}{4}\left(1 - \frac{B}{N}\right)^2 N^3 > \frac{2.629N^2g^2}{AB(B-A)^2}$$

(note that $A'(B-A)^2 \geq 4g^3$ and $N/B \geq 3.804 \cdot 5 \cdot 4 = 76.08$). These estimates yield

$$c^{-1} < 4 \cdot 2.051 \cdot 10^{13} \cdot \frac{10A'BN}{Ag^2}(2 \cdot 8.743 \cdot 10^{12})^{\lambda-1} < \frac{1.435 \cdot 10^{28}A'BN}{Ag^2}.$$

It is easy to see that inequality (2.1) follows from the assumption $N \geq 3.804A'B^2 \times (B-A)^2/g^4$, which completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. We use the notation in the proof of Theorem 2.1. If $j = 0$, then

$$2^{2k-1}A^kB^kN^k\binom{k+\frac{1}{2}}{h_0}C_{i0}^{-1} \in \mathbb{Z}.$$

If $j = 1$, then $|C_{i1}| = A^{k_{i0}+h_0+h_1-k}(B-A)^{k_{i2}+h_2}N^k/(BN_0+1)^{k-h_1}$. By $B-A \equiv 0 \pmod 2$, (2.3) and $k_{i2} + h_2 + h_1 + h_1' \leq 3k - 1$, we have

$$2^{k-1}A^k(B-A)^{2k}N^k\binom{k+\frac{1}{2}}{h_1}C_{i1}^{-1} \in \mathbb{Z}.$$

If $j = 2$, then similarly to the above we have

$$2^{k-1}B^k(B-A)^{2k}N^k\binom{k+\frac{1}{2}}{h_2}C_{i2}^{-1} \in \mathbb{Z}.$$

Therefore, we obtain

$$p_{ijk} := 2^{-1}\left\{2AB(B-A)^2N\right\}^k \Pi_2(k)^{-1}p_{ij}(1/N) \in \mathbb{Z}$$

and

$$|p_{ijk}| < pP^k, \quad \left|\sum_{j=0}^{2} p_{ijk}\theta_j\right| < lL^{-k},$$

where

$$p < 2.046 \cdot 10^{13}, \quad P < \frac{5A'BN}{A}, \quad l < 8.632 \cdot 10^{12}, \quad L > \frac{5.394N^2}{AB(B-A)^2}.$$

These bounds show that the inequalities $c^{-1} < 7.065 \cdot 10^{27}A'BN/A$ and (2.2) hold. This completes the proof of Theorem 2.2. □

3. Preparations for the proofs of the main results

Let $\{A, B, C, D\}$ be a Diophantine quadruple with $A < B < C < D$ and r, s, t, x, y, z the positive integers satisfying $AB + 1 = r^2$, $AC + 1 = s^2$, $BC + 1 = t^2$, $AD + 1 = x^2$, $BD + 1 = y^2$, $CD + 1 = z^2$. Eliminating D from these equations, we obtain the following system of generalized Pell equations

$$Az^2 - Cx^2 = A - C, \tag{3.1}$$

$$Bz^2 - Cy^2 = B - C, \tag{3.2}$$

whose solutions can be expressed as $z = v_M$ and respectively $z = w_N$ with nonnegative integers M and N, where

$$v_0 = z_0, \qquad v_1 = sz_0 + Cx_0, \qquad v_{M+2} = 2sv_{M+1} - v_M, \tag{3.3}$$

$$w_0 = z_1, \qquad w_1 = tz_1 + Cy_1, \qquad w_{N+2} = 2tw_{N+1} - w_N. \tag{3.4}$$

Following is a lower bound for the z-component of any solution to equations (3.1)–(3.2) in terms of the index of appearance in the linear recurrent sequence (3.4) and system coefficients.

Lemma 3.1. *Suppose that there exist positive integers m and n such that $z = v_{2m} = w_{2n}$ and $|z_0| = 1$, and that $C \geq B^2 \geq 100$. Then, $\log z > n\log(4BC)$.*

Proof. One can prove this lemma in the same way as Lemma 25 in [8]. □

The following result transfers the knowledge accumulated so far to an upper bound on n.

Lemma 3.2. *Let $A' = \max\{B - A, A\}$ and $g = \gcd(A, B)$. Suppose that there exist integers $m \geq 3$ and $n \geq 2$ such that $z = v_{2m} = w_{2n}$ and $|z_0| = 1$, and that $B/g \geq 5$ and $C \geq 3.804A'B(B - A)^2/(Ag^4)$. Then,*

$$n < \frac{4\log(8.4706 \cdot 10^{13}A^{1/2}(A')^{1/2}B^2Cg^{-1})\log(1.6215\,A^{1/2}B^{1/2}(B-A)^{-1}Cg)}{\log(4BC)\log(0.2629\,A(A')^{-1}B^{-1}(B-A)^{-2}Cg^4)}.$$

Proof. We apply Theorem 2.1 and the well-known inequality (see, for instance, [8, Lemma 12])

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} < \frac{C}{2Az^2}$$

with the choice $N = ABC$, $p_1 = sBx$, $p_2 = tAy$, $q = ABz$. Having in view that $\lambda < 2$, we obtain

$$g^2 z^{2-\lambda} < 7.175 \cdot 10^{27} A' A B^4 C^2.$$

Next we employ the explicit formula for λ and obtain an upper bound for $\log z$. Comparison with the lower bound provided by Lemma 3.1 gives the stated conclusion. □

In case when both A and B are odd, the role played by the previous lemma is taken by the next result, whose proof is completely analogous to the proof of Lemma 3.2.

Lemma 3.3. *Let A, B be odd integers and $A' = \max\{B - A, A\}$. Suppose that there exist integers $m \geq 3$ and $n \geq 2$ such that $z = v_{2m} = w_{2n}$ and $|z_0| = 1$. If $B > 2000$ and $C \geq 0.927 A' B (B-A)^2/A$ then*

$$n < \frac{4 \log(5.9435 \cdot 10^{13} A^{1/2} (A')^{1/2} B^2 C) \log(2.3225\, A^{1/2} B^{1/2} (B-A)^{-1} C)}{\log(4BC) \log(1.0788\, A (A')^{-1} B^{-1} (B-A)^{-2} C)}.$$

As mentioned in the introduction, the improvements brought by the present paper in comparison with the previous ones are partially due to an absolute lower bound for b in any Diophantine quadruple $\{a, b, c, d\}$ with $d > d_+$. The next result summarises the output of computations performed with the help of Mathematica. It is obtained by combining the approach described in the last section of [13] with the remark that the proof of Lemma 4 from [13] actually works for $b \leq 12\, a$ and not only for $b \leq 8\, a$.

Lemma 3.4. *Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $a < b < c < d_+ < d$.*

- *If $b < 2\, a$ then $b > 21000$.*
- *If $2\, a \leq b \leq 12\, a$ then $b > 130000$.*
- *If $b > 12\, a$ then $b > 4000$.*

4. Proof of Theorem 1.1

Part (1). We argue by reduction to absurd, so suppose that $\{a, b, c, d, e\}$ is a Diophantine quintuple with $a < b < c < d < e$ and $b \leq 3ag$, where $g = \gcd(a, b)$. Theorem 1.1 from [6] establishes the statement for $b \leq 3a$. From this result $b/g \geq 5$ is easily deduced. Therefore, for the rest of the proof we shall assume $b > 3a$, $b \geq 5g$ and $g \geq 2$.

We apply Lemma 3.2 for the Diophantine triple $\{a, b, d\}$. By Lemmas 2.2–2.4 in [17], there are integers $m \geq 3$ and $n \geq 2$ such that $z = v_{2m} = w_{2n}$ and $|z_0| = 1$. In our situation we get

$$a' = b - a \leq \frac{3g-1}{3g} b,$$
$$3.804 \frac{a'b(b-a)^2}{ag^4} = 3.804 \frac{b}{a} \cdot \frac{(b-a)^3}{g^4} \leq \frac{3.804}{9} \cdot \frac{(3g-1)^3}{g^6} b^3,$$
$$d > 4abc \geq \frac{4b^2}{3g}(a+b+2r) > \frac{4b^3}{9g^2}(1+\sqrt{3g}\,)^2,$$

so it remains to check that inequality $(1+\sqrt{3g}\,)^2 g^4 > 0.951(3g-1)^3$ holds for any integer $g \geq 2$. The verification is done numerically for $g = 2$, while for $g \geq 3$ it follows from the chain of inequalities $(1+\sqrt{3g}\,)^2 g^4 \geq 16g^4 > 28g^3 > 0.951(3g-1)^3$.

From $a^{1/2}(b-a)^{1/2} < b/2$, $(ab)^{1/2}(b-a)^{-1} \leq \sqrt{3}/2$, and Lemma 3.2, we deduce

$$n < \frac{4\log(4.2353 \cdot 10^{13} b^3 d g^{-1}) \log(1.4043 dg)}{\log(4bd) \log(0.2629 ab^{-1}(b-a)^{-3} dg^4)}.$$

Noting that the right side of the previous inequality increases when d decreases, from $d > 4b^3(1+\sqrt{3g}\,)^2 g^{-2}/9 > 4b^3 g^{-1}/3$ it results

$$n < \frac{4\log(5.6471 \cdot 10^{13} b^6 g^{-2}) \log(1.8724 b^3)}{\log(16b^4(3g)^{-1}) \log(0.11684 ab^2(b-a)^{-3} g^2(1+\sqrt{3g})^2)}.$$

Further simplification is possible by using the fact that the function $a \mapsto a(b-a)^{-3}$ is decreasing together with the hypothesis $3ag \geq b$:

$$n < \frac{4\log(5.6471 \cdot 10^{13} b^6 g^{-2}) \log(1.8724 b^3)}{\log(16b^4(3g)^{-1}) \log(1.0515 g^4(\sqrt{3g}+1)^2(3g-1)^{-3})}. \tag{4.1}$$

Note that one always has $1.8724b^3 < 16b^4(3g)^{-1}$ and consequently from (4.1) one obtains a simpler (and only marginally weaker) inequality, namely

$$n < \frac{4\log(5.6471 \cdot 10^{13} b^6 g^{-2})}{\log(3.1545 g^5(3g-1)^{-3})}. \tag{4.2}$$

We note, for future reference, that the function $g \mapsto g^5(3g-1)^{-3}$ is increasing.

A lower bound for n is obtained from Lemma 2.4 in [5]

$$n \geq 0.5m \geq 0.25 b^{-1/2} d^{1/2} > (1+\sqrt{3g}\,) b g^{-1}/6 > 0.5b(3g)^{-1/2}. \tag{4.3}$$

The presence of g in this relation is an annoyance. We can avoid it by using the condition $b \geq 5g$, with the price of a serious deterioration of strength — then n results greater than $0.5(5b/3)^{1/2}$. Better inequalities are derived if lower as well as upper bounds for g are available, for then n is at least a small constant times b.

Initially we use $n > 0.5(5b/3)^{1/2}$ in conjunction with relation (4.2) written for $g \geq 300$ to obtain

$$\frac{\sqrt{5b}}{2\sqrt{3}} < \frac{4\log(5.6471 \cdot 10^{13} \cdot 300^{-2} b^6)}{\log(3.1545 \cdot 300^5 \cdot 899^{-3})} < 2.591 \log(29.26b),$$

a relation which holds only for $b < 2000$. Since this contradicts Lemma 3.4 (which can be applied thanks to [9, Corollary 1]), we must have $g \leq 299$. Next we apply inequality (4.3) with $g \leq 299$ and obtain a contradiction to inequality (4.2) when $g \geq 200$. The same reasoning works well for g taking values in each of the intervals $[100, 199]$, $[50, 99]$, $[20, 49]$, $[10, 19]$ and $[6, 9]$. The remaining values $g = 2, 3, 4, 5$ can individually be explored using (4.1) instead of (4.2). If $g \in \{3, 4, 5\}$, then we obtain $b < 2000$, a contradiction. If $g = 2$, then $b < 2400$. Again by Lemma 3.4 we conclude that even this situation is contradictory.

The proof of the first part of Theorem 1.1 is thus complete.

Part (2). We next search for Diophantine quintuples $\{a, b, c, d, e\}$ with $a < b < c < d < e$, $c > a + b + 2\sqrt{ab+1}$ and $b \leq \max\{24\,a\,g, 2\,a^{3/2} g^2\}$. Thanks to Theorem 1.2 in [6] and Lemma 3.4, we know that one necessarily has

$$\max\{21\,a, 2\,a^{3/2}, 4000\} < b.$$

We proceed as in the proof of the first part. Start with $g = 1$. Having in view Lemma 4 in [21], in the present context one has

$$a' = b - a \leq \frac{23}{24} b, \quad \frac{\sqrt{ab}}{b-a} < \frac{\sqrt{21}}{20}, \quad \frac{a}{b(b-a)^3} \geq \frac{24^2}{23^3} b^{-3},$$

$$\frac{3.804\,a' b (b-a)^2}{a} = \frac{3.804\,b(b-a)^3}{a} \leq \frac{3.804 \cdot 23^3}{24^2} b^3,$$

$$d > 4abc > 16a^2b^2 \geq 16\left(\frac{b}{24}\right)^2 b^2 = \frac{b^4}{36},$$

so that Lemma 3.2 yields

$$n < \frac{22.4 \log(46.948\,b) \log(0.319\,b)}{\log(0.64439\,b) \log(2892.5^{-1} b)}.$$

However, for $b > 4000$ this upper bound on n is incompatible with the reverse inequality $n > 0.25\, d^{1/2} b^{-1/2} > a\, b^{1/2} \geq 24^{-1} b^{3/2}$ provided by Lemma 2.4 from [5]. Thus, the desired conclusion is valid for $g = 1$.

Suppose next that $g \geq 2$ and $b \leq 24\, a\, g$. The working hypotheses are hereafter

$$\max\{24\, a, 2\, a^{3/2}, 4000\} < b \leq 24\, a\, g.$$

Hence,

$$a' = b - a \leq \frac{24g - 1}{24g} b, \quad \frac{\sqrt{ab}}{b - a} < \frac{\sqrt{24}}{23}, \quad \frac{b(b-a)^3}{ag^4} \leq \frac{(24g-1)^3 b^3}{24^2 g^6},$$

and therefore Lemma 3.2 can be appplied provided that $16\, bg^4 > 3.804(24g - 1)^3$, which follows from $32\, b > 3.804 \cdot 24^3$. As above, comparing the upper bound on n in Lemma 3.2 with the lower bound $n > a\, b^{1/2}$ results in relation

$$b^{1/2} < \frac{4\, \log(4.80293 \cdot 10^{11} b^7 g^{-3}) \log(0.009594\, b^4 g^{-1})}{a\, \log(9^{-1} b^5 g^{-2}) \log(4.2064\, bg^4 (24g-1)^{-3})}, \tag{4.4}$$

which together with $a \geq g$ yields

$$b^{1/2} < \frac{4\, \log(4.80293 \cdot 10^{11} b^7 g^{-3})}{g\, \log(4.2064\, bg^4 (24g-1)^{-3})}.$$

In case $g \geq 4$, from the latter inequality one derives

$$b^{1/2} < \frac{7 \log(25.75\, b)}{\log(0.001255\, b)},$$

which is false for $b > 4000$. In case $g = 2$ or 3, inequality (4.4) with $a \geq (24\, g)^{-1} b$ shows that

$$b^{3/2} < \frac{96\, g\, \log(4.80293 \cdot 10^{11} b^7 g^{-3}) \log(0.009594\, b^4 g^{-1})}{\log(9^{-1} b^5 g^{-2}) \log(3184.8^{-1} bg)},$$

which does not hold for $b > 4000$. Therefore, we conclude that there exists no Diophantine quintuple with $c > a + b + 2\sqrt{ab+1}$ and $b \leq 24\, a\, g$.

To finish the proof of Theorem 1.1, it remains to look for Diophantine quintuples $\{a, b, c, d, e\}$ satisfying $a < b < c < d < e$, $c > a + b + 2\sqrt{ab+1}$ and $\max\{24\, a\, g, 4000\} < b \leq 2\, a^{3/2} g^2$. Since

$$\frac{3.804\, b(b-a)^3}{ag^4} < \frac{3.804\, b^4}{(0.5\, bg^{-2})^{2/3} g^4} = 3.804 \cdot 2^{2/3} b^{10/3} g^{-8/3},$$

$$d > 16(0.5\, bg^{-2})^{4/3} b^2 = 4 \cdot 2^{2/3} b^{10/3} g^{-8/3},$$

the hypotheses of Lemma 3.2 are fulfilled. We therefore obtain

$$n < \frac{4\log(1.09789\cdot 10^{14}b^{19/3}g^{-25/6})\log(50.43933(24g-1)^{-1}b^{10/3}g^{-7/6})}{\log(25.3984b^{13/3}g^{-8/3})\log(1.0516)}. \quad (4.5)$$

As $13\log(50.43933(24g-1)^{-1}b^{10/3}g^{-7/6}) < 10\log(25.3984b^{13/3}g^{-8/3})$, we get

$$n < 387.3205\log(164.79\,bg^{-25/38}).$$

In conjunction with

$$n > 0.25\,d^{1/2}b^{-1/2} > a\,b^{1/2} \geq g\,b^{1/2}, \quad (4.6)$$

this entails

$$b^{1/2} < 387.3205\,g^{-1}\log(164.79\,bg^{-25/38}).$$

Hence, for $g \geq 32$ one gets $b^{1/2} < 12.1038\log(16.854\,b)$, whence $b < 24500$, in contradiction with $b > 24\,g^2 \geq 24576$.

For $g \leq 31$ we shall work with the stronger relation

$$b^{7/6} < 614.833\,g^{4/3}\log(164.79\,bg^{-25/38}), \quad (4.7)$$

derived from $n > 0.25b^{-1/2}d^{1/2} > 2^{-2/3}g^{-4/3}b^{7/6}$. For a specific g we compute from relation (4.7) a bound $UB(g)$ for b, next we find the list of all Diophantine pairs $\{a,b\}$ with $g = \gcd(a,b)$ and $24\,ag < b \leq UB(g)$. For each of the resulting pairs we write down the inequality for d generated by comparing relation (4.6) to the converse inequality for n given by Lemma 3.2, and obtain an upper bound $UD(a,b,g)$ for the second largest entry in a hypothetical Diophantine quintuple satisfying all the requirements presently in force. We get a list of 671 quadruples of the type $[g,a,b,UD(a,b,g)]$. For each entry in the list we search the values c such that $\{a,b,c\}$ is a Diophantine triple, $c > 4\,a\,b$, and $4abc < UD(a,b,g)$. Since our program finds none, the proof of Theorem 1.1 is complete.

5. Proof of Theorem 1.2

Due to many similarities with the reasoning used in the previous section, the proof below will be more concise. We shall point out its salient points and the most significant details of it.

In the first part we are considering a Diophantine quintuple $\{a,b,c,d,e\}$ satisfying $a < b < c < d < e$ and

$$\max\{3\,a, 4000\} < b \leq \frac{40}{9}a \quad \text{and} \quad \gcd(a,b) = 1.$$

From

$$a' = b - a \le \frac{31}{40}b, \quad \frac{\sqrt{ab}}{b-a} < \frac{\sqrt{3}}{2}, \quad \frac{b(b-a)^3}{a} \le \frac{31^3 b^3}{9 \cdot 40^2},$$
$$d > 4abc > \frac{9(3+\sqrt{40})^2 b^3}{400},$$

we conclude that Lemma 3.3 applies for the Diophantine triple $\{a, b, d\}$. Therefore,

$$n < \frac{18 \log(194.9003\, b) \log(1.5788\, b)}{\log(1.67253\, b) \log(1.020133)}.$$

Since

$$n > 0.25\, d^{1/2} b^{-1/2} > \frac{3(3+\sqrt{40}\,)}{80} b,$$

it follows that

$$b < \frac{2582.5 \log(194.9003\, b) \log(1.5788\, b)}{\log(1.67253\, b)}.$$

This inequality forces $b \le 40821$. There are 1646 Diophantine pairs $\{a, b\}$ consisting of odd relatively prime integers with $3\,a < b \le 40821$ and $36000 < 9\,b \le 40\,a$. For each of them we obtain an inequality of the type

$$d^{1/2} < \frac{16 b^{1/2} \log(5.9435 \cdot 10^{13} a^{1/2} (b-a)^{1/2} b^2 d)}{\log(4bd)} \times \frac{\log(2.3225\, a^{1/2} b^{1/2} (b-a)^{-1} d)}{\log(1.0788\, ab^{-1}(b-a)^{-3} d)}, \quad (5.1)$$

which only holds for d bounded from above by an explicit number $UD(a, b)$. We next search for all prolongations of $\{a, b\}$ to a Diophantine triple $\{a, b, c\}$ with $b < c < 0.25 a^{-1} b^{-1} UD(a, b)$. However, none of these triples can be extended to a Diophantine quadruple by an integer d between $4abc$ and $\min\{4c(ab+1), UD(a, b)\}$.

Part (2). From now on $\{a, b, c, d, e\}$ is a Diophantine quintuple satisfying

$$a < b < c < d < e, \quad \gcd(a, b) = 1, \quad c > 4\,ab, \quad \max\{24\,a, 2\,a^{3/2}, 4000\} < b.$$

If $b \le 42\,a$, then Lemma 3.3 applies for the Diophantine triple $\{a, b, d\}$, because

$$a' = b - a \le \frac{41}{42}b, \quad \frac{\sqrt{ab}}{b-a} < \frac{\sqrt{24}}{23}, \quad \frac{b(b-a)^3}{a} \le \frac{41^3 b^3}{42^2}, \quad d > 16a^2b^2 \ge \frac{4b^4}{21^2},$$

and therefore one has

$$n < \frac{22.4 \log(37.675\, b) \log(0.259\, b)}{\log(0.5151\, b) \log(4293^{-1} b)}.$$

Together with $n > b^{3/2}/42$, this yields

$$b^{3/2} < \frac{940.8 \log(37.675\, b) \log(0.259\, b)}{\log(0.5151\, b) \log(4293^{-1} b)},$$

an inequality which is false for $b > 4450$. We now use a computer to find all Diophantine pairs $\{a, b\}$ satisfying the requirements in force. There are 4 of them: $\{105, 4199\}$, $\{115, 4433\}$, $\{145, 4239\}$, $\{153, 4183\}$. Using inequality (5.1) as seen above, we check that no pair can be prolongated to a Diophantine quintuple fulfilling all the hypotheses.

For the rest of the proof we shall suppose

$$\max\{42\, a, 4000\} < b \le 4\, a^{3/2}.$$

From $16a^3 \ge b^2$ it follows that $16a^2b^2 > 0.927a^{-1}b(b-a)^3$, so Lemma 3.3 can be applied for the triple $\{a, b, d\}$. As above, we obtain

$$n < \frac{760 \log(128.602\, b) \log(0.977\, b)}{39 \log(1.7043\, b) \log(1.0788)}.$$

The inequality resulting from comparison of this upper bound for n with the lower bound $n > ab^{1/2} \ge 4^{-2/3} b^{7/6}$ is false for $b > 4000$. This contradiction concludes the proof of Theorem 1.2.

6. Proof of Theorem 1.3

6.1. Case $b \ge a^3$, $c = a + b + 2r$. This case is very similar to the case $c = c_1^- = a + b - 2r$ in [14] so we will not give all details here, only the sketch of the proof. Here we have in the notation from [14] $a = a_1^- = b + c - 2t$, $t = \sqrt{bc+1}$ and $a < b < c$.

Let us now assume $b > 10^{14}$ and consider the equations

$$ay^2 - bx^2 = a - b, \qquad az^2 - cx^2 = a - c.$$

We have to solve $x = v_m = w_n$, where

$$v_0 = 1, \quad v_1 = r \pm a, \quad v_{m+2} = 2rv_{m+1} - v_m,$$

$$w_0 = 1, \quad w_1 = s \pm a, \quad w_{n+2} = 2sw_{n+1} - w_n.$$

We get the exact values for fundamental solutions in the same way as in [14, Lemma 3.1.(1)]. So we are actually solving only $v_{2m} = w_{2n}$. In the same way as in [14, Lemma 4.1.(1)] we get that if $v_{m'} = w_{n'}$ with $n' \geq 2$, then $m' > n'$. So we define $\nu' = m' - n' \geq 2$.

We now define a linear form in logarithms

$$\Lambda = m' \log \alpha_1 - n' \log \alpha_2 + \log \mu, \tag{6.1}$$

where

$$\alpha_1 = r + \sqrt{ab}, \quad \alpha_2 = s + \sqrt{ac}, \quad \mu = \frac{\sqrt{c}(\sqrt{b} \pm \sqrt{a})}{\sqrt{b}(\sqrt{c} \pm \sqrt{a})}.$$

And using the standard methods we prove

$$0 < \Lambda < \alpha_2^{1-2n'} \tag{6.2}$$

if $v_{m'} = w_{n'}$ for $n' \geq 2$.

Similarly as in [14, Lemmas 4.2, 4.3], using $b > 10^{14}$, we get

$$m' \log \alpha_1 - (n' + 2.796 \cdot 10^{-11}) \log \alpha_2 < 0$$

and

$$n' > 0.99999(\nu' - 2.796 \cdot 10^{-11}) b^{1/3} \log \alpha_1 - 2.796 \cdot 10^{-11}.$$

We now quote a theorem of Mignotte on estimates of linear forms in two logarithms.

Theorem 6.1 (see [22, Corollary of Theorem 2])**.** *Let γ_1 and γ_2 be multiplicatively independent positive real numbers. For positive integers b_1 and b_2, define $\Lambda = b_1 \log \gamma_1 - b_2 \gamma_2$. Put $D = [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]/[\mathbb{R}(\gamma_1, \gamma_2) : \mathbb{R}]$. Let ρ, κ and a_i $(i \in \{1, 2\})$ be positive real numbers with $\rho \geq 4$, $\kappa = \log \rho$,*

$$a_i \geq \max\{1, (\rho - 1) \log |\gamma_i| + 2Dh(\gamma_i)\}$$

and

$$a_1 a_2 \geq \max\left\{20, 4\kappa^2\right\}.$$

Suppose that h is a real number with

$$h \geq \max\left\{3.5, 1.5\kappa, D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log \kappa + 1.377\right) + 0.023\right\},$$

and put $\chi = h/\kappa$, $v = 4\chi + 4 + 1/\chi$. Then, we have

$$\log |\Lambda| \geq -(C_0 + 0.06)(\kappa + h)^2 a_1 a_2,$$

where

$$C_0 = \frac{1}{\kappa^3}\left\{\left(2 + \frac{1}{2\chi(\chi+1)}\right)\left(\frac{1}{3} + \sqrt{\frac{1}{9} + \frac{4\kappa}{3v}\left(\frac{1}{a_1} + \frac{1}{a_2}\right) + \frac{32\sqrt{2}(1+\chi)^{3/2}}{3v^2\sqrt{a_1 a_2}}}\right)\right\}^2.$$

In order to apply Theorem 6.1 to Λ, we rewrite (6.1) as

$$\Lambda = \log(\alpha_1^{\nu'}\mu) - n'\log\left(\frac{\alpha_2}{\alpha_1}\right),$$

and take

$$D = 4, \quad b_1 = 1,\, b_2 = n', \quad \gamma_1 = \alpha_1^{\nu'}\mu, \quad \gamma_2 = \alpha_2/\alpha_1.$$

Now we know that we can take a_2 to be larger than

$$4\log|\gamma_2| + 8h(\gamma_2) \le 4\log(\alpha_2/\alpha_1 \cdot \alpha_2^2),$$

and since $\alpha_2/\alpha_1 < 1.0001$, the right side of the previous inequality is less than

$$4\log(1.0001\alpha_2^2) < 8.0001\log\alpha_2.$$

So, we can choose

$$a_2 = 8.0001\log\alpha_2 \quad \text{and} \quad \rho = 5.$$

Because $b > 10^{14}$ and $c < 1.00005b$, we have

$$\begin{aligned}\mu &\le \frac{1+\sqrt{a/b}}{1+\sqrt{a/c}} \le \frac{1+\sqrt{b^{1/3}/b}}{1+\sqrt{b^{1/3}/c}} < \frac{1+b^{-1/3}}{1+1.00005^{-1/2}b^{-1/3}}\\ &< \frac{1+10^{-14/3}}{1+1.00005^{-1/2}10^{-14/3}} < 1.000001.\end{aligned}$$

Hence, $\log\mu + 4\log\alpha_2 < 4.0000001\log\alpha_2$, which enables us to take

$$a_1 = 8(\nu' + 2.0000001)\log\alpha_2.$$

Then,

$$\frac{b_1}{a_2} < 0.000003 \cdot \frac{b_2}{a_1},$$

and

$$h = 4\log\left(\frac{n'}{(\nu'+2.0000001)\log\alpha_2}\right) - 0.883.$$

Now, $h \ge 39.9$ yields $C_0 < 0.42847$. It follows from (6.2) and Theorem 6.1 that

$$\frac{n'}{(\nu'+2.0000001)\log\alpha_2} < 15.6313\left(4\log\left(\frac{n'}{(\nu'+2.0000001)\log\alpha_2}\right) + 0.72644\right)^2 + 0.00744.$$

Then we get

$$\frac{n'}{(\nu' + 2.0000001)\log\alpha_2} < 26968.$$

For $h < 39.9$ we get a slightly better bound:

$$\frac{n'}{(\nu' + 2.0000001)\log\alpha_2} < e^{(39.9+0.883)/4} < 26790 < 26968.$$

Combining it with the lower bound for n', we get that $b < 1.5692 \cdot 10^{14}$ and $a \leq 53936$. It furthermore implies $r < 2.91 \cdot 10^9$. To finish the proof we have done the Baker–Davenport reduction using Mathematica in the way that we fix r and go through all divisors of $r^2 - 1$ which satisfy the above bounds (together with the condition $b \geq a^3$). When we have a and b we also know $c = a + b + 2r$, and using the reduction, we find no extension of such Diophantine triple $\{a, b, c\}$ except to a regular quadruple.

6.2. Proof of Theorem 1.3 (2). Suppose $g = \gcd(b, c) > 1$. Put $r = \sqrt{ab+1}$. Since $\max\{3a, 4000\} < b < a^3$ implies $(a + 2r)^2 < 6.643ab$, we have

$$\frac{3.804b'c(c-b)^2}{bg^4} < \frac{3.804 \cdot 6.643^{3/2}a^{3/2}b^{1/2}c}{16} < 4abc < d,$$

where $b' = \max\{b, c-b\}$. Hence, we may apply Lemma 3.2 with the triple $\{b, c, d\}$, because Lemmas 2.2–2.4 in [17] again assure $z = v_{2m} = w_{2n}$ with $m \geq 3$, $n \geq 2$ and $|z_0| = 1$. Noting $\max\{3a, 4000\} < b < a^3$, $b^{1/2}c^{1/2}(c-b)^{-1} < 0.5a^{-1/2}c^{1/2}$, and $c = a + b + 2r < 2.48804b$, we see from Lemma 3.2 that

$$n < \frac{4\log(6.52314 \cdot 10^{14}b^3dg^{-1})\log(1.27884a^{-1/2}b^{1/2}dg)}{\log(4bd)\log(37.599^{-1}a^{-1}b^{-1}c^{-1}dg^4)}. \tag{6.3}$$

If $g \geq 4a^{1/2}b^{1/2}$ then $1.27884a^{-1/2}b^{1/2}g > 4b$, while if $g < 4a^{1/2}b^{1/2}$ then $1.27884a^{-1/2}b^{1/2}g > 37.599^{-1}a^{-1}b^{-1}c^{-1}g^4$. Since we always have

$$6.52314 \cdot 10^{14}b^3g^{-1} > \max\{4b, 37.599^{-1}a^{-1}b^{-1}c^{-1}g^4\},$$

the right-hand side of (6.3) is a decreasing function of d. Therefore, $d > 4abc$ together with $b^{4/3} < ac < 2.48804b^2/3$ imply

$$n < \frac{4\log(2.16398 \cdot 10^{15}b^6g^{-1})\log(7.34807b^3g)}{\log(16b^{10/3})\log(9.39975^{-1}g^4)}. \tag{6.4}$$

As the right-hand side of (6.4) is a decreasing function of g, combining (6.4) with

$$n \geq 0.25c^{-1/2}d^{1/2} > 0.5a^{1/2}b^{1/2} > 0.5b^{2/3}$$

shows $b < 3000$ if $g \geq 4$, a contradiction. In case $g = 3$ one can similarly obtain an upper bound $b < 4969$, while for $g = 2$ it results $b < 49852$. Searching those quintuples (g, a, b, c, d) satisfying $\max\{3a, 4000\} < b < \min\{a^3, UB(g)\}$, $c = a + b + 2r$, $g = \gcd\{b, c\} = 2$ or 3, $UB(2) = 49852$, $UB(3) = 4969$, $d = d_+$ and Lemma 3.2 with $n \geq 0.25c^{-1/2}d^{1/2}$, one can find 19 such quintuples, each of which satisfies $g = 2$ and $a \leq 28$. Now the reduction method easily leads us to a contradiction.

6.3. Case when both a and b are even. This is a direct consequence of Theorem 1.3 (2) because c is obviously even in this case.

Acknowledgements. The authors are grateful to the referees for their meticulous comments and pertinent suggestions incorporated in the final version of the paper.

References

[1] J. Arkin, V. E. Hoggatt and E. G. Strauss, On Euler's solution of a problem of Diophantus, *Fibonacci Quart.* **17** (1979), 333–339.

[2] A. Baker and H. Davenport, The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, *Quart. J. Math. Oxford Ser. (2)* **20** (1969), 129–137.

[3] M. A. Bennett, On the number of solutions of simultaneous Pell equations, *J. Reine Angew. Math.* **498** (1998), 173–199.

[4] Y. Bugeaud, A. Dujella and M. Mignotte, On the family of Diophantine triples $\{k - 1, k + 1, 16k^3 - 4k\}$, *Glasgow Math. J.* **49** (2007), 333–344.

[5] M. Cipu, Further remarks on Diophantine quadruples, *Acta Arith.* **168** (2015), 201–219.

[6] M. Cipu and Y. Fujita, Bounds for Diophantine quintuples, *Glas. Math. Ser. III* **50** (2015), 25–34.

[7] A. Dujella, The problem of the extension of a parametric family of Diophantine triples, *Publ. Math. Debrecen* **51** (1997), 311–322.

[8] A. Dujella, An absolute bound for the size of Diophantine m-tuples, *J. Number Theory* **89** (2001), 126–150.

[9] A. Dujella, There are only finitely many Diophantine quintuples, *J. Reine Angew. Math.* **566** (2004), 183–224.

[10] A. Dujella and A. Pethő, A generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford Ser. (2)* **49** (1998), 291–306.

[11] C. Elsholtz, A. Filipin and Y. Fujita, On Diophantine quintuples and $D(-1)$-quadruples, *Monats. Math.* **175** (2014), 227–239.

[12] A. Filipin and Y. Fujita, The number of Diophantine quintuples II, *Publ. Math. Debrecen* **82** (2013), 293–308.

[13] A. Filipin, Y. Fujita and A. Togbé, The extendibility of Diophantine pairs I: the general case, *Glas. Mat. Ser. III* **49** (2014), 25–36.

[14] A. Filipin, Y. Fujita and A. Togbé, The extendibility of Diophantine pairs II: examples, *J. Number Theory* **145** (2014), 604–631.

[15] Y. Fujita, The extensibility of Diophantine pairs $\{k-1, k+1\}$, *J. Number Theory* **128** (2009), 322–353.

[16] Y. Fujita, Any Diophantine quintuple contains a regular Diophantine quadruple, *J. Number Theory* **129** (2009), 1678–1697.

[17] Y. Fujita, The number of Diophantine quintuples, *Glas. Mat. Ser. III* **45** (2010), 15–29.

[18] P. E. Gibbs, *Computer Bulletin* **17** (1978), 16.

[19] B. He and A. Togbé, On a family of Diophantine triples $\{k, A^2k+2A, (A+1)^2k+2(A+1)\}$ with two parameters, *Acta Math. Hungar.* **124** (2009), 99–113.

[20] B. He and A. Togbé, On a family of Diophantine triples $\{k, A^2k+2A, (A+1)^2k+2(A+1)\}$ with two parameters II, *Period. Math. Hungar.* **64** (2012), 1–10..

[21] B. W. Jones, A second variation on a problem of Diophantus and Davenport, *Fibonacci Quart.* **16** (1978), 155–165.

[22] M. Mignotte, A corollary to a theorem of Laurent-Mignotte-Nesterenko, *Acta Arith.* **86** (1998), 101–111.

[23] The PARI Group, PARI/GP, version `2.3.5`, *Université de Bordeaux*, 2010, `http://pari.math.u-bordeaux.fr/`.

[24] J. H. Rickert, Simultaneous rational approximations and related Diophantine equations, *Proc. Cambridge Philos. Soc.* **113** (1993), 461–472.

[25] Wolfram Research, Inc., Mathematica, Version 8.0, *Champaign, IL*, 2010.

MIHAI CIPU
SIMION STOILOW INSTITUTE
OF MATHEMATICS
OF THE ROMANIAN ACADEMY
RESEARCH UNIT NR. 5
RO-014700 BUCHAREST, P.O. BOX 1-764
ROMANIA

E-mail: `mihai.cipu@imar.ro`

ALAN FILIPIN
FACULTY OF CIVIL ENGINEERING
UNIVERSITY OF ZAGREB
FRA ANDRIJE KAČIĆA-MIOŠIĆA 26
10000 ZAGREB
CROATIA

E-mail: `filipin@grad.hr`

YASUTSUGU FUJITA
DEPARTMENT OF MATHEMATICS
COLLEGE OF INDUSTRIAL TECHNOLOGY
NIHON UNIVERSITY
2-11-1 SHIN-EI
NARASHINO, CHIBA
JAPAN

E-mail: `fujita.yasutsugu@nihon-u.ac.jp`

(Received January 12, 2015; revised August 17, 2015)