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On the 2-groups whose abelianizations are of type $(2, 4)$ and applications

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Abstract. Let G be a metabelian 2-group satisfying the condition $G/G' \simeq \mathbb{Z}/2\mathbb{Z}\times \mathbb{Z}/4\mathbb{Z}$. In this paper, we give necessary and sufficient conditions for G to be metacyclic. We then apply these results to algebraic number fields $\mathbf{k}$ to study the capitulation of their 2-ideal classes of type $(2, 4)$. Particular examples are given to illustrate how these results can be applied to real quadratic and imaginary biquadratic number fields.

1. Introduction

Let G be a group. The *commutator* of two elements x and y in G is the element $[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y$, where $x^y = y^{-1}xy$. The following two properties are easily checked, for all x, y and $z \in G$:

$$[xy, z] = [x, z]^y[y, z]. \tag{1}$$

$$[x, yz] = [x, z][x, y]^z. \tag{2}$$

Let X and Y be two subsets of G, we denote by $[X, Y]$ the subgroup of G generated by the commutators $[x, y]$, where $x \in X$ and $y \in Y$. Let $G' = [G, G]$ denote the *derived group* of G, that is the subgroup of G generated by the commutators, and let $\gamma_i(G)$ be the i-th term of the lower central series of G defined inductively by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$. The group G is said to be *nilpotent* if there exists a positive integer c such that $\gamma_{c+1}(G) = 1$; the smallest integer c

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satisfying this equality is called *the nilpotency class* of G. We call *exponent* of an abelian group the greatest order of its elements. Recall that a group G is said to be *metabelian* if its derived group G' is abelian and is said to be *metacyclic* if there exists a normal cyclic subgroup H such that the quotient group G/H is cyclic. Denote by $d(G)$ the rank of G i.e. the smallest cardinality of a generating set for G. Finally, a maximal subgroup H of a group G is a proper subgroup, such that no proper subgroup K contains H strictly.

Let G be a metabelian 2-group satisfying the condition $G/G' \simeq \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$. In this paper, we give necessary and sufficient conditions for G to be metacyclic or not. We then apply these new results to study the capitulation problem of the 2-class groups of type $(2, 4)$. The structure of this paper is the following. In § 2, we summarize preliminary results on p-groups with abelianizations are of type (p^n, p^m), where p is a prime and n, m are positive integers. In § 3, the concept of the maximal subgroup of a group G, satisfying the condition $G/G' \simeq (2, 4)$, plays an essential role in proving the main results, and thus to determine the structure of G and in which cases is or not metacyclic. In § 4, we study the capitulation of the 2-ideal classes of a number field $\mathbf{k}$ in its unramified quadratic and biquadratic extensions according to the structure of the Galois group $G = \mathrm{G}al(\mathbf{k}_2^{(2)}/\mathbf{k})$ of the second Hilbert 2-class field $\mathbf{k}_2^{(2)}$ of $\mathbf{k}$, where $\mathbf{k}$ is a number field whose 2-class group is of type $(2, 4)$. In § 5, we illustrate some of our results by two examples: the first one is about a real quadratic number field whereas the second is about an imaginary bicyclic biquadratic number field.

Theorems 7 and 9 (below) imply the main result of this paper.

Theorem. *Let G be a 2-group such that G/G' has type $(2, 4)$, and let M be the maximal subgroup of G such that M/G' is of type $(2, 2)$. Then M/M' is of type $(2, 2, 2)$ or $(2, 2^m)$. More precisely, the following assertions are equivalent:*

1. *G is metacyclic,*
2. *M/M' is of type $(2, 2^m)$, with $m \geq 1$,*
3. *$d(M) = 2$.*

2. Preliminaries

Recall first the following definitions. The *Frattini subgroup*, $\Phi(G)$, of a group G is the intersection of all its maximal subgroups, and it is known that if G is a 2-group, then $\Phi(G) = G^2$ (see [6]). *A modular group*, M_{2^n}, is a group of order

2^n, where $n > 3$, with the following presentation:

$$\langle a, b : a^{2^{n-1}} = b^2 = 1, [a, b] = a^{2^{n-2}} \rangle.$$

The group M_{2^n} is metacyclic and $\gamma_2(M_{2^n}) = \langle a^{2^{n-2}} \rangle$ is of order 2; thus $M_{2^n}/\gamma_2(M_{2^n}) \simeq (2, 2^{n-2})$.

Let G be a 2-group satisfying $\gamma_1(G)/\gamma_2(G) \simeq (2,4)$ and $|G| = 2^n$, with $n \geq 1$, then G admits three subgroups of index 4 denote them by N_i, $i \in \{1,2,3\}$, and three subgroups of index 2 denote them by H, M and K. These subgroups are visualized in Figure 1.

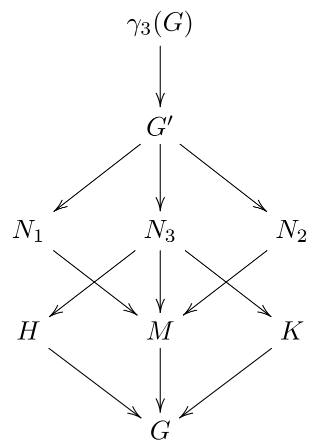

Figure 1. Subgroups of G/G'

In what follows, we will prove some new results for such group G. But first recall some results for the case where G' is cyclic not trivial. Let c be the nilpotency class of G, so

$$|G| = [G : G'] \prod_{i=2}^{c} [\gamma_i(G) : \gamma_{i+1}(G)]. \tag{3}$$

Thus Theorem 1 (see below) yields that $[\gamma_i(G) : \gamma_{i+1}(G)] = 2$, for $2 \leq i \leq c$, this in turn implies that $c = n - 2$ (the group G is said, in this case, *a group of almost maximal class*). On the other hand, C. Baginski and A. Konovalov have shown, in [7], some results which gave the complete list of these groups according to their generators and to certain relations. Among them, we find a theorem which describe all groups G that are metacyclic. The number of the metacyclic groups G of order 2^n, where $n \geq 5$ and $G/G' \simeq (2,4)$ is equal to:

$$\begin{cases} 3, & \text{if } n = 5; \\ 4, & \text{if } n > 5. \end{cases}$$

These groups have the following representation:

$$G_m = \langle a, b : a^{2^{n-2}} = 1, b^4 = z_1, a^b = a^{-1} z_2 \rangle,$$

where $1 \leq m \leq 4$ and the values of z_1, z_2 are given by the Table 1. (for G_4, we have $n > 5$).

	G_1	G_2	G_3	G_4
z_1	1	1	$a^{2^{n-3}}$	1
z_2	1	$a^{2^{n-3}}$	1	$a^{2^{n-4}}$

Table 1. The z_i values, $i = 1, 2$.

For the proof of this result see [25, Theorem 5.3, p. 352]. Finally, recall that if G is a metacyclic 2-group of order 16 such that $G/G' \simeq (2,4)$, then G is equal to

$$M_{16} = \langle a, b : a^8 = b^2 = 1, a^b = a^5 \rangle \text{ or } G_1 = \langle a, b : a^4 = b^4 = 1, a^b = a^{-1} \rangle.$$

We continue with some results on the p-groups whose abelianizations are of type (p^n, p^m), where n and m are positive integers and p is a prime. These groups are generated by two elements a and b such that $a^{p^n} \equiv b^{p^m} \equiv 1 \mod \gamma_2(G)$ (Burnside Basis Theorem).

Theorem 1 ([20]). *Let G be a p-group. If $G/G' \simeq (p^n, p^m)$, with $n \leq m$, then*

1. *$\gamma_2(G)/\gamma_3(G)$ is cyclic of order less than or equal to p^n;*
2. *The exponent of $\gamma_{i+1}(G)/\gamma_{i+2}(G)$ divides that of $\gamma_i(G)/\gamma_{i+1}(G)$.*

The following results are due to Blackburn [21, p. 334 and 335].

Theorem 2. *A p-group G is metacyclic if and only if $G/\Phi(G')\gamma_3(G)$ is metacyclic.*

Lemma 3. *If G is a nonabelian p-group generated by two elements, then $\Phi(G')\gamma_3(G)$ is the unique maximal subgroup of G' normal in G.*

Lemma 4. *If G is a nonmetacyclic p-group such that G/G' is of type (p^n, p^m), then G is generated by two elements a, b, and $G/\Phi(G')\gamma_3(G)$ is also generated by a and b modulo $\Phi(G')\gamma_3(G)$ such that*

$$[a,b]=c,\ a^{p^n}\equiv b^{p^m}\equiv c^p\equiv [a,c]\equiv [b,c]\equiv 1 \mod \Phi(G')\gamma_3(G)$$

Corollary 1. *Let G be a 2-group such that G/G' is of type $(2,4)$. Then*

1. *G is generated by two elements a and b satisfying $a^2\equiv b^4\equiv 1 \mod \gamma_2(G)$,*
2. *$\Phi(G')\gamma_3(G)=\gamma_3(G)$,*
3. *G is metacyclic if and only if $G/\gamma_3(G)$ is metacyclic,*
4. *If G is nonmetacyclic, then $a^2\equiv b^4\equiv c^2\equiv [a,c]\equiv [b,c]\equiv 1 \mod \gamma_3(G)$, where $c=[a,b]$.*

Proof. 1. The first assertion is an immediate consequence of the Burnside Basis Theorem.

2. The result is obvious if G is abelian. Assume G is not abelian, then Theorem 1 implies that $[G':\gamma_3(G)]=2$, hence $\gamma_3(G)$ is the maximal subgroup of G' normal in G; thus Lemma 3 yields that $\Phi(G')\gamma_3(G)=\gamma_3(G)$.

3. and 4. are obvious. □

E. Benjamin and C. Snyder showed, in [8], a lemma that gives information about the structure of a 2-group G, satisfying the condition G/G' is of type $(2,2^m)$, where $m>1$. Put $G^{(2,2)}=\Phi(G)^2[G,\Phi(G)]$, then the lemma is as follows:

Lemma 5. *Let G be a 2-group such that G/G' is of type $(2,2^m)$, where $m>1$. Then*

i) *If G is abelian, then $G/G^{(2,2)}$ is of type $(2,4)$,*

ii) *If G is modular, then $G/G^{(2,2)}$ is of type $(2,4)$,*

iii) *If G is metacyclic-nonmodular, then $G/G^{(2,2)}$ is the unique metacyclic-nonmodular group of order 16 whose abelianization is of type $(2,4)$,*

iv) *If G is nonmetacyclic, then $G/G^{(2,2)}$ is the unique nonmetacyclic group of order 16 whose abelianization is of type $(2,4)$.*

If G is a 2-group such that G/G' is of type $(2,4)$, then Lemma 5 is almost the same as the Corollary 1. The following remark shows this.

Remark 1. Let G be a 2-group such that G/G' is of type $(2,4)$, then $G^{(2,2)}=\gamma_3(G)$.

Proof. Let $G = \langle x, y\rangle$ such that $x^2 \equiv y^4 \equiv 1 \mod \gamma_2(G)$. Note that $G^{(2,2)} = (G^2)^2[G, G^2]$. According to [19, Corollary 2.3, p. 104] we have $G^{(2,2)} = G^4\gamma_2^2(G)\gamma_3(G)$, as $\gamma_2(G)/\gamma_3(G)$ is of order 2, so $\gamma_2^2(G) \subseteq \gamma_3(G)$, thus $G^{(2,2)} = G^4\gamma_3(G)$.

If G is a metacyclic metabelian group of order 16, then $G^4 = 1$, thus the remark is obvious. Assume G is metacyclic of order > 16. One can easily show that $\gamma_2(G) = \langle x^2\rangle$ (see Lemma 8 below), hence we get $\gamma_3(G) = \langle x^4\rangle$, which implies that $\gamma_3(G) \subseteq G^4$ and $G^{(2,2)} = G^4$. Moreover, as $\gamma_2(G)/\gamma_3(G)$ is of order 2 and $\gamma_3(G) \subseteq G^{(2,2)} \subseteq \gamma_2(G)$, so $G^{(2,2)} = \gamma_2(G)$ or $G^{(2,2)} = \gamma_3(G)$. The first case can not occur, for if $G^{(2,2)} = \gamma_2(G)$, then $G^4 = \gamma_2(G)$. Which is absurd, since $x^4 \in G^4$ and x^4 does not generate $\gamma_2(G)$.

If G is not metacyclic, then $x^2 \equiv y^4 \equiv 1 \mod \gamma_3(G)$ (Corollary 1). Thus $G^4 \subseteq \gamma_3(G)$, and hence $G^{(2,2)} = \gamma_3(G)$. □

3. Proof of the main result

The idea of these results is a consequence of the study that we have made for a particular case, where G is the Galois group of some number field extension (see [2]). To generalize these observations to any 2-group, we have based on the works of E. Benjamin, F. Lemmermeyer and C. Snyder, in particular the article [11].

Keep the notations used in [9, Lemma 1]. Let $G = \langle a, b\rangle$ be a metabelian-nonmetacyclic 2-group such that $a^2 \equiv b^4 \equiv 1 \mod \gamma_2(G)$. The terms c_i are defined as follows: $[a, b] = c = c_2$ and $c_{j+1} = [b, c_j]$. Thus $G' = \langle c_2, c_3, \dots\rangle$, $\gamma_3(G) = \langle c_2^2, c_3, \dots\rangle$ and $\gamma_4(G) = \langle c_2^4, c_3^2, c_4, \dots\rangle$ (see [9, Lemma 2]). As M is the maximal subgroup of G satisfying $M/G' \simeq (2, 2)$, so $M = \langle a, b^2, G'\rangle$. By a simple calculation based on [9, Lemma 2], we check that $M'\gamma_4(G) = \gamma_3(G)$. On the other hand, from [23, Theorem 2.49ii], we get $M' = \gamma_3(G)$. This allows us to cite the following lemma:

Lemma 6. *Let G be a metabelian-nonmetacyclic 2-group, satisfying G/G' is of type $(2, 4)$, and let M be the maximal subgroup of G such that M/G' is of type $(2, 2)$. Then $M' = \gamma_3(G)$.*

Corollary 2. *Let G be a nonmetacyclic 2-group of order 16 satisfying the condition G/G' is of type $(2, 4)$, and let M be the maximal subgroup of G such that M/G' is of type $(2, 2)$. Then M/M' is of type $(2, 2, 2)$.*

Proof. As G is the unique nonmetacyclic 2-group of order 16, satisfying G/G' is of type $(2,4)$, and as M is the maximal subgroup of G such that M/G' is of type $(2,2)$. So

$$G = \langle a, b : c = [a,b], a^2 = b^4 = c^2 = 1, [a,c] = [b,c] = 1\rangle.$$

Since $G' = \langle c\rangle$, then G is metabelian and

$$M = \langle a, b^2, c\rangle \quad \text{and} \quad M' = \langle [a,c], [a,b^2], [b^2,c]\rangle.$$

The Properties (1) and (2) yield that

$$[a,b^2] = [a,b]\cdot[a,b]^b = c\cdot c^b = c^2[c,b] = 1$$
$$[b^2,c] = [b,c]^b\cdot[b,c] = 1.$$

Hence M is an abelian group of order 8; but since $a^2 = b^4 = c^2 = 1$, so M/M' is of type $(2,2,2)$. □

The following theorem generalizes this result to any nonmetacyclic 2-group.

Theorem 7. *Let G be a metabelian 2-group such that G/G' has type $(2,4)$, and let M be the maximal subgroup of G such that M/G' is of type $(2,2)$. Then the following assertions are equivalent:*

1. *G is nonmetacyclic,*
2. *M/M' is of type $(2,2,2)$,*
3. *$d(M) = 3$.*

Proof. According to our hypotheses, we have $G = \langle a,b\rangle$ with $a^2 \equiv b^4 \equiv 1 \mod \gamma_2(G)$. Put $M = \langle a, b^2, \gamma_2(G)\rangle$, then by Schreier inequality we get $d(M)-1 \le [G:M](d(G)-1)$. Thus $d(M) \in \{1,2,3\}$; but M can not be cyclic, since M admits three maximal subgroups (N_1, N_2 and N_3 see Figure 1). From this, we conclude that $d(M) \in \{2,3\}$.

1. $\Longrightarrow$ 2. Assume that G is not metacyclic, then Lemma 6 implies that $M' = \gamma_3(G)$. As G/G' is of type $(2,4)$, so by Theorem 1 we get $\gamma_2(G)/\gamma_3(G)$ is of order 2. Therefore Corollary 1 yields that $a^2 \equiv b^4 \equiv c^2 \equiv 1 \mod \gamma_3(G)$, with $[a,b] = c$. This shows that the exponent of M/M' is 2. Finally, as $[M:M'] = [M:\gamma_3(G)] = [M:\gamma_2(G)]\cdot[\gamma_2(G):\gamma_3(G)] = 4\cdot 2 = 8$, so M/M' is of type $(2,2,2)$.

2. $\Longrightarrow$ 3. This implication is guaranteed by Burnside Basis Theorem.

3. $\Longrightarrow$ 1. If $d(M) = 3$, then G can not be metacyclic, for if G is metacyclic, then any subgroup M of a metacyclic p-group will satisfy $d(M) \le 2$. Which is a contradiction. □

Keep the previous notations. In the metacyclic case, we have the following lemma:

Lemma 8. *Let G be a metacyclic 2-groupe of order 2^n, where $n \geq 4$ and G/G' is of type $(2,4)$, and let M be the maximal subgroup of G satisfying the condition M/G' is of type $(2,2)$. Then M' is of order ≤ 2. Moreover*

$$M' = \begin{cases} 1, & \text{if } G = G_1, G_2, G_3 \text{ or } M_{16}; \\ \langle a^{2^{n-3}} \rangle, & \text{if } G = G_4. \end{cases}$$

Proof. There are two cases to distinguish:

1- Assume $n \geq 4$ and G not modular, then $G = G_m = \langle a, b : a^{2^{n-2}} = 1, b^4 = z_1, a^b = a^{-1} z_2 \rangle$, where $1 \leq m \leq 4$ and the values of z_1, z_2 are given by the Table 1 ($n = 4$ only for $G = G_1$, and for G_4 we have $n > 5$). As G is a metacyclic group, then G' is cyclic, which implies that $G' = \langle [a,b] \rangle$. Let us compute $[a,b]$:

$$[a,b] = a^{-1} a^b = a^{-2} z_2 = \begin{cases} a^{-2}, & \text{if } G = G_1 \text{ or } G_3; \\ a^{-2+2^{n-3}}, & \text{if } G = G_2; \\ a^{-2+2^{n-4}}, & \text{if } G = G_4. \end{cases}$$

$$= \begin{cases} a^{-2}, & \text{if } G = G_1 \text{ or } G_3; \\ a^{-2(1-2^{n-4})}, & \text{if } G = G_2; \\ a^{-2(1-2^{n-5})}, & \text{if } G = G_4. \end{cases} \tag{4}$$

If $G = G_2$ and $n \geq 5$, then $1 - 2^{n-4}$ is odd. If $G = G_4$ and $n \geq 6$, then $1 - 2^{n-5}$ is odd. Therefore we deduce that:

$$G' = \langle [a,b] \rangle = \langle a^{-2} \rangle = \langle a^2 \rangle.$$

As $a^2 \equiv b^4 \equiv 1 \mod \gamma_2(G)$, so $M = \langle a, b^2 \rangle$ and $M' = \langle [a, b^2] \rangle$. The Property (2) yields that $[a, b^2] = [a,b][a,b]^b$. Compute $[a,b]^b$:

$$[a,b]^b = \begin{cases} (a^{-2})^b, & \text{if } G = G_1 \text{ or } G_3; \\ (a^{-2+2^{n-3}})^b, & \text{if } G = G_2; \\ (a^{-2+2^{n-4}})^b, & \text{if } G = G_4. \end{cases}$$

$$= \begin{cases} a^2, & \text{if } G = G_1 \text{ or } G_3; \\ a^{(-1+2^{n-3})(-2-2^{n-3})}, & \text{if } G = G_2; \\ a^{(-1+2^{n-4})(-2-2^{n-4})}, & \text{if } G = G_4. \end{cases}$$

$$= \begin{cases} a^2, & \text{if } G = G_1 \text{ or } G_3; \\ a^{2-2^{n-3}-2^{n-2}+2^{2n-6}}, & \text{if } G = G_2; \\ a^{2-2^{n-4}-2^{n-3}+2^{2n-8}}, & \text{if } G = G_4. \end{cases}$$

The Equalities (4) imply that

$$[a, b^2] = \begin{cases} 1, & \text{if } G = G_1 \text{ or } G_3; \\ a^{-2^{n-2}+2^{2n-6}}, & \text{if } G = G_2; \\ a^{-2^{n-3}+2^{2n-8}}, & \text{if } G = G_4. \end{cases}$$

$$= \begin{cases} 1, & \text{if } G = G_1 \text{ or } G_3; \\ a^{2^{n-2}(-1+2^{n-4})}, & \text{if } G = G_2; \\ a^{2^{n-3}(-1+2^{n-5})}, & \text{if } G = G_4. \end{cases}$$

Since $a^{2^{n-2}} = 1$, then M' is of order ≤ 2. Moreover,

$$M' = \begin{cases} 1, & \text{if } G = G_1, G_2 \text{ or } G_3; \\ \langle a^{2^{n-3}} \rangle, & \text{if } G = G_4. \end{cases}.$$

2- Assume now G modular, then $G = M_{16} = \langle a, b : b^2 = a^8 = 1, a^b = a^5 \rangle$. It is easy to show that $[a, b] = a^4$ and $[a, b]^a = a^4$, hence $M' = \langle [a^2, b] \rangle = \langle a^8 \rangle = 1$. This completes the proof. □

Theorem 9. *Let G be a nonabelian group satisfying the condition G/G' is of type $(2,4)$, and let M be the maximal subgroup of G such that M/G' is of type $(2,2)$. Then the following assertions are equivalent:*

1. *G is metacyclic,*
2. *M/M' is of type $(2, 2^m)$, with $m \geq 2$,*
3. *$d(M) = 2$.*

Proof. 1. $\Longrightarrow$ 2. Suppose G is a metacyclic-nonmodular 2-group of order 2^n. If $n \geq 4$ and $G = G_1, G_2$ or G_3, then, as $M = \langle x, y^2 \rangle$, Lemma 8 yields that M/M' is of type $(2, 2^{n-2})$. If $n \geq 6$ and $G = G_4$, then M/M' is of type $(2, 2^{n-3})$. If G is a modular 2-group, then $G = \langle x, y : x^2 = y^8 = 1, y^x = y^5 \rangle$. In Lemma 8, we reported that $M = \langle x, y^2 \rangle$ is an abelian group, this implies that M/M' is of type $(2, 2^2)$.

2. $\Longrightarrow$ 3. This implication holds according to Burnside Basis Theorem.

3. $\Longrightarrow$ 1. See Theorem 7. □

From Theorems 7 and 9 we get:
– If the order of M/M' is > 8, then G is a metacyclic 2-group.
– If the order of M/M' is equal to 8, then G is a 2-group

$$\begin{cases} \text{modular or isomorphic to } G_1, & \text{if } \ M/M' \text{ is of type } (2,4), \\ \text{nonmetacyclic }, & \text{if } \ M/M' \text{ is of type } (2,2,2). \end{cases}$$

This allows us to cite the two following corollaries

Corollary 3. *Let G be a 2-group satisfying the condition G/G' is of type $(2,4)$, and let M be the maximal subgroup of G such that M/G' is of type $(2,2)$ and M/M' is of type $(2,4)$. Then G is a modular group or*

$$G = G_1 = \langle x, y : x^4 = y^4 = 1, y^x = y^{-1} \rangle.$$

Corollary 4. *Let G be a 2-group satisfying G/G' is of type $(2,4)$, and let M be the maximal subgroup of G such that M/G' is of type $(2,2)$ and M/M' is of order > 8. Then G is metacyclic-nonabelian-nonmodular.*

Theorem 10. *Let G be a 2-group satisfying G/G' is of type $(2,4)$, and let M (resp. H and K) be the maximal subgroup of G such that M/G' is of type $(2,2)$ (resp. H/G' and K/G' are cyclic of order 4). Then the following assertions are equivalent:*

1. *G is abelian,*
2. *M/M' is of type $(2,2)$,*
3. *H is cyclic of order 4,*
4. *K is cyclic of order 4.*

Proof. 1. $\Longleftrightarrow$ 2. Theorems 7 and 9.

1. $\Longrightarrow$ 3. and 4. obvious.

3. $\Longrightarrow$ 1. (resp. 4. $\Longrightarrow$ 1.) as $[H : G'] = 4$ (resp. $[K : G'] = 4$) and the order of H (resp. K) is 4, then $G' = 1$, thus G is abelian. □

4. Application: capitulation of the 2-ideal classes of type $(2,4)$

Throughout all this section, $\mathbf{k}$ denotes a number field whose 2-class group is of type $(2,4)$. Let $\mathrm{C}_{\mathbf{k},2}$ denote the 2-class group of $\mathbf{k}$, that is the 2-Sylow subgroup of the ideal class group $\mathrm{C}_{\mathbf{k}}$ of $\mathbf{k}$, in the wide sens. Let $\mathbf{k}_2^{(1)}$ be the Hilbert 2-class field of $\mathbf{k}$ in the wide sens. Then the Hilbert 2-class field tower of $\mathbf{k}$ is defined

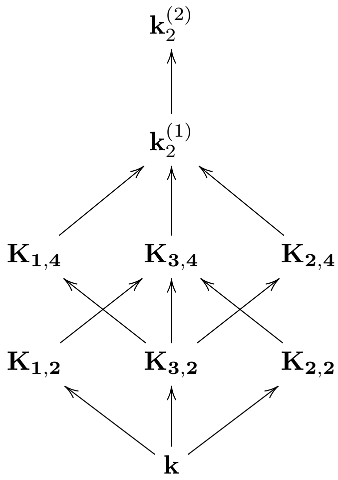

Figure 2. Subfields of $\mathbf{k}_2^{(1)}/\mathbf{k}$.

inductively by: $\mathbf{k}_2^{(0)} = \mathbf{k}$ and $\mathbf{k}_2^{(n+1)} = (\mathbf{k}_2^{(n)})^{(1)}$, where n is a positive integer. Let $\mathbb{M}$ be an unramified abelian extension of $\mathbf{k}$ and $\mathrm{C}_{\mathbb{M}}$ be the subgroup of $\mathrm{C}_{\mathbf{k}}$ associated to $\mathbb{M}$ by the class field theory. Denote by $j_{\mathbf{k}\to\mathbb{M}} : \mathrm{C}_{\mathbf{k}} \longrightarrow \mathrm{C}_{\mathbb{M}}$ the homomorphism that associate to the class of an ideal $\mathcal{A}$ of $\mathbf{k}$ the class of the ideal generated by $\mathcal{A}$ in $\mathbb{M}$, and by $\mathcal{N}_{\mathbb{M}/\mathbf{k}}$ the norm of the extension $\mathbb{M}/\mathbf{k}$.

According to class field theory G/G' is also of type $(2,4)$, where $G = \mathrm{G}al(\mathbf{k}_2^{(2)}/\mathbf{k})$; hence $G = \langle a, b\rangle$ such that $a^2 \equiv b^4 \equiv 1 \mod G'$ and $\mathrm{C}_{\mathbf{k},2} = \langle \mathfrak{c}, \mathfrak{d}\rangle \simeq \langle aG', bG'\rangle$, where $(\mathfrak{c}, \mathbf{k}_2^{(2)}/\mathbf{k}) = aG'$ and $(\mathfrak{d}, \mathbf{k}_2^{(2)}/\mathbf{k}) = bG'$, with $(\ .\ , \mathbf{k}_2^{(2)}/\mathbf{k})$ denotes the Artin symbol in $\mathbf{k}_2^{(2)}/\mathbf{k}$. Accordingly, there are three normal subgroups of G of index 2; denote theme by: $H_{1,2}$, $H_{2,2}$ and $H_{3,2}$ such that $H_{1,2} = \langle b, G'\rangle$, $H_{2,2} = \langle ab, G'\rangle$ and $H_{3,2} = \langle a, b^2, G'\rangle$. There are also three normal subgroups of G of index 4; denote theme by: $H_{1,4}$, $H_{2,4}$ and $H_{3,4}$ such that $H_{1,4} = \langle a, G'\rangle$, $H_{2,4} = \langle ab^2, G'\rangle$ and $H_{3,4} = \langle b^2, G'\rangle$.

It is well known that each subgroup $H_{i,j}$ of G is associated, by class field theory, to a unique unramified extension $\mathbf{K}_{i,j}$ in $\mathbf{k}_2^{(2)}$ such that $H_{i,j}/H'_{i,j} \simeq \mathrm{C}_{\mathbf{K}_{i,j},2}$. The situation is represented by Figure 2.

Our goal is to study the capitulation problem of the 2-ideal classes of $\mathbf{k}$ in its unramified quadratic extensions $\mathbf{K}_{1,2}$, $\mathbf{K}_{2,2}$ and $\mathbf{K}_{3,2}$, and in its unramified biquadratic extensions, if it is possible, $\mathbf{K}_{1,4}$, $\mathbf{K}_{2,4}$ and $\mathbf{K}_{3,4}$.

In what follows, we use the symbol (X_1, X_2, X_3), where $X_i \in \{4, 2, 2A, 2B\}$,

$i \in \{1,2,3\}$, with the following meanings. $X_i = 4$ or 2 means four or two ideal classes of $\mathbf{k}$ capitulate in $\mathbf{K}_{i,2}$, and $X_i = 2A$ (resp. $2B$) means two ideal classes of $\mathbf{k}$ capitulate in $\mathbf{K}_{i,2}$ and the unramified quadratic extension $\mathbf{K}_{i,2}$ of $\mathbf{k}$ satisfies the Taussky's condition (A) (resp. (B)) (see [22]).

We know, from [15], that if $\mathbf{L}$ is a number field and $G = \mathrm{Gal}(\mathbf{L}_2^{(2)}/\mathbf{L})$ is a 2-group satisfying the condition $G/G' \simeq (2,2)$, then the structure of G is completely determined based on the number of the 2-ideal classes of $\mathbf{L}$ that capitulate in its three unramified quadratic extensions, and specifying whether the unramified quadratic extensions of $\mathbf{L}$ are or not of type (A) or (B). More precisely, G is abelian of type $(2,2)$, quaternion, dihedral or semi-dihedral. Hence G is always a metacyclic 2-group. Note that in the case where G/G' is of type $(2, 2^m)$, with $m > 2$, E. Benjamin and C. Snyder proposed, in [8], a method to know whether G is or not metacyclic, this method is based on studying the same questions for $G/G^{(2,2)}$ as for G (see [8] p. 12). If we denote by $\mathbf{L}_1$, $\mathbf{L}_2$ and $\mathbf{L}_3$ the unramified quadratic extensions of $\mathbf{L}$, where $\mathbf{L}_3$ corresponds to $\mathbf{K}_{3,2}$ the non-cyclic maximal subgroup over G', then we can summarize these results in the following table:

$\mathbf{L}_1$	$\mathbf{L}_2$	$\mathbf{L}_3$	G
4	4	4	abelian
$2B$	$2B$	$2A$	modular or nonmetacyclic
$2A$	$2A$	4	metacyclic nonmodular
$2A$	$2A$	$2A$	metacyclic nonmodular or nonmetacyclic

In the other cases G is nonmetacyclic. Then in the cases $(2A, 2A, 2A)$ and $(2B, 2B, 2A)$, one cannot conclude anything about the structure of G. In [16], there is a characterization of $G/G^{(2,2)}$, but only in the case where $\mathbf{L}$ is an imaginary quadratic number field. In our case, we have $G/G^{(2,2)} = G/\gamma_3(G)$ (see Corollary 1 and Remark 1). This shows that Koch's characterization coincides with that of N. Blackburn. In Section 2, we gave a new method to determine if G is metacyclic or not, based on the structure of the abelian group $H_{3,2}/H'_{3,2}$, which is the structure of the 2-class group of $\mathbf{K}_{3,2}$.

By class field theory, the kernel of $j_{\mathbf{k}\to\mathbb{M}}$, $\ker j_{\mathbf{k}\to\mathbb{M}}$, is determined by the kernel of the transfer map $\mathrm{V}_{G\to H} : G/G' \to H/H'$, where $G = \mathrm{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k})$ and $H = \mathrm{Gal}(\mathbb{M}_2^{(2)}/\mathbb{M})$. To compute the kernel of the transfer map $\mathrm{V}_{G\to H}$, we use the following formula (see [17]): for $g \in G$, put $f = [\langle g\rangle.H : H]$ and let $\{x_1, x_2, \dots, x_t\}$ be a set of representatives of $G/\langle g\rangle H$, then

$$\mathrm{V}_{G\to H}(gG') = \prod_{i=1}^{t} x_i^{-1} g^f x_i.H'. \tag{5}$$

4.1. Modular case. In this subsection, we study the capitulation problem in the case where G is a modular 2-group. Then $G = \langle a, b : a^2 = b^8 = 1, b^a = b^5\rangle$. As a result of Formula (5), we get the following lemma:

Lemma 11. *Keep the previous notations. If G is a modular 2-group, then*

1. $\mathrm{V}_{G\to H_{i,2}}(xG') = \begin{cases} 1, & \text{if } x = a, \\ b^4, & \text{if } x = b^2. \end{cases}$ *with* $i = 1, 2$.
2. $\mathrm{V}_{G\to H_{3,2}}(xG') = \begin{cases} b^{-4}, & \text{if } x = a, \\ b^4, & \text{if } x = b^2. \end{cases}$
3. $\mathrm{V}_{G\to H_{i,4}}(xG') = \begin{cases} 1, & \text{if } x = a, \\ b^4, & \text{if } x = b. \end{cases}$ *with* $i = 1, 2$ *or* 3.

Proof. As $G = \langle a, b : a^2 = b^8 = 1, b^a = b^5\rangle$, so $[b, a] = b^{-1}b^a = b^4$, thus $G' = \langle b^4\rangle$, $H_{1,2} = \langle b\rangle$, $H_{2,2} = \langle ab, b^4\rangle$, $H_{3,2} = \langle a, b^2\rangle$, $H_{1,4} = \langle a, b^4\rangle$, $H_{2,4} = \langle ab^2, b^4\rangle$ and $H_{3,4} = \langle b^2\rangle$. Moreover, $ba = ab^5$, $b^8 = 1$ and $a^2 = 1$, hence

$$(ab)^4 = abababab = a^2b^5ba^2b^5b = b^{12} = b^4, \quad (ab^2)^2 = abbab^2 = abab^7 = a^2b^{12} = b^4.$$

From which we deduce that $H_{2,2} = \langle ab\rangle$ and $H_{2,4} = \langle ab^2\rangle$.

We will only show the first result (i.e. 1.); for the others, we proceed similarly. If $i = 1$ or 2, then the Formula (5) yields that

$$\mathrm{V}_{G\to H_{i,2}}(xG') = \begin{cases} a^2H'_{i,2}, & \text{if } x = a, \\ b^4[b^2, a]H'_{i,2}, & \text{if } x = b^2, \end{cases}$$

since $a \notin H_{i,2}$ and $b^2 \in H_{i,2}$. As $[b^2, a] = [b, a]^b[b, a] = (b^4)^b b^4 = b^8 = 1$ and $H_{i,2}$ is a cyclic subgroup of G, which completes the prove of the result 1. □

Theorem 12. *Keep the previous notations. Then G is a modular group if and only if the 2-class group of $\mathbf{K_{3,2}}$ is of type $(2, 4)$ and only two ideal classes capitulate in $\mathbf{K_{3,2}}$. In this situation, we have:*

1. $\ker j_{\mathbf{k}\to\mathbf{K}_{i,2}} = \{1, \mathfrak{c}\}$ *with* $i = 1, 2$.
2. $\ker j_{\mathbf{k}\to\mathbf{K}_{3,2}} = \{1, \mathfrak{c}\mathfrak{d}^2\}$.
3. $\ker j_{\mathbf{k}\to\mathbf{K}_{i,4}} = \{1, \mathfrak{c}, \mathfrak{d}^2, \mathfrak{c}\mathfrak{d}^2\}$ *with* $i = 1, 2$ *or* 3.
4. *The capitulation of the 2-ideal classes of* $\mathbf{k}$ *is of type* $(2B, 2B, 2A)$.

Proof. Suppose that the 2-class group of $\mathbf{K_{3,2}}$ is of type $(2, 4)$, then the quotient group $H_{3,2}/H'_{3,2}$ is of type $(2, 4)$. Hence, according to Corollary 3, G is a modular group or

$$G = G_1 = \langle a, b : a^4 = b^4 = 1, a^b = a^{-1}\rangle.$$

If $G = G_1$, then $[a,b] = a^{-1}a^b = a^{-2}$ and $H_{3,2} = \langle a, b^2\rangle$. As G' is a cyclic group generated by a^2, so $H'_{3,2}$ is also a cyclic group generated by

$$[b^2, a] = [b,a]^b[b,a] = (a^2)^b a^2 = (a^b)^2 a^2 = a^{-2}a^2 = 1,$$

this yields that $H'_{3,2} = 1$; from which we deduce that

$$\mathrm{V}_{G\to H_{3,2}}(xG') = \begin{cases} a^2[a,b] = 1, & \text{if } \ x = a, \\ b^4[b^2,b] = 1, & \text{if } \ x = b^2. \end{cases}$$

Therefore $\ker \mathrm{V}_{G\to H_{3,2}} = \{aG', b^2G', ab^2G', G'\}$ i.e. there are four classes capitulate in $\mathbf{K_{3,2}}$; which contradicts the fact that only two classes capitulate in $\mathbf{K_{3,2}}$. Thus G is a modular group.

Conversely, if G is a modular group, then according to the proof of Theorem 9, we have $H_{3,2}/H'_{3,2}$ is of type $(2,4)$, from which we deduce that the 2-class group of $\mathbf{K_{3,2}}$ is of type $(2,4)$. Moreover, Lemma 11 yields that $\ker \mathrm{V}_{G\to H_{3,2}} = \{ab^2, 1\}$ i.e. two classes capitulate in $\mathbf{K_{3,2}}$, precisely, $\ker j_{\mathbf{k}\to\mathbf{K}_{3,2}} = \{1, \mathfrak{c}\mathfrak{d}^2\}$.

To prove the assertions 1. and 3., it suffices to compute the kernel of $\mathrm{V}_{G\to H_{i,j}}$ using Lemma 11. For the assertion 4., it is a consequence of the Formula (5). □

4.2. Nonmodular metacyclic case. In this subsection, we study the capitulation problem in the case where G is a metacyclic-nonmodular 2-group of order 2^n and $G/G' \simeq (2,4)$, with $n \geq 4$. We begin by the following lemma that will allows us to compute the derived group of $H_{i,j}$; after this, we calculate the images of aG', bG' and b^2G' by $\mathrm{V}_{G\to H_{i,j}}$.

Lemma 13. *Keep the previous notations. If $G = G_m$, then*

(i) $[a,b] = \begin{cases} a^{-2}, & \text{if } \ G = G_1 \text{ or } G_3; \\ a^{-2(1-2^{n-4})}, & \text{if } \ G = G_2; \\ a^{-2(1-2^{n-5})}, & \text{if } \ G = G_4. \end{cases}$

(ii) $[a,b^2] = \begin{cases} 1, & \text{if } \ G = G_1, G_2 \text{ or } G_3; \\ a^{2^{n-3}(-1+2^{n-5})}, & \text{if } \ G = G_4. \end{cases}$

(iii) $[a^2,b] = \begin{cases} a^{-4}, & \text{if } \ G = G_1, G_2 \text{ or } G_3; \\ a^{-4+2^{n-3}}, & \text{if } \ G = G_4. \end{cases}$

(iv) $[a^2, b^2] = 1$.

Proof. (i) and (ii). See the proof of Lemma 8.

(iii) It suffices to use the equalities $[a^2,b] = [a,b]^a[a,b]$ and $[a,b] = a^{-2}z_2$.

(iv) We have

$$[a^2,b^2]=[a,b^2]^a[a,b^2]=\begin{cases}a^{2^{n-2}(-1+2^{n-5})}, & \text{if } G=G_4,\\ 1, & \text{if not,}\end{cases} \quad \text{since } a^{2^{n-2}}=1.$$

The result is then obvious. □

Corollary 5. *Keep the previous notations. If $G=G_m$, then*

(i) $G'=\langle a^2\rangle$;

(ii) $H'_{2,2}=H'_{1,2}=\langle a^4\rangle$ *and* $H'_{3,2}=\begin{cases}1, & \textit{if } G=G_1,G_2 \textit{ or } G_3;\\ \langle a^{2^{n-3}}\rangle, & \textit{if } G=G_4.\end{cases}$

(iii) $H'_{1,4}=H'_{2,4}=H'_{3,4}=1$.

Proof. (i) As $G=\langle a,b\rangle$ is a metacyclic group, then the derived group G' is cyclic generated by $[a,b]$. If $G=G_4$ and $n\geq 6$, then $1-2^{n-5}$ is odd. Therefore $G'=\langle [a,b]\rangle=\langle a^{-2}\rangle=\langle a^2\rangle$.

(ii) The result in (i) allows us to conclude that $H_{1,2}=\langle b,a^2\rangle$, $H_{2,2}=\langle ab,a^2\rangle$, $H_{3,2}=\langle a,b^2\rangle$, $H_{1,4}=\langle a\rangle$, $H_{2,4}=\langle ab^2,a^2\rangle$, and $H_{3,4}=\langle a^2,b^2\rangle$. On the other hand, by using Properties (1) and (2), we get

$$[ab^2,a^2]=[a,a^2]^{b^2}[b^2,a^2]=[b^2,a^2] \text{ et } [ab,a^2]=[a,a^2]^b[b,a^2]=[b,a^2].$$

Our corollary is then an immediate consequence of the Lemma 13. □

Theorem 14. *Keep the previous notations and assume that G is of order >16, then the following properties are equivalent:*

1. *G is metacyclic,*
2. *The 2-class group of $\mathbf{K_{3,2}}$ is of type $(2,2^m)$, with $m\geq 3$,*
3. *The 2-class number of $\mathbf{K_{3,2}}$ is >8,*
4. *The 2-class group of $\mathbf{K_{1,4}}$ is cyclic of order >2.*

Proof. The equivalences 1. $\Longleftrightarrow$ 2. $\Longleftrightarrow$ 3. are direct consequences of Theorem 9 and Corollary 4.

1. $\Longrightarrow$ 4. From the Corollary 5, we get $H_{1,4}=\langle a\rangle$ and the order of a is >2, hence $H'_{1,4}=1$. Given the fact that $\langle a\rangle=H_{1,4}/H'_{1,4}\simeq \mathrm{C}_{2,K_{1,4}}$, we get the result.

4. $\Longrightarrow$ 1. As $H_{1,4}/H'_{1,4}\simeq \mathrm{C}_{2,K_{1,4}}$ and $\mathrm{C}_{2,K_{1,4}}$ is a cyclic group, so the Burnside Basis Theorem implies that $H_{1,4}$ is a normal cyclic subgroup of G of index 4. Note that $G/H_{1,4}$ is a cyclic group, then G is metacyclic. □

Corollary 6. *Keep the previous notations. If $G=\mathrm{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k})$ is a metacyclic-nonabelian group, then the Hilbert 2-class field tower of $\mathbf{k}$ stops at $\mathbf{k}_2^{(2)}$.*

Proof. Put $\mathcal{H} = \mathrm{Gal}(\mathbf{k}_2^{(3)}/\mathbf{k}_2^{(1)})$, then, by class field theory, the derived group $\mathcal{H}'$ of $\mathcal{H}$ is equal to $\mathrm{Gal}(\mathbf{k}_2^{(3)}/\mathbf{k}_2^{(2)})$. Thus the quotient group $\mathcal{H}/\mathcal{H}'$ is isomorphic to G'. By the Corollary 5, the group G' is cyclic, hence the Burnside Basis Theorem implies that $\mathcal{H}$ is a cyclic group; this in turn yields that $\mathcal{H}' = 1$. Therefore $\mathbf{k}_2^{(3)} = \mathbf{k}_2^{(2)}$. □

Lemma 15. *Keep the previous notations. If G is a metacyclic-nonabelian 2-group, then*

1. $\mathrm{V}_{G\to H_{i,2}}(xG') = \begin{cases} a^2H'_{i,2}, & \text{if } x = a, \\ H'_{i,2}, & \text{if } x = b^2. \end{cases}$ *with* $i = 1, 2$.
2. $\mathrm{V}_{G\to H_{3,2}}(xG') = \begin{cases} H'_{3,2}, & \text{if } x = a \text{ and } G = G_1 \text{ or } G_3, \\ a^{2^{n-3}}H'_{3,2}, & \text{if } x = a \text{ and } G = G_2, \\ a^{2^{n-4}}H'_{3,2}, & \text{if } x = a \text{ and } G = G_4, \\ a^{2^{n-3}}H'_{3,2}, & \text{if } x = b^2 \text{ and } G = G_3, \\ H'_{3,2}, & \text{if } x = b^2 \text{ and } G = G_1,\, G_2 \text{ or } G_4. \end{cases}$
3. $\mathrm{V}_{G\to H_{i,4}}(xG') = \begin{cases} a^{2^{n-3}}H'_{i,4}, & \text{if } x = a \text{ and } G = G_4, \\ H'_{i,4}, & \text{if } x = a \text{ and } G = G_j \text{ with } j \neq 4, \\ H'_{i,4}, & \text{if } x = b \text{ and } G \neq G_3 \text{ with } i \neq 3, \\ a^{2^{n-3}}H'_{i,4}, & \text{if } x = b \text{ and } G = G_3 \text{ with } i \neq 3, \\ H'_{3,4}, & \text{if } x = b \text{ and } G = G_1,\, G_2, \\ a^{2^{n-3}}H'_{3,4}, & \text{if } x = b \text{ and } G = G_3, \\ a^{2^{n-3}(-1+2^{n-5})}H'_{3,4}, & \text{if } x = b \text{ and } G = G_4; \end{cases}$

   with $i = 1, 2$ *or* 3.

Proof. Show only the first result, the others are proved similarly.

We know that $H_{1,2} = \langle b, a^2\rangle$ and $H_{2,2} = \langle ab, a^2\rangle$, hence $a \notin H_{i,2}$, where $i = 1, 2$. Which implies that $\mathrm{V}_{G\to H_{i,2}}(aG') = a^2H'_{i,2}$. As $b^2 \in H_{i,2}$, for $i \in \{1, 2\}$, and $G/H_{i,2} = \langle aH'_{i,2}\rangle$, so

$$\mathrm{V}_{G\to H_{i,2}}(b^2G') = b^4[b^2, a]H'_{i,2} = \begin{cases} H'_{i,2}, & \text{if } G = G_1, G_2; \\ a^{-2^{n-3}}H'_{i,2} & \text{if } G = G_3; \\ a^{-2^{n-3}(-1+2^{n-5})}H'_{i,2}, & \text{if } G = G_4. \end{cases}$$

If $G = G_3$ or G_4, then $n \geq 5$ and $H'_{i,2} = \langle a^4\rangle$. From what we conclude that $a^{-2^{n-3}}$ and $a^{-2^{n-3}(-1+2^{n-5})}$ are in $H'_{i,2}$. Finally, we get

$$\mathrm{V}_{G\to H_{i,2}}(b^2G') = H'_{i,2}.$$

□

Theorem 16. *Keep the previous notations. If G is metacyclic, nonmodular and nonabelian, then*

1. $\ker j_{\mathbf{k}\to\mathbf{K}_{i,2}} = \{1, \mathfrak{d}^2\}$, *for* $i = 1$ *or* 2.
2. $$\ker j_{\mathbf{k}\to\mathbf{K}_{3,2}} = \begin{cases} \{1, \mathfrak{c}, \mathfrak{d}^2, \mathfrak{c}\mathfrak{d}^2\}, & \text{if } G = G_1, \\ \{1, \mathfrak{d}^2\}, & \text{if } G = G_2 \text{ or } G_4, \\ \{1, \mathfrak{c}\}, & \text{if } G = G_3. \end{cases}$$
3. $$\ker j_{\mathbf{k}\to\mathbf{K}_{i,4}} = \begin{cases} \mathrm{C}_{\mathbf{k},2}, & \text{if } G = G_1 \text{ or } G_2, \\ \{1, \mathfrak{c}, \mathfrak{d}^2, \mathfrak{c}\mathfrak{d}^2\}, & \text{if } G = G_3, \\ \{1, \mathfrak{d}, \mathfrak{d}^2, \mathfrak{d}^3\}, & \text{if } G = G_4, \end{cases} \quad \text{for } i = 1, 2.$$
4. $$\ker j_{\mathbf{k}\to\mathbf{K}_{3,4}} = \begin{cases} \mathrm{C}_{\mathbf{k},2}, & \text{if } G = G_1 \text{ or } G_2, \\ \{1, \mathfrak{c}, \mathfrak{d}^2, \mathfrak{c}\mathfrak{d}^2\}, & \text{if } G = G_3 \text{ or } G = G_4. \end{cases}$$
5. *The capitulation of the* 2*-ideal classes of* $\mathbf{k}$ *is of type* $(2A, 2A, 2A)$ *or* $(2A, 2A, 4)$.

Proof. We prove assertion 2; for assertions 1, 3, and 4 we proceed similarly. According to the Lemma 15 we have

$$\mathrm{V}_{G\to H_{3,2}}(xG') = \begin{cases} H'_{3,2}, & \text{if } x = a \text{ and } G = G_1 \text{ or } G_3, \\ a^{2^{n-3}} H'_{3,2}, & \text{if } x = a \text{ and } G = G_2, \\ a^{2^{n-4}} H'_{3,2}, & \text{if } x = a \text{ and } G = G_4, \\ a^{2^{n-3}} H'_{3,2}, & \text{if } x = b^2 \text{ and } G = G_3, \\ H'_{3,2}, & \text{if } x = b^2 \text{ and } G = G_1,\ G_2 \text{ or } G_4, \end{cases}$$

on the other hand, the Corollary 5 yields that $H'_{3,2} = 1$ for G_1, G_2, and G_3, hence

$$\mathrm{V}_{G\to H_{3,2}}(xG') = \begin{cases} 1, & \text{if } x = a \text{ and } G = G_1 \text{ or } G_3, \\ a^{2^{n-3}}, & \text{if } x = a \text{ and } G = G_2, \\ a^{2^{n-4}}, & \text{if } x = a \text{ and } G = G_4, \\ a^{2^{n-3}}, & \text{if } x = b^2 \text{ and } G = G_3, \\ 1, & \text{if } x = b^2 \text{ and } G = G_1,\ G_2 \text{ or } G_4. \end{cases}$$

The definitions of G_2, G_3 and G_4 require that $n \geq 5$, thus $a^{2^{n-3}}$ and $a^{2^{n-4}}$ are different from 1. This allows us to conclude that

$$\ker \mathrm{V}_{G\to H_{3,2}} = \begin{cases} \{G', aG', b^2G', ab^2G'\}, & \text{if } G = G_1, \\ \{G', b^2G'\}, & \text{if } G = G_2 \text{ or } G_4, \\ \{G', aG'\}, & \text{if } G = G_3. \end{cases}$$

By the Artin's reciprocity law, we get the assertion 2.

5. The results 1. and 2. of Lemma 15 imply that: if $i = 1$ or 2, then

$\ker \mathrm{V}_{G\to H_{i,2}} \cap H_{i,2}/G' = \{G', b^2G'\} \cap \langle bG'\rangle = \{G', b^2G'\}$ or $\{G', b^2G'\} \cap \langle ab, G'\rangle$,

and

$$\ker \mathrm{V}_{G\to H_{3,2}} \cap H_{3,2}/G' = \begin{cases} \langle aG', b^2G'\rangle \cap \langle aG', b^2G'\rangle = \langle aG', b^2G'\rangle, & \text{if } G = G_1,\\ \{G', aG'\} \cap \langle aG', b^2G'\rangle = \{G', aG'\}, & \text{if } G = G_3,\\ \{G', b^2G'\} \cap \langle aG', b^2G'\rangle = \{G', b^2G'\}, & \text{if not.} \end{cases}$$

By the Artin's reciprocity law, we get

$$|\ker j_{\mathbf{k}\to K_{i,2}} \cap \mathcal{N}_{K_{i,2}/\mathbf{k}}(C_{K_{i,2}})| > 1,$$

where $i = 1, 2$ and 3. The results 1. and 2. of this theorem yield that the capitulation of the 2-ideal classes of $\mathbf{k}$ is of type $(2A, 2A, 2A)$ or $(2A, 2A, 4)$. □

Corollary 7. *Keep the previous notations. If G is a metacyclic-nonmodular-nonabelian group and only $\mathfrak{d}^2$ and its square capitulate in $\mathbf{K}_{3,2}$, then the following assertions are equivalent:*

1. $G = G_2$,
2. *The Hilbert 2-class field tower of $\mathbf{K}_{3,2}$ stops at the first stage,*
3. $h(\mathbf{K}_{i,4}) = \frac{h(\mathbf{K}_{3,2})}{2}$, *where $i = 1, 2$ or 3.*

Proof. 2. $\Longleftrightarrow$ 3. As G is a metacyclic-nonmodular-nonabelian group and only $\mathfrak{d}^2$ and its square capitulate in $\mathbf{K}_{3,2}$, then the order of G is > 16 and the rank of the 2-class group of $\mathbf{K}_{3,2}$ is 2 (Theorem 14). Thus according to [10, Proposition 7], we have the Hilbert 2-class field tower of $\mathbf{K}_{3,2}$ stops at the first stage if and only if $h(\mathbf{K}_{i,4}) = \frac{h(\mathbf{K}_{3,2})}{2}$, where $i = 1, 2$ or 3.

1. $\Longleftrightarrow$ 3. Since only the class of $\mathfrak{d}^2$ and its square capitulate in $\mathbf{K}_{3,2}$, so $G = G_2$ or $G = G_4$. If the order of G is 2^n, then we have on the first hand (see p. 101),

$$H_{3,2}/H'_{3,2} \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n-2}\mathbb{Z}, & \text{if } G = G_2,\\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n-3}\mathbb{Z}, & \text{if } G = G_4. \end{cases}$$

This implies that

$$\frac{h(\mathbf{K}_{3,2})}{2} = \begin{cases} 2^{n-2}, & \text{if } G = G_2,\\ 2^{n-3}, & \text{if } G = G_4. \end{cases}$$

On the other hand and for $i = 1, 2, 3$, we have

$$h(\mathbf{K}_{i,4}) = [H_{i,4} : H'_{i,4}] = |H_{i,4}| = 2^{n-2},$$

since $H'_{i,4} = 1$ (see Corollary 5). Finally, $G = G_2$ if and only if $h(\mathbf{K}_{i,4}) = \frac{h(\mathbf{K}_{3,2})}{2}$ with $i = 1, 2$ or 3. □

4.3. Nonmetacyclic case. In this subsection, we study the capitulation problem in the case where G is a nonmetacyclic 2-group.

Theorem 17. *Keep the previous notations. Then the following assertions are equivalent:*

1. *G is nonmetacyclic,*
2. *The 2-class group of $\mathbf{K_{3,2}}$ is of type $(2, 2, 2)$,*
3. *The rank of the 2-class group of $\mathbf{K_{3,2}}$ is equal to 3.*

PROOF. Direct consequence of Theorem 7 p. 99. □

Lemma 18. *Keep the previous notations. If G is a nonmetacyclic 2-group, then*

$$\mathrm{V}_{G\to H_{3,2}}(xG') = \begin{cases} [a, b]\gamma_3(G), & \text{if } x = a, \\ \gamma_3(G), & \text{if } x = b^2. \end{cases}$$

PROOF. According to Corollary 1, result 1, p. 97, we have: if G is nonmetacyclic, then

$$a^2 \equiv b^4 \equiv c^2 \equiv [a, c] \equiv [b, c] \equiv 1 \mod \gamma_3(G), \quad \text{where } [a, b] = c.$$

By applying Formula (5), we get

$$\mathrm{V}_{G\to H_{3,2}}(xG') = \begin{cases} a^2[a, b]H'_{3,2}, & \text{if } x = a, \\ H'_{3,2}, & \text{if } x = b^2, \end{cases}$$

since $a \in H_{3,2}$ and $b^2 \in H_{3,2}$. As $H'_{3,2} = \gamma_3(G)$ (Lemma 6, p. 98) and $a^2 \equiv 1 \mod \gamma_3(G)$, then

$$\mathrm{V}_{G\to H_{3,2}}(xG') = \begin{cases} [a, b]\gamma_3(G), & \text{if } x = a, \\ \gamma_3(G), & \text{if } x = b^2. \end{cases}$$

□

Corollary 8. *Keep the previous notations. If G is a nonmetacyclic 2-group, then*

1. $\ker j_{\mathbf{k}\to\mathbf{K}_{3,2}} = \{1, \mathfrak{d}^2\}$,
2. *The capitulation of the 2-ideal classes of $\mathbf{k}$ in $\mathbf{K}_{3,2}$ is of type $2A$.*

PROOF. 1. Let $G = \langle a, b\rangle$ such that $a^2 \equiv b^4 \equiv 1 \mod \gamma_2(G)$. The terms c_i are defined as follows: $[a, b] = c = c_2$ and $c_{j+1} = [b, c_j]$. We have $G' = [c_2, c_3, \dots]$, $\gamma_3(G) = [c_2^2, c_3, \dots]$ see [9, Lemma 2]. This explains why $c = [a, b] \notin \gamma_3(G)$, then

$$\ker \mathrm{V}_{G\to H_{3,2}} = \{G', b^2G'\}.$$

By the Artin's reciprocity law, we get

$$\ker j_{\mathbf{k}\to\mathbf{K}_{3,2}} = \{1, \mathfrak{d}^2\}$$

2. It suffices to note that

$$\ker \mathrm{V}_{G\to H_{3,2}} \cap H_{3,2}/G' = \{G', b^2G'\} \cap \langle aG', b^2G', c^2G'\rangle.$$

By the Artin's reciprocity law, we have

$$|\ker j_{\mathbf{k}\to K_{3,2}} \cap \mathcal{N}_{K_{i,2}/\mathbf{k}}(C_{K_{3,2}})| > 1.$$

This allows us to state that the capitulation of the 2-ideal classes of $\mathbf{k}$ in $\mathbf{K}_{3,2}$ is of type $2A$. □

4.4. Abelian case. In this subsection, we prove two theorems about the capitulation problem in the case where G is an abelian 2-group.

Theorem 19. *Keep the previous notations. Then the following assertions are equivalent:*

1. *G is abelian,*
2. *The 2-class group of $\mathbf{K_{3,2}}$ is of type $(2,2)$,*
3. *The 2-class number of $\mathbf{K_{3,2}}$ is 4,*
4. *The 2-class group of $\mathbf{K_{i,2}}$ is cyclic of order 4, with $i = 1$ or 2,*
5. *The 2-number class of $\mathbf{K_{i,2}}$ is 4, with $i = 1$ or 2,*
6. *The Hilbert 2-class field tower of $\mathbf{k}$ stops at $\mathbf{k}_2^{(1)}$,*
7. *The capitulation of the 2-ideal classes of $\mathbf{k}$ is of type $(4,4,4)$.*

Proof. By the Artin's reciprocity law and Theorem 10 p. 102, we get 1. $\Longleftrightarrow$ 2. $\Longleftrightarrow$ 3. $\Longleftrightarrow$ 4. $\Longleftrightarrow$ 5.

1. $\Longleftrightarrow$ 6. Obvious.

6. $\Longleftrightarrow$ 7. See [12, Lemma 1 p. 105]. □

Corollary 9. *Keep the previous notations. If G is abelian, then*

1. $\ker j_{\mathbf{k}\to\mathbf{K}_{i,2}} = \{1, \mathfrak{c}, \mathfrak{d}^2, \mathfrak{c}\mathfrak{d}^2\}$, *with $i = 1, 2$ or 3;*
2. $\ker j_{\mathbf{k}\to\mathbf{K}_{i,4}} = \mathrm{C}_{\mathbf{k},2}$, *with $i = 1, 2$ or 3.*

Proof. As the Hilbert 2-class field tower of $\mathbf{k}$ stops at $\mathbf{k}_2^{(1)}$, then it suffices to apply [14, Theorem p. 193 and Hilbert's Theorem 94]. □

5. Examples

In this section, we give two examples that illustrate some of our results: the first one is about a real quadratic number field and the second one is about an imaginary bicyclic biquadratic number field.

5.1. First example. Let $\mathbf{k}=\mathbb{Q}(\sqrt{p_1p_2p_3})$ be a real quadratic field, such that p_i are different primes congruent to 1 (mod 4) and the 2-class group of $\mathbf{k}$ is of type $(2,4)$. Note that, in [13], E. Benjamin did not give clarification about the structure of the group $G=\mathrm{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k})$ if the capitulation is of type $(2A,2A,2A)$ or $(2B,2B,2A)$. In what follows, we give necessary and sufficient condition to have the group G nonmetacyclic. But let us first establish the following lemma:

Lemma 20. *Let p, q and r be different primes, such that $p\equiv q\equiv r\equiv 1$ (mod 4) and $\left(\frac{p}{q}\right)=1$, and let l be the rank of the 2-class group of $\mathbb{Q}(\sqrt{p},\sqrt{qr})$. We have the following properties:*

(1) *If* $\left(\frac{p}{r}\right)=1$, *then* $l=\begin{cases}3, & \text{if } \left(\frac{p}{q}\right)_4=\left(\frac{q}{p}\right)_4 \text{ and } \left(\frac{p}{r}\right)_4=\left(\frac{r}{p}\right)_4,\\ 2, & \text{if not.}\end{cases}$

(2) *If* $\left(\frac{p}{r}\right)=-1$, *then* $l=\begin{cases}2, & \text{if } \left(\frac{p}{q}\right)_4=\left(\frac{q}{p}\right)_4,\\ 1, & \text{if not.}\end{cases}$

Proof. It is an immediate consequence of Theorem 2 of [1]. □

Theorem 21. *Let p_1, p_2 and p_3 be different primes such that $p_1\equiv p_2\equiv p_3\equiv 1$ (mod 4). If the 2-class group of $\mathbf{k}=\mathbb{Q}(\sqrt{p_1p_2p_3})$ is of type $(2,4)$, then the group $G=\mathrm{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k})$ is nonmetacyclic if and only if the following assertions hold:*

(1) $\left(\frac{p_i}{p_j}\right)=\left(\frac{p_i}{p_k}\right)=1$,

(2) $\left(\frac{p_i}{p_j}\right)_4=\left(\frac{p_j}{p_i}\right)_4$ and $\left(\frac{p_i}{p_k}\right)_4=\left(\frac{p_k}{p_i}\right)_4$.

Proof. As the 2-class group of $\mathbf{k}=\mathbb{Q}(\sqrt{p_1p_2p_3})$ is of type $(2,4)$, then $\mathbf{k}$ admits a real unramified cyclic extension of degree 4. Hence by [18], [5] we conclude that

$$\left(\frac{p_i}{p_j}\right)=\left(\frac{p_i}{p_k}\right)=1 \quad\text{and}\quad \left(\frac{p_i}{p_jp_k}\right)_4=\left(\frac{p_jp_k}{p_i}\right)_4. \tag{6}$$

In this situation $\mathbf{K_{3,2}} = \mathbb{Q}(\sqrt{p_i}, \sqrt{p_j p_k})$. Thus Theorem 17 yields that G is not metacyclic if and only if the rank of the 2-class group of $\mathbf{K_{3,2}}$ is equal to 3, and this is equivalent, by Lemma 20, to

$$\left(\frac{p_i}{p_j}\right)_4 = \left(\frac{p_j}{p_i}\right)_4 \quad \text{and} \quad \left(\frac{p_i}{p_k}\right)_4 = \left(\frac{p_k}{p_i}\right)_4. \qquad \square$$

Note that, if G is a metacyclic group, then the last equalities are not true. Hence the equalities (6) implies that

$$\left(\frac{p_i}{p_j}\right)_4 = -\left(\frac{p_j}{p_i}\right)_4 \quad \text{and} \quad \left(\frac{p_i}{p_k}\right)_4 = -\left(\frac{p_k}{p_i}\right)_4. \tag{7}$$

Corollary 10. *Let i, j and k be the positive integers in the equalities* (6). *Let ε be the fundamental unit of the field $\mathbf{k} = \mathbb{Q}(\sqrt{p_1p_2p_3})$. If the 2-class group of $\mathbf{k} = \mathbb{Q}(\sqrt{p_1p_2p_3})$ is of type $(2,4)$ and the group $G = \mathrm{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k})$ is metacyclic, then*

(a) *G is abelian if and only if $\left(\frac{p_j}{p_k}\right) = -1$ and $N(\varepsilon) = -1$.*

(b) *G is nonabelian-nonmodular if and only if $\left(\frac{p_j}{p_k}\right) = 1$.*

Proof. Under our conditions; on one hand, P. Kaplan in [24] states that $\left(\frac{p_j}{p_k}\right) = -1$ or $\left(\frac{p_j}{p_k}\right) = 1$. On the other hand, we have

$$\left(\frac{p_i}{p_j}\right) = \left(\frac{p_i}{p_k}\right) = 1, \quad \left(\frac{p_i}{p_j}\right)_4 = -\left(\frac{p_j}{p_i}\right)_4 \quad \text{and} \quad \left(\frac{p_i}{p_k}\right)_4 = -\left(\frac{p_k}{p_i}\right)_4.$$

(a) According to [10, Theorem 1], the Hilbert 2-class field tower of $\mathbf{k}$ stops at $\mathbf{k}_2^{(1)}$ if and only if $\left(\frac{p_j}{p_k}\right) = -1$ and $N(\varepsilon) = -1$. Thus G is abelian if and only if $\left(\frac{p_j}{p_k}\right) = -1$ and $N(\varepsilon) = -1$.

(b) If G is nonabelian-nonmodular, then the case $\left(\frac{p_j}{p_k}\right) = -1$ can not occur. For if $\left(\frac{p_j}{p_k}\right) = -1$, then by Lemma 20(2) $H_{1,2}$, the 2-class group of $K_{1,2} = \mathbb{Q}(\sqrt{p_i}, \sqrt{p_j p_k})$, is cyclic. As G is of type $(2,4)$, so by [4, Theorem 1.2] G is abelian or modular, which contradicts our hypotheses. Therefore $\left(\frac{p_j}{p_k}\right) = 1$.

Conversely, if $\left(\frac{p_j}{p_k}\right) = 1$, then the Equality (7) and Lemma 20 show that the ranks of the 2-class groups of $K_{1,2}$ and $K_{2,2}$ are equal to 2. Hence G does not admit a maximal cyclic subgroup, thus G is nonabelian-nonmodular. $\square$

5.2. Second example. Let now $\mathbf{k} = \mathbb{Q}(\sqrt{2p}, i)$ and assume its 2-class group is of type $(2,4)$. The following theorem is almost the main results in [2], and we cite it here as an example which illustrates some of our previous results.

Theorem 22. *Let* $\mathbf{k} = \mathbb{Q}(\sqrt{2p}, i)$*, where* p *is a prime such that* $p \equiv 1 \mod 8$ *and* $\left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4 = -1$*. Denote by* 2^n *the 2-class number of* $\mathbb{Q}(\sqrt{-p})$*. Then* $G = \mathrm{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k})$ *is metacyclic-nonmodular, moreover*

$$G = \langle a, b : a^{2^n} = 1, b^4 = 1, a^b = a^{-1+2^{n-1}} \rangle.$$

Proof. Note that the 2-class group of $\mathbf{k}$ is of type $(2, 4)$, since $p \equiv 1 \mod 8$ and $\left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4 = -1$ (see [3]). In [2], we have constructed an unramified cyclic extension of $\mathbf{k}$ of order 4, which contains the field $\mathbf{k}(\sqrt{2})$. With the notations above, $\mathbf{k}(\sqrt{2})$ is the field $\mathbf{K_{3,2}}$. According to [2, Theorem 9], the rank of the 2-class group of $\mathbf{K_{3,2}}$ is 2, then Theorem 14 yields that G is metacyclic-nonmodular of order 2^{n+2}, since the 2-class number of $\mathbf{K_{3,2}}$ is 2^{n+1} with $n \geq 3$. Thus the order of G is divisible by 32. Finally, Theorem 12 and Proposition 9 of [2], and the Corollary 7 imply that

$$G = \langle a, b : a^{2^n} = 1, b^4 = 1, a^b = a^{-1+2^{n-1}} \rangle. \qquad \square$$

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