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Strong 2-commutativity preserving maps on prime rings

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Abstract. Let $\mathcal{R}$ be a unital prime ring and $k \geq 1$ a positive integer. A map $f : \mathcal{R} \to \mathcal{R}$ is called preserving strong k-commutativity if $[f(x), f(y)]_k = [x, y]_k = [[x, y]_{k-1}, y]$ for all $x, y \in \mathcal{R}$. In this paper, it is shown that, if $\mathcal{R}$ contains a nontrivial idempotent, char$\mathcal{R} \neq 2$ and f is surjective, then f is strong 2-commutativity preserving if and only if $f(x) = \beta x + \mu(x)$ for all $x \in \mathcal{R}$, where β is in the extended centroid of $\mathcal{R}$ with $\beta^3 = 1$ and μ is a central valued map. Based on this, a characterization of general strong 2-commutativity preserving maps on factor von Neumann algebras is obtained.

1. Introduction

Let $\mathcal{R}$ be a ring with the center $\mathcal{Z}(\mathcal{R})$. Then $\mathcal{R}$ is a Lie ring under the Lie product $[a, b] = ab - ba$. Recall that a map $f : \mathcal{R} \to \mathcal{R}$ preserves commutativity if $[f(a), f(b)] = 0$ whenever $[a, b] = 0$ for a, $b \in \mathcal{R}$; preserves strong commutativity if $[f(a), f(b)] = [a, b]$ for all $a, b \in \mathcal{R}$. Obviously, a strong commutativity preserving map must be commutativity preserving; but the inverse is not true.

The problem of characterizing commutativity preserving maps had been studied intensively on various rings and algebras (for example, see [2], [5], [16], [19] and the references therein). For strong commutativity preserving maps (SCPM), Brešar and Miers in [4] proved that every additive SCPM f on a semiprime ring $\mathcal{R}$ has the form $f(a) = \lambda a + \mu(a)$, where $\lambda \in \mathcal{C}$, the extended centroid of $\mathcal{R}$,

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$\lambda^2 = 1$ and $\mu : \mathcal{R} \to \mathcal{C}$ is an additive map. Let $\mathcal{L}$ be a noncentral Lie ideal of a prime ring $\mathcal{R}$. Lin and Liu in [13] proved that every additive SCPM $f : \mathcal{L} \to \mathcal{R}$ is of the form $f(a) = \lambda a + \mu(a)$, where $\lambda \in \mathcal{C}$ with $\lambda^2 = 1$ and $\mu : \mathcal{L} \to \mathcal{Z}(\mathcal{R})$ is an additive map, unless $\operatorname{char} \mathcal{R} = 2$ and $\mathcal{R}$ satisfies the standard identity of degree 4. Qi and Hou [18] discussed general SCPM on prime rings. It was shown in [18] that, if $\mathcal{R}$ is a unital prime ring with a nontrivial idempotent, then a surjective map $f : \mathcal{R} \to \mathcal{R}$ is a SCPM if and only if $f(a) = \alpha a + \mu(a)$ for all $a \in \mathcal{R}$, where $\alpha \in \{1, -1\}$ and $\mu : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ is an arbitrary map. Recently, Lee and Wong in [12] generalized above results. Assume that $\mathcal{R}$ is a prime ring with $\mathcal{L}$ a noncentral Lie ideal of $\mathcal{R}$. They proved that, if $f : \mathcal{L} \to \mathcal{R}$ is a map satisfying $[f(x), f(y)] - [x, y] \in \mathcal{C}$ for all $x, y \in \mathcal{L}$, then $f(a) = \alpha a + \mu(a)$ for all $a \in \mathcal{L}$, where $\alpha \in \{1, -1\}$ and $\mu : \mathcal{L} \to \mathcal{C}$ is a map, unless $\operatorname{char} \mathcal{R} = 2$ and $\mathcal{R}$ satisfies the standard identity of degree 4. For other results about SCPMs, see [1], [7], [8], [14], [15], [20] and the references therein.

For any elements $a, b \in \mathcal{R}$, define $[a, b]_0 = a$, $[a, b]_1 = ab - ba$, and inductively $[a, b]_k = [[a, b]_{k-1}, b]$, where $k \geq 1$ is a positive integer. Thus, we can introduce the concept of strong k-commutativity preserving maps. A map $f : \mathcal{R} \to \mathcal{R}$ is said to preserve strong k-commutativity if $[f(a), f(b)]_k = [a, b]_k$ for all $a, b \in \mathcal{R}$. Obviously, strong k-commutativity preserving maps are usual SCPMs if $k = 1$. A natural problem is how to characterize strong k-commutativity preserving maps for $k > 1$. The purpose of the present paper is to consider the problem of characterizing strong 2-commutativity preserving maps on prime rings.

Let $\mathcal{R}$ be a ring with center $\mathcal{Z}(\mathcal{R})$. We say that $\mathcal{R}$ is prime if for any $a, b \in \mathcal{R}$, $a\mathcal{R}b = \{0\}$ implies either $a = 0$ or $b = 0$; or equivalently, if for any two left (or right) ideals $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{R}$, $\mathcal{AB} = \{0\}$ implies $\mathcal{A} = \{0\}$ or $\mathcal{B} = \{0\}$. Denote by $\mathcal{Q} = \mathcal{Q}_{mr}(\mathcal{R})$ the maximal right ring of quotients. If $\mathcal{R}$ is prime, then the center $\mathcal{C}$ of $\mathcal{Q}$ is a field, which is called the extended centroid of $\mathcal{R}$. Moreover, $\mathcal{Z}(\mathcal{R}) \subseteq \mathcal{C}$. For more details about prime rings, see [3].

The following is our main result in this paper.

Theorem 1.1. *Let $\mathcal{R}$ be a unital prime ring containing a nontrivial idempotent. Assume that $f : \mathcal{R} \to \mathcal{R}$ is a surjective map and the characteristic of $\mathcal{R}$ is not 2. Then f is strong 2-commutativity preserving if and only if there exist a map $\mu : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ and an element $\beta \in \mathcal{C}$ with $\beta^3 = 1$ such that $f(x) = \beta x + \mu(x)$ for all $x \in \mathcal{R}$.*

Remark 1.1. From Theorem 1.1, we see that a strong 2-commutativity preserving map may not be a strong commutativity preserving map, and vice versa.

Recall that a von Neumann algebra $\mathcal{M}$ is a subalgebra of some $\mathcal{B}(H)$, the algebra of all bounded linear operators acting on a complex Hilbert space H, which satisfies the double commutant property: $\mathcal{M}'' = \mathcal{M}$, where $\mathcal{M}' = \{T : T \in \mathcal{B}(H)$ and $TA = AT\ \forall A \in \mathcal{M}\}$ and $\mathcal{M}'' = \{\mathcal{M}'\}'$. $\mathcal{M}$ is called a factor if its center, $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$, is trivial (i.e., $\mathcal{Z}(\mathcal{M}) = \mathbb{C}I$).

It is well-known that every factor von Neumann algebra must be prime. So, as an application of Theorem 1.1 to the factor von Neumann algebra case, the following corollary is immediate.

Corollary 1.2. *Let $\mathcal{A}$ be a factor von Neumann algebra. Assume that $\Phi : \mathcal{A} \to \mathcal{A}$ is a surjective map. Then Φ is strong 2-commutativity preserving if and only if there exist a scalar $\alpha \in \{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$ and a functional $g : \mathcal{A} \to \mathbb{C}$ such that $\Phi(A) = \alpha A + g(A)I$ for all $A \in \mathcal{A}$.*

2. Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1.

Lemma 2.1 ([3, Theorem A.7])**.** *Let $\mathcal{A}$ be a prime ring, and let $A_i, B_i, C_j, D_j \in \mathcal{Q}_{ml}(\mathcal{A})$ be such that $\sum_{i=1}^{n} A_i X B_i = \sum_{j=1}^{m} C_j X D_j$ for all $X \in \mathcal{A}$. If $A_1, \dots, A_n$ are linearly independent over $\mathcal{C}$, then each B_i is a $\mathcal{C}$-linear combination of $D_1, \dots, D_m$. Similarly, if $B_1, \dots, B_n$ are linearly independent over $\mathcal{C}$, then each A_i is a $\mathcal{C}$-linear combination of $C_1, \dots, C_m$. In particular, if $AXB = BXA$ for all $X \in \mathcal{A}$, then A and B are $\mathcal{C}$-linearly dependent.*

Lemma 2.2. *Let $\mathcal{R}$ be a prime ring. Then $\mathcal{Z}(\mathcal{R})_2 = \{z \in \mathcal{R} : [z, x]_2 = 0$ for all $x \in \mathcal{R}\} = \mathcal{Z}(\mathcal{R})$.*

Proof. This is a direct consequence of [10, Theorem 1] (also see [15, Theorem 2]). $\square$

Lemma 2.3. *Let $\mathcal{R}$ be a prime ring of characteristic not 2 and $s \in \mathcal{R}$. If $xs^2 + s^2x = 2sxs$ holds for all $x \in \mathcal{R}$, then $s \in \mathcal{Z}(\mathcal{R})$.*

Proof. Since $xs^2 + s^2x = 2sxs$, we have $[s, [s, x]] = 0$ for all $x \in \mathcal{R}$. It is easily seen that $s \in \mathcal{Z}(\mathcal{R})$ by the primeness of $\mathcal{R}$ and $\operatorname{char} \mathcal{R} \neq 2$. $\square$

Lemma 2.4. *Assume that $\mathcal{R}$ is a unital prime ring with a nontrivial idempotent e. Then $e\mathcal{R}e$ is also a prime ring with $\mathcal{Z}(e\mathcal{R}e) \subseteq \mathcal{C}e$.*

Proof. For any elements $exe, eye \in e\mathcal{R}e$, if $exe\mathcal{R}eye = \{0\}$, by using the primeness of $\mathcal{R}$, we have either $exe = 0$ or $eye = 0$. So $e\mathcal{R}e$ is prime. Now, taking any $eze \in \mathcal{Z}(e\mathcal{R}e)$; then $ezexe = exeze$ holds for all $x \in \mathcal{R}$. By Lemma 2.1, there exists some $\lambda \in \mathcal{C}$ such that $eze = \lambda e$, which implies $\mathcal{Z}(e\mathcal{R}e) \subseteq \mathcal{C}e$. $\square$

Now we are at a position to our proof of Theorem 1.1.

Proof of Theorem 1.1. The "if" part of the theorem is obvious. In the sequel, we always assume that $f:\mathcal{R}\to\mathcal{R}$ is a surjective strong 2-commutativity preserving map. We will check the "only if" part by several steps.

Step 1. $f(\mathcal{Z}(\mathcal{R}))=\mathcal{Z}(\mathcal{R})$.

Take any $z\in\mathcal{Z}(\mathcal{R})$. Then for any $x\in\mathcal{R}$, we have $[f(z),f(x)]_2=[z,x]_2=0$. By the surjectivity of f and Lemma 2.2, $f(z)\in\mathcal{Z}(\mathcal{R})$. On the other hand, if $f(x)=z$, then $[x,y]_2=[f(x),f(y)]_2=[z,f(y)]_2=0$ for all $y\in\mathcal{R}$. Hence $x\in\mathcal{Z}(\mathcal{R})$ by Lemma 2.2, that is, $\mathcal{Z}(\mathcal{R})\subseteq f(\mathcal{Z}(\mathcal{R}))$, completing the proof of the step.

Let $e\in\mathcal{R}$ be a nontrivial idempotent. Write $e_1=e$ and $e_2=1-e$. Then $\mathcal{R}$ can be decomposed into $\mathcal{R}=\mathcal{R}_{11}+\mathcal{R}_{12}+\mathcal{R}_{21}+\mathcal{R}_{22}$, where $\mathcal{R}_{ij}=e_i\mathcal{R}e_j$ $(i,j\in\{1,2\})$.

Step 2. For any $x\in\mathcal{R}$, we have $[x,e_1]_2=e_1xe_2+e_2xe_1\in\mathcal{R}_{12}+\mathcal{R}_{21}$.

Obvious by a simple and direct calculation.

Step 3. There exist two elements $\lambda,\mu\in\mathcal{C}$ with $\lambda\neq 0$ such that $f(e_1)=\lambda e_1+\mu 1$.

For any $x\in\mathcal{R}$, it is easily checked that $[x,e_1]=[x,e_1]_3$. Then $[x,e_1]_2=[x,e_1]_4=\big[[[x,e_1]_2,e_1],e_1\big]$, and so $[f(x),f(e_1)]_2=\big[[[f(x),f(e_1)]_2,e_1],e_1\big]$. It follows from the surjectivity of f that

$$[x,f(e_1)]_2=\big[[[x,f(e_1)]_2,e_1],e_1\big]=[[x,f(e_1)]_2,e_1]_2\quad\text{for all } x\in\mathcal{R}.$$

By Step 2, we have $[[x,f(e_1)]_2,e_1]_2\in\mathcal{R}_{12}+\mathcal{R}_{21}$, and so

$$[x,f(e_1)]_2\in\mathcal{R}_{12}+\mathcal{R}_{21}. \tag{1}$$

Write $f(e_1)=s_{11}+s_{12}+s_{21}+s_{22}$. We will first show the following claim.

Claim. $s_{12}=s_{21}=0$.

On the contrary, assume that $s_{21}\neq 0$. Taking $x=x_{11}$; then $e_2[x_{11},f(e_1)]_2e_2=2s_{21}x_{11}s_{12}=0$ by equation (1). Since the characteristic of $\mathcal{R}$ is not 2, we have $s_{21}e_1xe_1s_{12}=0$ for all $x\in\mathcal{R}$. It follows from the primeness of $\mathcal{R}$ that $s_{12}=0$.

Taking $x=x_{12}$; then

$$[x_{12},f(e_1)]_2=x_{12}(s_{21}s_{11}+s_{22}s_{21})-2s_{11}x_{12}s_{21}+(s_{21}s_{11}+s_{22}s_{21})x_{12}$$
$$-2s_{21}x_{12}s_{22}+s_{11}^2x_{12}+x_{12}s_{22}^2-2s_{11}x_{12}s_{22}+s_{12}s_{21}x_{12}+x_{12}s_{21}s_{12}-2s_{21}x_{12}s_{21}.$$

This and equation (1) imply

$$x_{12}(s_{21}s_{11}+s_{22}s_{21})=2s_{11}x_{12}s_{21} \quad \text{and} \quad (s_{21}s_{11}+s_{22}s_{21})x_{12}=2s_{21}x_{12}s_{22},$$

that is,

$$e_1xe_2(s_{21}s_{11}+s_{22}s_{21})=2s_{11}e_1xe_2s_{21}=s_{11}e_1x(2e_2s_{21}) \tag{2}$$

and

$$(s_{21}s_{11}+s_{22}s_{21})e_1xe_2=2s_{21}e_1xe_2s_{22} \tag{3}$$

for all $x\in\mathcal{R}$. By Lemma 2.2, equation (2) implies that e_1 and s_{11} are $\mathcal{C}$-linearly dependent as $2e_2s_{21}\neq 0$. If $s_{11}=\lambda e_1$ for some $\lambda\in\mathcal{C}$, by equation (2) and the primeness of $\mathcal{R}$, we get $s_{21}s_{11}+s_{22}s_{21}=2\lambda e_2s_{21}$, which, combining equation (3) and the primeness of $\mathcal{R}$, implies $s_{22}=\lambda e_2$. Thus we get $f(e_1)=\lambda 1+s_{21}$.

For any $x_{12}\in\mathcal{R}_{12}$, by the surjectivity of f, there exists some $y=\sum_{i,j=1}^{2}y_{ij}\in\mathcal{R}$ such that $f(y)=x_{12}$. Then

$$y_{21}+y_{12}=[y,e_1]_2=[x_{12},f(e_1)]_2=[x_{12}s_{21}-s_{21}x_{12},s_{21}]=-2s_{21}x_{12}s_{21}.$$

It follows that $y_{12}=0$. On the other hand, since

$$[f(e_1),x_{12}]_2=[s_{21}x_{12}-x_{12}s_{21},x_{12}]=-2x_{12}s_{21}x_{12}$$

and

$$[e_1,y]_2=[-y_{21},y]=-y_{21}y_{11}+y_{22}y_{21},$$

one can obtain $x_{12}s_{21}x_{12}=0$ as $[f(e_1),x_{12}]_2=[e_1,y]_2$ and $\operatorname{char}\mathcal{R}\neq 2$. Then we have $(e_1xe_2s_{21})^2=0$ for all $x\in\mathcal{R}$. Note that $e_1\mathcal{R}e_2s_{21}$ is a left ideal of the ring $e_1\mathcal{R}e_1$ and $e_1\mathcal{R}e_1$ is also prime. By the definition of prime rings, we get $e_1\mathcal{R}e_2s_{21}=\{0\}$. It follows from the primeness of $\mathcal{R}$ that $s_{21}=0$, a contradiction.

By a similar argument to the above, one can check that $s_{12}=0$. The claim holds.

Next, we will consider s_{11} and s_{22}. Taking $x=x_{11}$ in equation (1); then, by Claim and equation (1), one gets

$$[x_{11},f(e_1)]_2=x_{11}s_{11}^2-2s_{11}x_{11}s_{11}+s_{11}^2x_{11}=0.$$

Note that, by Lemma 2.4, $\mathcal{R}_{11}$ is prime and the characteristic of $\mathcal{R}_{11}$ is not 2. It follows from Lemma 2.3 that $s_{11}\in\mathcal{Z}(\mathcal{R}_{11})$, that is, $s_{11}e_1xe_1=e_1xe_1s_{11}$ holds for all $x\in\mathcal{R}$. Now, by Lemma 2.1, there exists some $\lambda_1\in\mathcal{C}$ such that $s_{11}=\lambda_1e_1$. A similar argument to that for s_{11} can obtain $s_{22}=\mu e_2$ for some $\mu\in\mathcal{C}$. Hence $f(e_1)=\lambda_1e_1+\mu e_2=\lambda e_1+\mu 1$, where $\lambda=\lambda_1-\mu$.

Finally, we still need to prove that $\lambda \neq 0$. On the contrary, if $\lambda = 0$, then $f(e_1) = \mu 1 \in \mathcal{C}$. Since $f(e_1) \in \mathcal{R}$, it follows that $f(e_1) \in \mathcal{Z}(\mathcal{R})$. By Step 2, one has $e_1 \in \mathcal{Z}(\mathcal{R})$, which is impossible. The proof of the step is completed.

Since $\mathcal{R}$ is prime, by [3, Theorem A.6], $\mathcal{C}$ is a field. So $\lambda \in \mathcal{C}$ is invertible. In the sequel, let $\alpha = \lambda^{-1}$.

Step 4. For any $x_{ij} \in \mathcal{R}_{ij}$, there exists some element $\nu_{ij}(x_{ij}) \in \mathcal{C}$ such that $f(x_{ij}) = \alpha^2 x_{ij} + \nu_{ij}(x_{ij})1$, $1 \leq i \neq j \leq 2$.

Here, we only give the proof for x_{12}. The proof for x_{21} is similar.

For any $x_{12} \in \mathcal{R}_{12}$, let $f(x_{12}) = s_{11} + s_{12} + s_{21} + s_{22}$. Since

$$x_{12} = [x_{12}, e_1]_2 = [f(x_{12}), f(e_1)]_2 = \lambda^2(s_{21} + s_{12}),$$

we get $\lambda^2 s_{21} = 0$ and $\lambda^2 s_{12} = x_{12}$, which implies that $s_{21} = 0$ and $s_{12} = \lambda^{-2}x_{12} = \alpha^2 x_{12}$.

For any $y = y_{11} + y_{21} + y_{22} \in \mathcal{R}$, by the surjectivity of f, there exists an element $t = t_{11} + t_{12} + t_{21} + t_{22} \in \mathcal{R}$ such that $f(t) = y$. Since $[y, f(x_{12})]_2 = [f(t), f(x_{12})]_2 = [t, x_{12}]_2$, we have

$$\begin{aligned} & y_{11}s_{11}^2 + y_{21}s_{11}^2 - 2s_{11}y_{11}s_{11} - \alpha^2 x_{12}y_{21}s_{11} - s_{22}y_{21}s_{11} + \alpha^2 y_{11}s_{11}x_{12} \\ &\quad + \alpha^2 y_{21}s_{11}x_{12} - \alpha^2 s_{11}y_{11}x_{12} - \alpha^4 x_{12}y_{21}x_{12} - \alpha^2 s_{22}y_{21}x_{12} + \alpha^2 y_{11}x_{12}s_{22} \\ &\quad + \alpha^2 y_{21}x_{12}s_{22} + y_{22}s_{22}^2 - \alpha^2 x_{12}y_{22}s_{22} - s_{22}y_{22}s_{22} - \alpha^2 s_{11}y_{11}x_{12} \\ &\quad + s_{11}^2 y_{11} + \alpha^2 s_{11}x_{12}y_{21} + \alpha^2 s_{11}x_{12}y_{22} - \alpha^2 x_{12}y_{21}s_{11} \\ &\quad - \alpha^4 x_{12}y_{21}x_{12} - \alpha^2 x_{12}y_{22}s_{22} + \alpha^2 x_{12}s_{22}y_{21} + \alpha^2 x_{12}s_{22}y_{22} \\ &\quad - s_{22}y_{21}s_{11} - \alpha^2 s_{22}y_{21}x_{12} - s_{22}y_{22}s_{22} + s_{22}^2 y_{21} + s_{22}^2 y_{22} = -2x_{12}t_{21}x_{12}. \end{aligned}$$

Multiplying by e_1 from both sides in the above equation gives

$$\begin{aligned} & y_{11}s_{11}^2 - 2s_{11}y_{11}s_{11} + s_{11}^2 y_{11} \\ &\qquad = \alpha^2(x_{12}y_{21}s_{11} - s_{11}x_{12}y_{21} + x_{12}y_{21}s_{11} - x_{12}s_{22}y_{21}) \end{aligned} \tag{4}$$

for all y_{11} and y_{21}; multiplying by e_2 from both sides in the above equation gives

$$\begin{aligned} & \alpha^2(y_{21}s_{11}x_{12} - s_{22}y_{21}x_{12} + y_{21}x_{12}s_{22} - s_{22}y_{21}x_{12}) \\ &\qquad = 2s_{22}y_{22}s_{22} - y_{22}s_{22}^2 - s_{22}^2 y_{22} \end{aligned} \tag{5}$$

for all y_{21} and y_{22}. Particularly, letting $y_{21} = 0$ in equations (4)–(5), one obtains

$$y_{11}s_{11}^2 + s_{11}^2 y_{11} = 2s_{11}y_{11}s_{11} \quad \text{and} \quad 2s_{22}y_{22}s_{22} = y_{22}s_{22}^2 + s_{22}^2 y_{22}$$

for all y_{11} and y_{22}. By Lemmas 2.3–2.4, the above two equations imply $s_{ii}\in\mathcal{Z}(\mathcal{R}_{ii})$; and furthermore, there exist some $\nu_1,\nu_2\in\mathcal{C}$ such that $s_{ii}=\nu_i e_i$ $(i=1,2)$. These and equations (4)–(5) imply $(\nu_1-\nu_2)x_{12}y_{21}=0$ and $(\nu_1-\nu_2)y_{21}x_{12}=0$, that is, $x_{12}e_2y(\nu_1-\nu_2)e_1=0$ and $(\nu_1-\nu_2)e_2ye_1x_{12}=0$ for all $y\in\mathcal{R}$. It follows from the primeness of $\mathcal{R}$ that $(\nu_1-\nu_2)e_1=0$ and $(\nu_1-\nu_2)e_2=0$, and so $\nu_1=\nu_2$. Now letting $\nu_{12}(x_{12})=\nu_1=\nu_2$; then $f(x_{12})=\alpha^2x_{12}+\nu_{12}(x_{12})1$, as desired.

Step 5. For any $x_{ii}\in\mathcal{R}_{ii}$, there exists some element $\nu_{ii}(x_{ii})\in\mathcal{C}$ such that $f(x_{ii})=e_if(x_{ii})e_i+\nu_{ii}(x_{ii})e_j$, $1\leq i\neq j\leq 2$.

Still, we only prove the assertion of Step 5 for the case $i=1$. The case when $i=2$ is dealt with similarly.

Take any $x_{11}\in\mathcal{R}_{11}$ and let $f(x_{11})=s_{11}+s_{12}+s_{21}+s_{22}$. For any $y_{12}\in\mathcal{R}_{12}$, by Step 4, we have $[\alpha^2y_{12},f(x_{11})]_2=[f(y_{12}),f(x_{11})]_2=[y_{12},x_{11}]_2$, that is,

$$\begin{aligned}\alpha^2(&y_{12}s_{21}s_{11}+y_{12}s_{21}s_{12}+y_{12}s_{22}s_{21}+y_{12}s_{22}^2-s_{11}y_{12}s_{21}-s_{11}y_{12}s_{22}\\&-s_{21}y_{12}s_{21}-s_{21}y_{12}s_{22}-s_{11}y_{12}s_{21}-s_{11}y_{12}s_{22}+s_{11}^2y_{12}+s_{12}s_{21}y_{12}\\&-s_{21}y_{12}s_{21}-s_{21}y_{12}s_{22}+s_{21}s_{11}y_{12}+s_{22}s_{21}y_{12})=x_{11}^2y_{12}.\end{aligned}$$

Multiplying by e_2 and e_1 from the left and the right in the above equation, respectively, one gets $2s_{21}y_{12}s_{21}=0$, that is, $2s_{21}e_1ye_2s_{21}=0$ for all $y\in\mathcal{R}$. Since $\mathcal{R}$ is prime and the characteristic of $\mathcal{R}$ is not 2, we get $s_{21}=0$.

For any $y_{21}\in\mathcal{R}_{21}$, by Step 4, we have

$$[\alpha^2y_{21},f(x_{11})]_2=[f(y_{21}),f(x_{11})]_2=[y_{21},x_{11}]_2=y_{21}x_{11}^2,$$

and so $e_1[\alpha^2y_{21},f(x_{11})]_2e_2=0$. Note that $f(x_{11})=s_{11}+s_{12}+s_{22}$. A direct calculation obtains $2s_{12}y_{21}s_{12}=0$, which implies $s_{12}=0$.

Now, take any $y_{22}\in\mathcal{R}_{22}$. By the surjectivity of f, there exists some $t=\sum_{i,j=1}^2 t_{ij}\in\mathcal{R}$ such that $f(t)=y_{22}$. Since $[y_{22},s_{11}+s_{22}]_2=[y_{22},f(x_{11})]_2=[f(t),f(x_{11})]_2=[t,x_{11}]_2$, we have

$$y_{22}s_{22}^2-2s_{22}y_{22}s_{22}+s_{22}^2y_{22}=t_{11}x_{11}^2+t_{21}x_{11}^2-2x_{11}t_{11}x_{11}+x_{11}^2t_{11}+x_{11}^2t_{12}.$$

Multiplying by e_2 from both sides in the above equation achieves $y_{22}s_{22}^2-2s_{22}y_{22}s_{22}+s_{22}^2y_{22}=0$ for all $y_{22}\in\mathcal{R}_{22}$. It follows from Lemmas 2.3–2.4 that $s_{22}=\nu_{11}(x_{11})e_2$ for some $\nu_{11}(x_{11})\in\mathcal{C}$.

Step 6. For any $x_{12}\in\mathcal{R}_{12}$ and any $x_{21}\in\mathcal{R}_{21}$, there exists an element $t=\alpha^4(x_{12}+x_{21})+\mu_1e_1+\mu_2e_2\in\mathcal{R}$ with $\mu_1,\mu_2\in\mathcal{C}$ such that $f(t)=x_{12}+x_{21}$.

Take any $x_{12}, y_{12} \in \mathcal{R}_{12}$ and any $x_{21} \in \mathcal{R}_{21}$. By the surjectivity of f, there exists some $t = t_{11} + t_{12} + t_{21} + t_{22} \in \mathcal{R}$ such that $f(t) = x_{12} + x_{21}$. By Step 4, we have

$$[x_{12} + x_{21}, \alpha^2 y_{12}]_2 = [f(t), f(y_{12})]_2 = [t, y_{12}]_2,$$

that is, $-2\alpha^4 y_{12}x_{21}y_{12} = -2y_{12}t_{21}y_{12}$. Since the characteristic of $\mathcal{R}$ is not 2, we have $y_{12}(\alpha^4 x_{21} - t_{21})y_{12} = 0$, that is, $e_1 r(\alpha^4 x_{21} - t_{21})re_2 = 0$ for all $r \in \mathcal{R}$ and hence for all $x \in \mathcal{RC}$ (see [6, Theorem2]). In view of [17, Lemma 2], $\alpha^4 x_{21} = t_{21}$.

Similarly, by using the relation $[x_{12}+x_{21}, \alpha^2 y_{21}]_2 = [f(t), f(y_{21})]_2 = [t, y_{21}]_2$ for each y_{21}, one can show that $\alpha^4 x_{12} = t_{12}$.

Now, for any $y_{11} \in \mathcal{R}_{11}$, by Step 5, one has

$$[x_{12} + x_{21}, s_{11} + \nu_{11}(y_{11})e_2]_2 = [f(t), f(y_{11})]_2 = [t, y_{11}]_2.$$

Here $s_{11} = e_1 f(y_{11})e_1$. A simple calculation gets

$$\begin{aligned} x_{21}s_{11}^2 + \nu_{11}(y_{11})^2 x_{12} + s_{11}^2 x_{12} + \nu_{11}(y_{11})^2 x_{21} - 2\nu_{11}(y_{11})x_{21}s_{11} - 2\nu_{11}(y_{11})s_{11}x_{12} \\ = \alpha^4 x_{21}y_{11}^2 + t_{11}y_{11}^2 - 2y_{11}t_{11}y_{11} + \alpha^4 y_{11}^2 x_{12} + y_{11}^2 t_{11}. \end{aligned}$$

Multiplying e_1 from both sides in the above equation, we have $t_{11}y_{11}^2 - 2y_{11}t_{11}y_{11} + y_{11}^2 t_{11} = 0$ for all $y_{11} \in \mathcal{R}_{11}$. It follows from Lemmas 2.3–2.4 that $t_{11} = \mu_1 e_1$ for some $\mu_1 \in \mathcal{C}$.

Similarly, by using the relation $[f(t), f(y_{22})]_2 = [t, y_{22}]_2$ for each $y_{22} \in \mathcal{R}_{22}$, one can check $t_{22} = \mu_2 e_2$ for some $\mu_2 \in \mathcal{C}$, completing the proof of the step.

Step 7. For any $x_{ii} \in \mathcal{R}_{ii}$, we have $f(x_{ii}) = \alpha^8 x_{ii} + \nu_{ii}(x_{ii})1$, $i = 1, 2$.

By Step 5, we only need to prove $e_i f(x_{ii})e_i = \alpha^8 x_{ii} + \nu_{ii}(x_{ii})e_i$ for $i = 1, 2$. Write $s_{ii} = e_i f(x_{ii})e_i$. For any $x_{ij} \in \mathcal{R}_{ij}$ $(1 \le i \ne j \le 2)$, by Step 6, we have

$$[s_{ii} + \nu_{ii}(x_{ii})e_j, x_{ij} + x_{ji}]_2 = [f(x_{ii}), x_{ij} + x_{ji}]_2 = [x_{ii}, \alpha^4(x_{ij} + x_{ji}) + \mu_i e_i + \mu_j e_j]_2.$$

A calculation achieves

$$\begin{aligned} &\alpha^8(x_{ii}x_{ij}x_{ji} - 2x_{ji}x_{ii}x_{ij} + x_{ij}x_{ji}x_{ii}) + \alpha^4(\mu_j - \mu_i)(x_{ii}x_{ij} + x_{ji}x_{ii}) \\ &\quad = s_{ii}x_{ij}x_{ji} + x_{ij}x_{ji}s_{ii} - 2\nu_{ii}(x_{ii})x_{ij}x_{ji} + 2\nu_{ii}(x_{ii})x_{ji}x_{ij} - 2x_{ji}s_{ii}x_{ij}. \end{aligned}$$

Multiplying by e_j from both sides in the above equation, one gets $x_{ji}(\alpha^8 x_{ii} + \nu_{ii}(x_{ii})e_i - s_{ii})x_{ij} = 0$, that is,

$$e_j \mathcal{R} e_i(\alpha^8 x_{ii} + \nu_{ii}(x_{ii})e_i - s_{ii})e_i \mathcal{R} e_j = \{0\}.$$

By using the primeness of $\mathcal{R}$ for two times, we achieve $\alpha^8 x_{ii} + \nu_{ii}(x_{ii})e_i = s_{ii}$, completing the proof.

Step 8. For any $x, y \in \mathcal{R}$, there exists an element $z_{x,y} \in \mathcal{Z}(\mathcal{R})$ depending on x, y such that $f(x+y) = f(x) + f(y) + z_{x,y}$.

For any $x, y, t \in \mathcal{R}$, we have

$$\begin{aligned}[f(x+y) - f(x) - f(y), f(t)]_2 &= [f(x+y), f(t)]_2 - [f(x), f(t)]_2 - [f(y), f(t)]_2 \\ &= [x+y, t]_2 - [x, t]_2 - [y, t]_2 = 0.\end{aligned}$$

By the surjectivity of f and Lemma 2.2, the above equation implies $f(x+y) - f(x) - f(y) \in \mathcal{Z}(\mathcal{R})$.

Step 9. $\alpha^3 = 1$ and there exist a map $\mu : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ such that $f(x) = \alpha^2 x + \mu(x)$ for all $x \in \mathcal{R}$, that is, Theorem 1.1 is true.

For any $x_{12} \in \mathcal{R}_{12}$ and $x_{21} \in \mathcal{R}_{21}$, by the definition of f and Step 4, we have

$$\alpha^6 [x_{12}, x_{21}]_2 = [\alpha^2 x_{12}, \alpha^2 x_{21}]_2 = [f(x_{12}), f(x_{21})]_2 = [x_{12}, x_{21}]_2.$$

It follows that $(\alpha^6 - 1)[x_{12}, x_{21}]_2 = -2(\alpha^6 - 1)x_{21}x_{12}x_{21} = 0$, which implies that

$$(\alpha^6 - 1)x_{21}x_{12}x_{21} = 0 \quad \text{for all } x_{12} \in \mathcal{R}_{12},\ x_{21} \in \mathcal{R}_{21}. \tag{6}$$

If $\alpha^6 - 1 \neq 0$, then $\alpha^6 - 1$ is invertible as $\mathcal{C}$ is a field. So equation (6) yields $x_{21}x_{12}x_{21} = 0$ for all $x_{12} \in \mathcal{R}_{12}$ and $x_{21} \in \mathcal{R}_{21}$. First fix x_{21}. Then the above equation implies $x_{21} = 0$, that is, $e_2 x e_1 = 0$ for all $x \in \mathcal{R}$. Obviously, this is impossible. Hence $\alpha^6 = 1$.

Thus, combining Step 4 and Step 7, we have proved that $f(x_{ij}) = \alpha^2 x_{ij} + \mu_{ij}(x_{ij})1$ for all $x_{ij} \in \mathcal{R}_{ij}$, where $\mu_{ij}(x_{ij}) \in \mathcal{C}$ $(i, j = 1, 2)$. Particularly, $f(e_1) = \alpha^2 e_1 + \mu_{11}(e_1)1$. Also note that, by Step 3, $f(e_1) = \alpha^{-1} e_1 + \mu 1$. So $\alpha^2 e_1 + \mu_{11}(e_1)1 = \alpha^{-1} e_1 + \mu 1$, which implies $(\alpha^2 - \alpha^{-1})e_1 \in \mathcal{Z}(\mathcal{R})$. This forces $\alpha^2 - \alpha^{-1} = 0$, and so $\alpha^3 = 1$.

Now, for any $x = x_{11} + x_{12} + x_{21} + x_{22} \in \mathcal{R}$, we have

$$\begin{aligned}f(x_{11}) + f(x_{12}) + f(x_{21}) + f(x_{22}) &= \alpha^2 x_{11} + \mu_{11}(x_{11})1 \\ &\quad + \alpha^2 x_{12} + \mu_{12}(x_{12})1 + \alpha^2 x_{21} + \mu_{21}(x_{21})1 + \alpha^2 x_{22} + \mu_{22}(x_{22})1 \\ &= \alpha^2 x + (\mu_{11}(x_{11}) + \mu_{12}(x_{12}) + \mu_{21}(x_{21}) + \mu_{22}(x_{22}))1.\end{aligned} \tag{7}$$

On the other hand, by Step 8, there exists $z_x \in \mathcal{Z}(\mathcal{R})$ such that

$$\begin{aligned}&f(x) - (f(x_{11}) + f(x_{12}) + f(x_{21}) + f(x_{22})) \\ &= f(x_{11} + x_{12} + x_{21} + x_{22}) - f(x_{11}) - f(x_{12}) - f(x_{21}) - f(x_{22}) = z_x.\end{aligned} \tag{8}$$

Define a map $\mu : \mathcal{R} \to \mathcal{C}$ by $\mu(x) = (\mu_{11}(x_{11})+\mu_{12}(x_{12})+\mu_{21}(x_{21})+\mu_{22}(x_{22}))1+z_x$ for each $x \in \mathcal{R}$. Then by Eqs.(7)-(8), one has $f(x) = \alpha^2 x + \mu(x)$. Since $f(x) \in \mathcal{R}$ and $\alpha^2 x \in \mathcal{R}$, it is clear that $\mu(x) \in \mathcal{R}$ for all $x \in \mathcal{R}$. Hence μ in fact maps into $\mathcal{Z}(\mathcal{R})$. Finally, let $\beta = \alpha^2$. Then $\beta \in \mathcal{C}$ and $\beta^3 = \alpha^6 = 1$, completing the proof of Theorem 1.1. $\square$

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