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Hardy's inequality and Hausdorff operators on rearrangement-invariant Morrey spaces

By KWOK-PUN HO (Hong Kong)

Abstract. We generalize the Minkowski inequality, the Hardy–Littlewood–Pólya inequalities and the Hardy inequalities to rearrangement-invariant Morrey spaces. We obtain these results by extending the notion of Boyd's indices to the rearrangement-invariant Morrey spaces. Our method also applied to establish the boundedness of Hausdorff operators on rearrangement-invariant Morrey spaces.

1. Introduction

The main theme of this paper is the extension of several classical and important inequalities to Morrey type spaces. More precisely, we generalize the Minkowski inequalities, the Hardy–Littlewood–Pólya inequalities, the Hardy inequalities and the Hilbert inequalities to Morrey spaces built on rearrangement-invariant Banach function spaces.

The Hardy inequality plays an important role in analysis. For instance, it is related to the studies of partial differential equations, interpolation theory and theory of function spaces. For a more complete account on the generalizations and applications of Hardy inequalities, the reader is referred to [13], [18], [43].

In particular, the Hardy inequalities had been generalized to several function spaces used in analysis such as the Lorentz–Karamata spaces [13], [43] which include the Lorentz spaces, the Lorentz–Zygmund spaces and the generalized

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Lorentz–Zygmund spaces. Furthermore, for the study of the Hardy inequalities on rearrangement-invariant Banach function spaces, the reader is referred to [9], [36].

In this paper, we aim to establish the Hardy inequalities on Morrey spaces built on rearrangement-invariant Banach function spaces. The classical Morrey spaces were introduced by Morrey [38] to study the solutions of quasi-linear elliptic partial differential equations. Since the introduction of the classical Morrey space, it provides an important research direction on the theory of function spaces and partial differential equations, see [12], [23], [25], [26], [28], [41], [44], [48].

For the classical Morrey space, it is built on Lebesgue spaces. Recently, the classical Morrey spaces have been extended to Morrey spaces associated with non-Lebesgue spaces. In [21], [42], [45], several important results in harmonic analysis such as the boundedness of maximal operator, the singular integral operators and the fractional integral operators had been generalized to Orlicz–Morrey spaces. Similarly, we also have the Sobolev type embedding for the Morrey–Lorentz spaces in [22]. For the studies of Morrey spaces built on general function spaces, the reader is referred to [20], [24].

The Orlicz spaces and the Lorentz spaces are members of rearrangement-invariant Banach function spaces (r.i.B.f.s.). Therefore, in this paper, we study the Hardy inequalities on the Morrey spaces built on rearrangement-invariant Banach function spaces. We call it rearrangement-invariant (r.-i.) Morrey spaces. In fact, the r.-i. Morrey spaces were already studied in [19].

To establish the Hardy inequalities in r.-i. Morrey spaces, we follow the approach given in [18]. We first extend the Hardy–Littlewood–Pólya inequalities for integral operators to r.-i. Morrey spaces and, then, the Hardy inequalities and the Hilbert inequalities are consequences of these inequalities.

Notice that we obtain our results by extending the notion of Boyd index to r.-i. Morrey spaces. The notion of Boyd index was introduced by Boyd in [3], [4], [5], [6] for the study of operators on r.i.B.f.s., see [2, Chapter 3, Theorems 5.16 and 5.18]. Roughly speaking, the Boyd index is used to measure the quantitative behaviours of the dilation operators on r.i.B.f.s.

Even though the classical Boyd index was defined for r.i.B.f.s. only and r.-i. Morrey spaces is not necessary a r.i.B.f.s., we can extend the notion of Boyd's index to Morrey type spaces by studying the operator norms of dilation operators.

Our method is not only applied to Hardy's inequalities. We find that we can also apply our method to study the Hausdorff operator on r.-i. Morrey spaces.

The study of Hausdorff operators recently provides several research direction for the operator theory and the theory of function spaces. It is impossible to give a detail review here. the reader is referred to [1], [7], [8], [10], [15], [27], [29], [30], [32], [33], [34], [40], [47] for the studies of Hausdorff operators.

In this paper, our method to establish the Hardy inequality for r.-i. Morrey spaces can also apply to the boundedness of the Hausdorff operator on r.-i. Morrey spaces. We find that the boundedness of the Hausdorff operator is related to the mapping properties of the dilation operator on r.-i. Morrey spaces. That is, it is related to the Boyd index.

This paper is organized as follows. In Section 2, we introduce the r.-i. Morrey spaces and study the mapping properties of the dilation operator on r.-i. Morrey spaces. In Section 3, we establish the Minkowski inequalities, the Hardy–Littlewood–Pólya inequalities, the Hardy inequalities and the Hilbert inequalities to r.-i. Morrey spaces. At the end of this paper, we apply our method to generalize the boundedness result of the Hausdorff operator to r.-i. Morrey spaces in Section 4.

2. Definitions and preliminaries

In this section, we first recall the definition of rearrangement-invariant Banach function spaces. Then, we introduce the r.-i. Morrey spaces and study the mapping properties of the dilation operator on r.-i. Morrey spaces.

For any $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $\mathbb{B} = \{B(x,r) : x \in \mathbb{R}^n,\ r > 0\}$.

For any $t \in (0,\infty)$ and $r > 0$, let $I(t,r) = \{s \in (0,\infty) : |t-s| < r\}$ and $\mathbb{I} = \{I(t,r) : t \in (0,\infty),\ r > 0\}$.

Let $\mathcal{M}(\mathbb{R}^n)$ and $L_{\mathrm{loc}}(\mathbb{R}^n)$ denote the space of Lebesgue measurable functions and the space of locally integrable functions on $\mathbb{R}^n$, respectively.

For any $f \in \mathcal{M}(\mathbb{R}^n)$, let

$$\mu_f(\lambda) = |\{t \in (0,\infty) : |f(t)| > \lambda\}|, \quad \lambda \geq 0$$

denote the distribution function of f. Two functions $f, g \in \mathcal{M}(\mathbb{R}^n)$ are said to be equimeasurable if

$$\mu_f(\lambda) = \mu_g(\lambda), \quad \forall \lambda \geq 0.$$

We recall the definition of rearrangement-invariant Banach function space (r.i.B.f.s.) from [2, Chapter 1, Defintions 1.1 and 1.3].

Definition 2.1. A Banach space $X \subset \mathcal{M}(\mathbb{R}^n)$ is said to be a rearrangement-invariant Banach function space on $\mathbb{R}^n$ if it satisfies

(1) $\|f\|_X = 0 \iff f = 0$ a.e.,

(2) $|g| \le |f|$ a.e. $\implies \|g\|_X \le \|f\|_X$,

(3) $0 \le f_n \uparrow f$ a.e. $\implies \|f_n\|_X \uparrow \|f\|_X$,

(4) $\chi_E \in \mathcal{M}(\mathbb{R}^n)$ and $|E| < \infty \implies \chi_E \in X$,

(5) $\chi_E \in \mathcal{M}(\mathbb{R}^n)$ and $|E| < \infty \implies \int_E |f(t)|dx < C_E\|f\|_X$, $\forall f \in X$ for some $C_E > 0$.

(6) If f and g are equimeasurable, then $\|f\|_X = \|g\|_X$.

We recall the definition of associate space from [2, Chapter 1, Definitions 2.1 and 2.3].

Definition 2.2. Let X be a r.i.B.f.s. The associate space of X, X', consists of all Lebesgue measurable function f such that

$$\|f\|_{X'} = \sup\left\{\left|\int f(t)g(t)dt\right| : g \in X,\ \|g\|_X \le 1\right\} < \infty.$$

We use the dilation operator D_s with $s < 0$ for the study of the Hausdorff operators, therefore, we slightly modify the definition given in [35, Volume II, Definition 2.b.1].

Definition 2.3. For each $s \in \mathbb{R}\backslash\{0\}$ and for any Lebesgue measurable function f on $\mathbb{R}^n$, let D_s be the dilation operator defined by

$$(D_s f)(x) = f(x/s), \quad x \in \mathbb{R}^n.$$

The Boyd indices of a r.-i.B.f.s. X on $\mathbb{R}^n$ are the numbers defined by

$$p_X = \sup_{s>1} \frac{n\log|s|}{\log\|D_s\|}, \quad q_X = \inf_{0<s<1} \frac{n\log|s|}{\log\|D_s\|}$$

where $\|D_s\|$ is the operator norm of the linear operator, $D_s : X \to X$.

We have

$$\frac{1}{p_X} + \frac{1}{q_{X'}} = 1 \quad \text{and} \quad \frac{1}{p_{X'}} + \frac{1}{q_X} = 1. \tag{2.1}$$

The Lorentz–Luxemburg theorem [2, Chapter 1, Theorem 2.6] and [2, Chapter 1, Lemma 2.8] yield the following lemma.

Lemma 2.1. *Let X be a r.-i.B.f.s. Then the norms $\|f\|_X$ and*

$$\|f\|_{X''} = \sup\left\{\left|\int f(t)g(t)dt\right| : g \in X',\ \|g\|_{X'} \le 1\right\}$$

are mutually equivalent.

We are now ready to introduce the rearrangement-invariant (r.-i.) Morrey space.

Definition 2.4. Let $X \subset \mathcal{M}(\mathbb{R}^n)$ be a r.i.B.f.s. on $\mathbb{R}^n$ and $u(y,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. The *rearrangement-invariant Morrey space* $M_X^u(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{M}(\mathbb{R}^n)$ satisfying

$$\|f\|_{M_X^u(\mathbb{R}^n)} = \sup_{y\in\mathbb{R}^n,\, r>0} \frac{1}{u(y,r)} \|\chi_{B(y,r)} f\|_X < \infty. \tag{2.2}$$

Similarly, we also define the rearrangement-invariant Morrey spaces on the interval $(0,\infty)$, $M_X^u(0,\infty)$, with the set of balls $\mathbb{B}$ replaced by the set of intervals $\mathbb{I}$ on (2.2).

When $1 \le q \le p < \infty$, $X = L^q(\mathbb{R}^n)$ and $u(x,r) = |B(x,r)|^{\frac{1}{q}-\frac{1}{p}}$, $M_X^u(\mathbb{R}^n)$ is the classical Morrey space $M_q^p(\mathbb{R}^n)$.

Notice that $M_X^u(\mathbb{R}^n)$ is not necessarily rearrangement-invariant. That is, whenever f and g are equimeasurable, $\|f\|_{M_X^u(\mathbb{R}^n)}$ is not necessarily equal to $\|g\|_{M_X^u(\mathbb{R}^n)}$. On the other hand, for simplicity, we use the abused terminology "rearrangement-invariant Morrey space" to name $M_X^u(\mathbb{R}^n)$.

Lemma 2.2. *Let X be a r.i.B.f.s. on $\mathbb{R}^n$ and $u(y,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. Suppose that there exist $\alpha_u, \beta_u \in \mathbb{R}$ such that u satisfies*

$$u(y/s, r/s) \le C_0 s^{-n\alpha_u} u(y,r), \quad \text{for all } 1 < s < \infty \tag{2.3}$$

$$u(y/s, r/s) \le C_0 s^{-n\beta_u} u(y,r), \quad \text{for all } 0 < s < 1 \tag{2.4}$$

for some $C_0 > 0$, then, for any $p < p_X$ and $q_X < q$ there exists $C > 0$ such that

$$\|D_t f\|_{M_X^u(\mathbb{R}^n)} \le C|t|^{\frac{n}{p}-n\alpha_u} \|f\|_{M_X^u(\mathbb{R}^n)}, \quad \text{for all } 1 < |t| < \infty, \tag{2.5}$$

$$\|D_t f\|_{M_X^u(\mathbb{R}^n)} \le C|t|^{\frac{n}{q}-n\beta_u} \|f\|_{M_X^u(\mathbb{R}^n)}, \quad \text{for all } 0 < |t| \le 1. \tag{2.6}$$

Proof. For any $1 < |t| < \infty$ and $B(z,R) \in \mathbb{B}$, we have

$$\|\chi_{B(z,R)} D_t f\|_X = \|D_t(\chi_{B(z/|t|, R/|t|)} f)\|_X.$$

According to Definition 2.3 and (2.3), we obtain

$$\begin{aligned}\frac{1}{u(z,R)} \|\chi_{B(z,R)} D_t f\|_X &= \frac{u(z/|t|, R/|t|)}{u(z,R)} \frac{1}{u(z/|t|, R/|t|)} \|D_t(\chi_{B(z/|t|,R/|t|)} f)\|_X \\ &\le \frac{u(z/|t|, R/|t|)}{u(z,R)} \|D_t\| \|f\|_{M_X^u(\mathbb{R}^n)} \le C|t|^{\frac{n}{p}-n\alpha_u} \|f\|_{M_X^u(\mathbb{R}^n)}.\end{aligned}$$

By taking supremum over $B(z,R)\in\mathbb{B}$ on the left hand side of the above inequality, we establish (2.5). The proof for (2.6) is similar, therefore, for simplicity, we skip the details. $\square$

We call $\frac{1}{\frac{1}{p}-\alpha_u}$ and $\frac{1}{\frac{1}{q}-\beta_u}$ in (2.5) and (2.6) the Boyd indices of $M^u_X(\mathbb{R}^n)$. These are generalizations of the notion of Boyd's index to r.-i. Morrey spaces.

For instance, when $X = L^s(\mathbb{R}^n)$ and $1\le s\le t<\infty$ and $u(x,r) = |B(x,r)|^{\frac{1}{s}-\frac{1}{t}}$, we have $p_X = q_X = s$ and $\alpha_u = \beta_u = \frac{1}{s}-\frac{1}{t}$. Moreover, the p and q in (2.5) and (2.6) can be taken to be s. Therefore, both of the Boyd indices of the classical Morrey space $M^t_s(\mathbb{R}^n)$ are equal to t.

Furthermore, whenever $s=t$, $M^t_s(\mathbb{R}^n)$ is the Lebesgue space $L^t(\mathbb{R}^n)$. Thus, our Boyd indices reduce to the Boyd indices for Lebesgue spaces defined in [35, Volume II, Definition 2.b.1].

We have the analogue of Lemma 2.2 for $M^u_X(0,\infty)$.

In the next section, we apply the above result to extend those classical inequalities such as the Minkowski inequalities and the Hardy inequalities to r.-i. Morrey spaces.

3. Minkowski's inequality and Hardy's inequality

In this section, we extend the Minkowski inequalities, the Hardy–Littlewood–Pólya inequalities and the Hardy inequalities to r.-i. Morrey spaces. Roughly speaking, the Hardy–Littlewood–Pólya inequalities and the Hardy inequalities are consequences of the Minkowski inequalities and Lemma 2.2. Furthermore, as another application of the Minkowski inequalities, we also obtain a mapping properties for the Mellin convolution on r.-i. Morrey spaces.

We first establish the Minkowski inequalities on r.-i. Morrey spaces.

Theorem 3.1 (Minkowski's inequality)**.** *Let μ be a signed σ-finite Borel measure on $\mathbb{R}$. Let $f(x,s)$ be a Lebesgue measurable function on $\mathbb{R}^n\times\mathbb{R}$. Let X be a r.i.B.f.s. on $\mathbb{R}^n$ and $u(t,r):\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ be a Lebesgue measurable function. We have*

$$\Big\| \int_{\mathbb{R}} f(\cdot,s)d\mu\Big\|_{M^u_X(\mathbb{R}^n)} \le \int_{\mathbb{R}} \|f(\cdot,s)\|_{M^u_X(\mathbb{R}^n)} d|\mu|. \tag{3.1}$$

Proof. For any $B(z,R)\in\mathbb{B}$, write

$$h(x) = \chi_{B(z,R)}(x)\int_{\mathbb{R}} f(x,s)d\mu.$$

Let $g \in X'$ with $\|g\|_{X'} \leq 1$. Fubini's theorem yields

$$\int_{\mathbb{R}^n} |g(x)h(x)|dx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}} \chi_{B(z,R)}(x)|g(x)||f(x,s)|d|\mu|dx$$
$$\leq C \int_{\mathbb{R}} \|\chi_{B(z,R)}(\cdot)f(\cdot,s)\|_X d|\mu|.$$

By taking supremum over those $g \in X'$ with $\|g\|_{X'} \leq 1$, Lemma 2.1 gives

$$\left\|\chi_{B(z,R)}(\cdot) \int_{\mathbb{R}} f(\cdot,s)d\mu\right\|_X \leq C \int_{\mathbb{R}} \|\chi_{B(z,R)}(\cdot)f(\cdot,s)\|_X d|\mu|.$$

Consequently,

$$\frac{1}{u(z,R)}\left\|\chi_{B(z,R)}(\cdot) \int_{\mathbb{R}} f(\cdot,s)d\mu\right\|_X \leq C \int_{\mathbb{R}} \frac{1}{u(z,R)}\|\chi_{B(z,R)}(\cdot)f(\cdot,s)\|_X d|\mu|$$
$$\leq C \int_{\mathbb{R}} \|f(\cdot,s)\|_{M_X^u(\mathbb{R}^n)} d|\mu|$$

for some $C > 0$ independent of $z \in \mathbb{R}^n$ and $R > 0$. Therefore, by taking supreme over $B(z,R) \in \mathbb{B}$ on the left hand side of the above inequality, we obtain (3.1). $\square$

Particularly, the Minkowski inequality is also valid on $M_X^u(0,\infty)$. In the rest of this section, we apply the Minkowski inequality on $M_X^u(0,\infty)$ to extend some classical inequalities in analysis to $M_X^u(0,\infty)$.

For any Lebesgue measurable functions f, g, the Mellin convolution of f and g is defined by

$$(f \overset{M}{*} g)(t) = \int_0^\infty f(t/s)g(s)\frac{ds}{s}.$$

As an application of Minkowski's inequality on $M_X^u(0,\infty)$, the following presents a property of Mellin convolution on $M_X^u(0,\infty)$.

Theorem 3.2. *Let X be a r.i.B.f.s. and $u(t,r) : (0,\infty) \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function satisfying* (2.3) *and* (2.4)*. If g is a Lebesgue measurable function satisfying*

$$\int_1^\infty s^{\frac{1}{p}-\alpha_u-1}|g(s)|ds + \int_0^1 s^{\frac{1}{q}-\beta_u-1}|g(s)|ds < \infty$$

for some $p < p_X$ and $q_X < q$, then,

$$\|f \overset{M}{*} g\|_{M_X^u(0,\infty)} \leq C\|f\|_{M_X^u(0,\infty)}$$

for some $C > 0$.

Proof. By the Minkowski's inequality, we have

$$\|f \overset{M}{*} g\|_{M_X^u(0,\infty)} = \left\|\int_0^\infty f(\cdot/s)g(s)\frac{ds}{s}\right\|_{M_X^u(0,\infty)} \le \int_0^\infty \|D_s f\|_{M_X^u(0,\infty)}|g(s)|\frac{ds}{s}.$$

In view of (2.5) and (2.6), we have

$$\begin{aligned}\|f \overset{M}{*} g\|_{M_X^u(0,\infty)} &\le C\|f\|_{M_X^u(0,\infty)}\left(\int_1^\infty s^{\frac{1}{p}-\alpha_u-1}|g(s)|ds + \int_0^1 s^{\frac{1}{q}-\beta_u-1}|g(s)|ds\right)\\ &\le C\|f\|_{M_X^u(0,\infty)}\end{aligned}$$

for some $C > 0$. □

By using the idea of the proof from Theorem 3.2, we can also generalize the Hardy–Littlewood–Pólya inequalities. These inequalities offer the mapping properties of some integral operators on $M_X^u(0,\infty)$ [18, Theorem 319].

Theorem 3.3. *Let X be a r.i.B.f.s. with $1 < p_X \le q_X < \infty$ and $u(t,r)$: $(0,\infty)\times(0,\infty) \to (0,\infty)$ be a Lebesgue measurable function satisfying* (2.3) *and* (2.4).

Let $K(\cdot,\cdot)$ be a Lebesgue measurable function on $(0,\infty)\times(0,\infty)$. If K satisfies

(1) $K(\lambda s, \lambda t) = \lambda^{-1}K(s,t)$,
(2) $\int_1^\infty |K(v,1)|v^{-\frac{1}{q}+\beta_u}dv + \int_0^1 |K(v,1)|v^{-\frac{1}{p}+\alpha_u}dv < \infty$ *for some $p < p_X$ and $q_X < q$,*

then, the linear operator

$$Tf(t) = \int_0^\infty K(s,t)f(s)ds$$

is bounded on $M_X^u(0,\infty)$.

Proof. Let $v = s/t$. We have

$$|Tf(t)| \le \int_0^\infty |K(vt,t)||(D_{1/v}f)(t)|tdv = \int_0^\infty |K(v,1)||(D_{1/v}f)(t)|dv.$$

Applying the norm $\|\cdot\|_{M_X^u(0,\infty)}$ on both sides of the above inequality, Theorem 3.1 yields

$$\|Tf\|_{M_X^u(0,\infty)} \le \int_0^\infty |K(v,1)|\|(D_{1/v}f)\|_{M_X^u(0,\infty)}dv.$$

For the p and q given by Item (2), (2.5) and (2.6) guarantee that

$$\begin{aligned}\|Tf\|_{M_X^u(0,\infty)} &\le C\|f\|_{M_X^u(0,\infty)}\left(\int_1^\infty |K(v,1)|v^{-\frac{1}{q}+\beta_u}dv + \int_0^1 |K(v,1)|v^{-\frac{1}{p}+\alpha_u}dv\right)\\ &\le C\|f\|_{M_X^u(0,\infty)}\end{aligned}$$

for some $C > 0$. Hence, T is bounded on $M_X^u(0,\infty)$. □

We now apply the Hardy–Littlewood–Pólya inequalities to extend the Hardy inequalities to $M_X^u(0,\infty)$.

Theorem 3.4 (Hardy's Inequality). *Let X be a r.i.B.f.s. with $1 < p_X \le q_X < \infty$ and $u(t,r) : (0,\infty)\times(0,\infty) \to (0,\infty)$ be a Lebesgue measurable function satisfying (2.3) and (2.4).*

(1) *If*

$$1 + \alpha_u > \frac{1}{p_X}, \tag{3.2}$$

then

$$Tf(t) = \frac{1}{t}\int_0^t f(s)ds$$

is bounded on $M_X^u(0,\infty)$.

(2) *If*

$$\frac{1}{q_X} > \beta_u, \tag{3.3}$$

then

$$Sf(t) = \int_t^\infty \frac{f(s)}{s}ds$$

is bounded on $M_X^u(0,\infty)$.

Proof. Let $K(s,t) = t^{-1}\chi_E(s,t)$ where $E = \{(s,t) : s < t\}$. It satisfies Item (1) of Theorem 3.3.

Since $1 < p_X$ and $1 + \alpha_u > \frac{1}{p_X}$, there exists a $1 < p < p_X$ such that $1 + \alpha_u > \frac{1}{p} > \frac{1}{p_X}$. Hence, for any $q > q_X$, we have

$$\begin{aligned}\int_1^\infty |K(v,1)|v^{-\frac{1}{q}+\beta_u}dv + \int_0^1 |K(v,1)|v^{-\frac{1}{p}+\alpha_u}dv &= \int_0^1 v^{-\frac{1}{p}+\alpha_u}dv\\ &= \left.\frac{v^{-\frac{1}{p}+\alpha_u+1}}{-\frac{1}{p}+\alpha_u+1}\right|_0^1 < \infty.\end{aligned}$$

According to Theorem 3.3, we find that

$$\|Tf\|_{M^u_X(0,\infty)} \leq C\|f\|_{M^u_X(0,\infty)}.$$

To establish the boundedness of the operator S, let $K(s,t) = s^{-1}\chi_E(s,t)$ where $E = \{(s,t) : s > t\}$. It also satisfies Item (1) of Theorem 3.3. Similarly, we have a $q > q_X$ such that $\beta_u < \frac{1}{q} < \frac{1}{q_X}$. Hence, for any $p > p_X$, we have

$$\int_1^\infty |K(v,1)|v^{-\frac{1}{q}+\beta_u}dv + \int_0^1 |K(v,1)|v^{-\frac{1}{p}+\alpha_u}dv = \int_1^\infty v^{-\frac{1}{q}+\beta_u-1}dv < \infty.$$

Thus, Theorem 3.3 concludes that

$$\|Sf\|_{M^u_X(0,\infty)} \leq C\|f\|_{M^u_X(0,\infty)}$$

for some $C > 0$. □

For instance, when $X = L^p(0,\infty)$, $1 \leq p \leq \infty$ and $u \equiv 1$, $M^u_X(0,\infty)$ reduces to $L^p(0,\infty)$. We have $p_X = q_X = p$ and $\alpha_u = \beta_u = 0$. Therefore, the conditions (3.2) and (3.3) become $1 < p \leq \infty$ and $1 \leq p < \infty$ which are the well known conditions for the classical Hardy inequalities [14, Corollary 6.21].

As a special case of the previous theorem, we have the Hardy inequality for the classical Morrey spaces $M^p_q(0,\infty)$.

Corollary 3.5. *Let $1 \leq q \leq p < \infty$.*

(1) *If $1 < p \leq \infty$, then*

$$\left\|\frac{1}{t}\int_0^t f(s)ds\right\|_{M^p_q(0,\infty)} \leq C\|f(t)\|_{M^p_q(0,\infty)}. \tag{3.4}$$

(2) *If $1 \leq p < \infty$, then*

$$\left\|\int_t^\infty \frac{f(s)}{s}ds\right\|_{M^p_q(0,\infty)} \leq C\|f(t)\|_{M^p_q(0,\infty)}. \tag{3.5}$$

Proof. Let $X = L^q(0,\infty)$ and $u(z,R) = |I(z,R)|^{\frac{1}{q}-\frac{1}{p}}$, we find that $p_X = q_X = q$ and $\alpha_u = \beta_u = \frac{1}{q} - \frac{1}{p}$. Therefore, when $1 < p \leq \infty$, we have

$$1 + \alpha_u = 1 + \frac{1}{q} - \frac{1}{p} > \frac{1}{q} = \frac{1}{p_X}.$$

We are allowed to apply Theorem 3.4 to obtain (3.4).

Similarly, when $1 \leq p < \infty$, we have $\frac{1}{q_X} = \frac{1}{q} > \frac{1}{q} - \frac{1}{p} = \beta_u$. Thus, Theorem 3.4 yields (3.5). □

According to the above corollary, we see that even those the index q comes from the Lebesgue space $L^q(0,\infty)$ which is used to built $M_q^p(0,\infty)$ but the validity of the Hardy inequality does not rely on q. It relies on the other index p from $u(z,R) = |I(z,R)|^{\frac{1}{q}-\frac{1}{p}}$. For example, the inequality

$$\left\| \frac{1}{t} \int_0^t f(s)ds \right\|_{M_1^p(0,\infty)} \le C\|f(t)\|_{M_1^p(0,\infty)}, \quad 1 < p \le \infty,$$

is invalid but we do not have the Hardy's inequality on $L^1(0,\infty)$.

Theorem 3.4 extends the results in [9, 36] to the r.-i. Morrey spaces. We also have the generalization of Hilbert's inequality on $M_X^u(0,\infty)$.

Theorem 3.6 (Hilbert's inequality)**.** *Let X be a r.i.B.f.s. with $1 < p_X \le q_X < \infty$ and $u(t,r) : (0,\infty)\times(0,\infty) \to (0,\infty)$ be a Lebesgue measurable function satisfying* (2.3) *and* (2.4)*. If*

$$1 + \alpha_u > \frac{1}{p_X} \quad \textit{and} \quad \beta_u < \frac{1}{q_X},$$

then

$$Tf(t) = \int_0^\infty \frac{f(s)}{t+s} ds$$

is bounded on $M_X^u(0,\infty)$.

Proof. Let $K(s,t) = (s+t)^{-1}$. It obviously fulfills Item (1) of Theorem 3.3. Moreover, there exist $q > q_X$ and $p < p_X$ such that

$$1 + \alpha_u > \frac{1}{p} > \frac{1}{p_X} \quad \text{and} \quad \beta_u < \frac{1}{q} < \frac{1}{q_X}.$$

Therefore, we have

$$\int_1^\infty (1+v)^{-1} v^{-\frac{1}{q}+\beta_u} dv + \int_0^1 (1+v)^{-1} v^{-\frac{1}{p}+\alpha_u} dv \\ \le \int_1^\infty v^{-\frac{1}{q}+\beta_u-1} dv + \int_0^1 v^{-\frac{1}{p}+\alpha_u} dv < \infty.$$

Hence, Theorem 3.3 assures the boundedness of T on $M_X^u(0,\infty)$. □

In particular, we have the Hilbert inequality on the classical Morrey spaces $M_q^p(0,\infty)$. If $1 < q \le p < \infty$, then

$$\left\| \int_0^\infty \frac{f(s)}{t+s} ds \right\|_{M_q^p(0,\infty)} \le C\|f(t)\|_{M_q^p(0,\infty)}.$$

4. Hausdorff operator

In this section, we study the boundedness of the Hausdorff operator on r.-i. Morrey spaces on $\mathbb{R}^n$. There are many results on the boundedness of Hausdorff operators on several important function spaces arising in harmonic analysis. For instance, we have the boundedness of the Hausdorff operator on Lebesgue spaces, Hardy spaces and the spaces of bounded mean oscillation in [1], [7], [8], [10], [15], [27], [29], [30], [32], [33], [34], [40], [47]. The Hausdorff operator is also an extension of the study of the Cesáro operator [16], [39], [46].

We begin with the definition of Hausdorff operator associated with a signed σ-finite Borel measure. For any signed σ-finite Borel measure μ on $\mathbb{R}$, the Hausdorff operator associated with μ is given by

$$H_\mu f(x) = \int_{\mathbb{R}} f(tx) d\mu(t), \quad x \in \mathbb{R}^n.$$

The adjoint operator of H_μ is defined as

$$H_\mu^* f(x) = \int_{\mathbb{R}} |t|^{-n} f(x/t) d\mu(t), \quad x \in \mathbb{R}^n.$$

We now ready to establish the mapping properties of the Hausdorff operator and its adjoint operator on r.-i. Morrey spaces. We present a notation related to the measure μ. Let $a, b \in \mathbb{R}$. For any signed σ-finite Borel measure μ on $\mathbb{R}^n$, write

$$\|\mu\|_{a,b} = \int_{|t|\le 1} |t|^a d|\mu| + \int_{|t|>1} |t|^b d|\mu|.$$

Theorem 4.1. *Let X be a r.i.B.f.s. on $\mathbb{R}^n$ with $1 < p_X \le q_X < \infty$ and $u(y,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function satisfying* (2.3) *and* (2.4).

(1) *If there exist $p < p_X \le q_X < q$ such that*

$$\|\mu\|_{-\frac{n}{p}+n\alpha_u, -\frac{n}{q}+n\beta_u} < \infty,$$

then

$$\|H_\mu f\|_{M_X^u(\mathbb{R}^n)} \le C \|\mu\|_{-\frac{n}{p}+n\alpha_u, -\frac{n}{q}+n\beta_u} \|f\|_{M_X^u(\mathbb{R}^n)}$$

for some $C > 0$ independent of μ and f.

(2) *If there exist $p < p_X \le q_X < q$ such that*

$$\|\mu\|_{\frac{n}{q}-n\beta_u-n, \frac{n}{p}-n\alpha_u-n} < \infty,$$

then

$$\|H_\mu^* f\|_{M_X^u(\mathbb{R}^n)} \le C \|\mu\|_{\frac{n}{q}-n\beta_u-n, \frac{n}{p}-n\alpha_u-n} \|f\|_{M_X^u(\mathbb{R}^n)}$$

for some $C > 0$ independent of μ and f.

Proof. In view of the Minkowski inequality for $M_X^u(\mathbb{R}^n)$, we find that

$$\|H_\mu f\|_{M_X^u(\mathbb{R}^n)} \leq \int_{\mathbb{R}} \|D_{1/t} f\|_{M_X^u(\mathbb{R}^n)} d|\mu|(t)$$
$$\leq C\biggl(\int_{|t|\leq 1} |t|^{-\frac{n}{p}+n\alpha_u} d|\mu|(t) + \int_{|t|>1} |t|^{-\frac{n}{q}+n\beta_u} d|\mu|(t)\biggr)\|f\|_{M_X^u(\mathbb{R}^n)}$$
$$\leq C\|\mu\|_{-\frac{n}{p}+n\alpha_u,-\frac{n}{q}+n\beta_u}\|f\|_{M_X^u(\mathbb{R}^n)}$$

for some $C>0$ independent of μ and f.

Similarly, we have

$$\|H_\mu^* f\|_{M_X^u(\mathbb{R}^n)} \leq \int_{\mathbb{R}} \|D_t f\|_{M_X^u(\mathbb{R}^n)} |t|^{-n} d|\mu|(t)$$
$$\leq C\biggl(\int_{|t|\leq 1} |t|^{\frac{n}{q}-n\beta_u-n} d|\mu|(t) + \int_{|t|>1} |t|^{\frac{n}{p}-n\alpha_u-n} d|\mu|(t)\biggr)\|f\|_{M_X^u(\mathbb{R}^n)}$$
$$\leq C\|\mu\|_{\frac{n}{q}-n\beta_u-n,\frac{n}{p}-n\alpha_u-n}\|f\|_{M_X^u(\mathbb{R}^n)}$$

for some $C>0$ independent of μ and f. □

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KWOK-PUN HO
DEPARTMENT OF MATHEMATICS
AND INFORMATION TECHNOLOGY
THE HONG KONG INSTITUTE OF EDUCATION
10 LO PING ROAD
TAI PO, HONG KONG
CHINA

E-mail: vkpho@ied.edu.hk

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