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# Functional equations in the spectral theory of random fields II.

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## 1. Introduction

Let $S_\infty$ denote the unit sphere in $l^2$, that is, $S_\infty = \{t \mid t \in l^2;\ t = (t_1, \dots, t_n, \dots);\ \sum_{k=1}^\infty t_k^2 = 1\}$. A random field $X(t)$ on $S_\infty$ is called isotropic in the wide sence if $MX^2(t) < +\infty$, or $MX(t)$ is independent of $t$, and if $MX(t)\,X(s) = B(\cos\Theta)$ depends on $\cos\Theta = \sum_{k=1}^\infty t_k\, s_k$ $(t = (t_k),\ s = (s_k))$.

Next let us consider Markov random fields. Let $L(t_1^0)$ be the set of vectors from $S_\infty$, whose first component equals $t_1^0$. A random field $X(t)$ on $S_\infty$ is called Markov if for any $-1 \le t_1^0 \le 1$ and any $t'$, $t''$, $t_1' > t_1^0 > t_1''$, the random variables $X(t')$, $X(t'')$ are conditionally independent given $X(t)$ on $L(t_1^0)$.

The following theorem is known.

**Theorem** (see [3]). *The correlation function of a Gaussian isotropic random field of the Markov type on $S_\infty$ is given by*

$$B(\cos\Theta) = (\cos\Theta)^m,$$

*where $m$ is an integer.*

The proof based on the fact that, in this case, the correlation function $B$ satisfies the functional equation

$$(1)\quad B(\cos\Theta'')\,B(\cos^2\Theta) = B(\cos\Theta''\cos\Theta)\,B(\cos\Theta), \qquad \Theta'' > \Theta;\ \Theta'', \Theta \in [0,\pi],$$

and differentiable in $(0,1)$.

The general solution of this functional equation is presented here.

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## 2. The non zero solutions of (1)

Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$.

**Lemma 1.** *If the function $B : [-1,1] \to \mathbb{R}$ satisfies the functional equation* (1) *then $B$ satisfies the functional equation*

$$B(t^2)\,B(s) = B(ts)\,B(t); \quad t > s;\; t,s \in [-1,1]. \tag{2}$$

Proof. By the transformation

$$s = \cos\Theta'',\; t = \cos\Theta \qquad (\Theta'' > \Theta;\; \Theta'',\Theta \in [0,\pi]), \tag{3}$$

it follows immediately from (1) that the functional equation (3) is valid on the $(t,s)$ domain of transformation (3), i.e., for all $(t,s) \in T = \{(t,s) \mid t > s;\; t,s \in [-1,1]\}$.

Before stating the main theorem of this section, we will prove the following lemma.

**Lemma 2.** *If the function $B : (0,1) \to \mathbb{R}_0$ satisfies the functional equation*

$$B(t^2)\,B(s) = B(ts)\,B(t); \qquad t > s,\; t,s \in (0,1), \tag{2'}$$

*then there exists an additive function $A : \mathbb{R} \to \mathbb{R}$ such that*

$$B(t) = C\exp[A(\ln t)], \qquad t \in (0,1), \tag{4}$$

*where $C \in \mathbb{R}_0$ is an arbitrary constant.*

Proof. By the transformation

$$t = e^{-\frac{u}{2}},\; s = e^{-v} \qquad (t > s,\; t,s \in (0,1)), \tag{5}$$

we obtain from (2′) that the function

$$f : \mathbb{R}_+ \to \mathbb{R}_0, \quad f(u) = B(\exp(-u)), \tag{6}$$

satisfies the functional equation

$$f(u)\,f(v) = f\left(\frac{u}{2} + v\right)\,f\left(\frac{u}{2}\right) \qquad \left(v > \frac{u}{2} > 0\right). \tag{7}$$

Using Lemma 3 in [2], it follows that the function

$$h : (-1,\infty) \to \mathbb{R}, \quad h(x) = \ln\frac{f(x+1)}{f(1)}, \tag{8}$$

satisfies the functional equation

$$h(x+y) = h(x) + h(y) \qquad (x \in (0,1),\; y > x - 1), \tag{9}$$

i.e., $h$ is additive on the open connected domain $\{(x,y) \mid x \in (0,1),\ y > x-1\}$.

Applying now Lemma 4 in [2] we get that there exists an additive function $\bar{A} : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) = \bar{A}(x), \qquad x \in (-1,\infty). \tag{10}$$

From (6), (8) and (10) we obtain the form

$$B(t) = f(1)\exp[\bar{A}(-1)]\exp[\bar{A}(-\ln t)], \qquad t \in (0,1), \tag{11}$$

for the function $B$, where $f(1)$, $\bar{A}(-1)$ are arbitrary real constants. Finally we get from (11) the form (4) for $B$, where $C = f(1)\exp[\bar{A}(-1)] \in \mathbb{R}_0$ is arbitrary constant and the additive function $A$ is defined by $A(x) = \bar{A}(-x)$ $(x \in \mathbb{R})$.

**Theorem 1.** *If the function $B : [-1,1] \to \mathbb{R}$ satisfies the functional equation (1) and $B(t) \neq 0$ $(t \in (0,1))$, then either*

$$B(t) = \begin{cases} 0 & t = 0 \\ c_1 \exp[A(\ln t)] & t \in (0,1) \\ c_2 & t = 1 \\ c_3 \exp[A(\ln |t|)] & t \in [-1,0), \end{cases} \tag{12}$$

*or*

$$B(t) = \begin{cases} c_3 & t \in [-1,0] \\ c_1 & t \in (0,1) \\ c_2 & t = 1, \end{cases} \tag{13}$$

*where $c_1,\ c_3 \in \mathbb{R}_0,\ c_2 \in \mathbb{R}$ are arbitrary constants and $A : \mathbb{R} \to \mathbb{R}$ is an additive function on $\mathbb{R}^2$.*

Proof. If the function $B$ satisfies the functional equation (1) and $B \neq 0$ on $(0,1)$, then $B$ satisfies (2) and (2′) too. Thus, by Lemma 2,

$$B(t) = c_1 \exp[A(\ln(t)] \qquad t \in (0,1), \tag{14}$$

where $c_1 \in \mathbb{R}_0$ is an arbitrary constant and $A : \mathbb{R} \to \mathbb{R}$ is an additive function on $\mathbb{R}^2$.

Substitute $s = -1$ in (2), then we get

$$B(t^2)\, B(-1) = B(-t)\, B(t), \qquad t \in (-1,1]. \tag{15}$$

Since $B(t) \neq 0$ for $t \in (0,1)$, we obtain from (15) that

$$B(t) = B(-1)\, \frac{B(t^2)}{B(-t)}, \qquad t \in (-1,0). \tag{16}$$

If $t \in (-1,0)$, then $-t, t^2 \in (0,1)$ and (16) with (14) implies

$$B(t) = B(-1)\exp[A(\ln|t|)], \qquad t \in (-1,0). \tag{17}$$

Let us write in (2) $t=1$, then we get the identity

$$B(1)\,B(s) = B(s)\,B(1), \qquad s \in [-1,1).$$

On the other hand if $t, s \in (-1,1)$ then $t^2, ts \in (-1,1)$. Thus the value of $B$ on $[-1,1)$ does not depend on $B(1)$ and so

$$B(1) = c_2 \qquad (c_2 \in \mathbb{R} \text{ arbitrary}), \tag{18}$$

Put in (2) $t=0$, then we get

$$B(0)\,B(s) = B^2(0), \qquad s \in [-1,0). \tag{19}$$

If $B(0)=0$, then, according to (14), (17) and (18), we obtain the form (12) for $B$, where $c_3 = B(-1) \in \mathbb{R}_0$ is an arbitrary constant.

If $c_3 = B(0) \neq 0$, then (19) implies that

$$B(s) = c_3, \qquad s \in [-1,0]. \tag{20}$$

But in this case (17) shows that $A=0$ on $(0,1)$ and so

$$B(t) = c_1, \qquad t \in (0,1),$$

which together with (18) and (20) implies the form (13) for $B$.

It is easy to see that the functions (12) and (13) satisfy the functional equation (1).

## 3. The not almost everywhere zero solutions of (1)

**Lemma 3.** *If the function $B : [-1,1] \to \mathbb{R}$ satisfies the functional equation* (1) *and there exists a subset $E \subset (0,1)$ of positive Lebesgue-measure, such that $B(t) \neq 0$ for all $t \in E$, then $B(t) \neq 0$ for all $t \in (0,1)$.*

Proof. Under the conditions of the Lemma we get that $B$ satisfies the functional equation (2′). Let us write in (2′) $\sqrt{t}$ for $t$, then we get the functional equation

$$B(t)\,B(s) = B(s\sqrt{t})\,B(\sqrt{t}), \qquad (t,s) \in D = \{(t,s) \mid \sqrt{t} > s;\ t,s \in (0,1)\}. \tag{21}$$

Suppose $B(t) \neq 0$ for $t \in E$, where $E$ has positive measure, then there exists a compact subset $E_1 \subset E$ of positive measure such that $E_1 \subset [\alpha,\beta] \subset (0,1)$ for some closed interval $[\alpha,\beta]$. Further there exists a natural number $n$ such that $\sqrt[2n]{\alpha} > \beta$. It is obvious that one of the intervals

$(\alpha, \sqrt{\alpha}), \dots, (\sqrt[2n-2]{\alpha}, \sqrt[2n-1]{\alpha})$ $(\sqrt[2n-1]{\alpha}, \beta)$ contains a subset $E_2 \subset E_1$ of positive measure with $E_2 \times E_2 \subset D$ and then $B(t)\,B(s) \neq 0$ for $(t,s) \in E_2 \times E_2$.

Thus, by equation (21), $B(s\sqrt{t})\,B(\sqrt{t}) \neq 0$ if $t, s \in E_2$, which implies that $B(u) \neq 0$, whenever $u \in E_2\sqrt{E_2}$.

Since $E_2$ and $\sqrt{E_2}$ have positive Lebesgue-measure, by a theorem of Steinhaus (see [1]), the set $E_2\sqrt{E_2}$ contains an interval of positive length $[a,b] \subset (0,1)$ and so $B(u) \neq 0$ $(u \in [a,b])$.

Substitute $s = t$ in (21) (this is posibile because of $\sqrt{t} > t$ for $t \in (0,1)$), then we get

$$B^2(t) = B(t^{3/2})\,B(\sqrt{t}), \qquad t \in (0,1), \tag{22}$$

which implies $B(t^{3/2}) \neq 0,\ B(\sqrt{t}) \neq 0$ for all $t \in [a,b]$. Thus $B(u) \neq 0$ if $u \in [a^{3/2}, \sqrt{b}]$. It follows by induction that

$$B(u) \neq 0,\ u \in \left[a^{(\frac{3}{2})^n},\, b^{(\frac{1}{2})^n}\right], \qquad n \in N, \tag{23}$$

where $N$ is the set of natural numbers.

Since $a^{(\frac{3}{2})^n} \to 0$ and $b^{(\frac{1}{2})^n} \to 1$, we get from (23) that $B(u) \neq 0$ for all $u \in (0,1)$.

This completes the proof of Lemma 3.

Now it is easy to prove our main result.

**Theorem 2.** *If the function $B : [-1,1] \to \mathbb{R}$ satisfies the functional equation* (1) *and there exists a subset $E \subset (0,1)$ of positive Lebesgue-measure such that $B(t) \neq 0$ for all $t \in E$, then $B$ has the form* (12) *or* (13), *where $c_1, c_3 \in \mathbb{R}_0$, $c_2 \in \mathbb{R}$ arbitrary constants and $A : \mathbb{R} \to \mathbb{R}$ additive function on $\mathbb{R}^2$.*

Proof. By Lemma 3, $B(t) \neq 0$ for all $t \in (0,1)$. Then Theorem 1 implies the statement of our Theorem 2.

## 4. Remarks

*Remark 1.* It is not too difficult to determine the continuous solution of (1). We have the following

**Theorem 3.** *A function $B : [-1,1] \to \mathbb{R}$ is a continuous solution of equation* (1) *if and only if either $B$ is a constant function, or $f$ has the form*

$$B(t) = \begin{cases} c_1\, t^a & t \in [0,1] \\ c_3\, t^a & t \in [-1,0), \end{cases} \tag{24}$$

*where* $c_1, c_3 \in \mathbb{R}_0$ *and* $a \in \mathbb{R}_+ = \{x \mid x \in \mathbb{R},\ x > 0\}$ *is an arbitrary constants.*

Proof. If $B$ is continuous in $[-1,1]$ and there exists a value $t_0 \in (0,1)$, such that $B(t_0) \neq 0$, then there exists an internal $I = [t_0 - r,\ t_0 + r] \subset (0,1)$, such that $B(t) \neq 0$ for all $t \in I$ and $I \times I \subset D$.

Then, using the last part of the proof of Lemma 3, it follows that $B(t) \neq 0$ for all $t \in (0,1)$.

Since $B$ is continuous, functions $f$, $h$ and $A$ in Lemma 2 are continuous, thus $A(t) = at$ with some $a \in \mathbb{R}$.

Then we get from (12) and (13) that either

$$B(t) = \begin{cases} 0 & t = 0 \\ c_1 t^a & t \in (0,1) \\ c_2 & t = 1 \\ c_3 |t|^a & t \in [-1,0) \end{cases} \tag{12'}$$

or

$$B(t) = \begin{cases} c_3 & t \in [-1,0] \\ c_1 & t \in (0,1) \\ c_2 & t = 1 \end{cases} \tag{13'}$$

respectively.

Using again the continuity of $B$, $B = c$ results from (13′) and from (12′) for $a = 0$.

If $a < 0$, then $B$ given by (12′) satisfies $\lim_{x \to o+0} B(t) = \infty$ and thus $B$ cannot be continuous at zero.

If $a > 0$, then (24) follows from (12′).

*Remark 2.* A similar calculation shows, that if $B$ is continuous on the set $[-1,1] \setminus \{0\}$, then either

$$B(t) = \begin{cases} 0 & t = 0 \\ c_1 t^a & t \in (0,1] \\ c_3 |t|^a & t \in [-1,0), \end{cases}$$

or

$$B(t) = \begin{cases} c_3 & t \in [-1,0] \\ c_1 & t \in (0,1], \end{cases}$$

where $a \in \mathbb{R}$, $c_1, c_3 \in \mathbb{R}_0$ arbitrary constants.

*Remark 3.* For measurable solutions of (1) the situation is more complicated. For example the function (12′) and (13′) are measurable, satisfy equation (1) but usually are not continuous on the interval $[-1,1]$.

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