Publ. Math. Debrecen
88/1-2 (2016), 249–259
DOI: 10.5486/PMD.2016.7457

On projectively related spherically symmetric Finsler metrics

By XIAOHUAN MO (Beijing) and XIAOYANG WANG (Beijing)

Abstract. Spherically symmetric metrics and Randers metrics form two rich classes of Finsler metrics. In this paper, we find an equation that characterizes pointwise projectively related spherically symmetric metrics. Some applications are given. In particular, by using known spherically symmetric metrics, we produce a lot of Randers metrics of quadratic Weyl curvature which are non-trivial in the sense that they are not of Weyl type.

1. Introduction

Two Finsler metrics on a manifold are said to be *(pointwise) projectively related* (*projectively equivalent* in an alternative terminology in [2]) if they have the same geodesics as point sets.

For an example, consider the Funk metric F on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ given by

$$F(x,y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2} + \frac{\langle x,y\rangle}{1-|x|^2}, \tag{1.1}$$

where $y \in T_x\mathbb{B}^n \approx \mathbb{R}^n$. Then F is pointwise projectively related to the standard Euclidean metric. Metrics of this type are called *projectively flat*. More generally, Shen and Yu proved that two Randers metrics are projectively related if and only if the corresponding Riemannian metrics are projectively related and they have the same Douglas tensor [15]. A simple equation along geodesics for pointwise projectively related Einstein metrics has been obtained by Shen Z. [16]. Similar

---

Mathematics Subject Classification: 53B40, 53C60.
Key words and phrases: Finsler metric, projectively related, spherically symmetric, Weyl curvature.
This paper is supported by the National Natural Science Foundation of China 11371032.

results on pointwise projectively related (α, β)-type Finsler metrics have been discussed in [20], [17], [1], [4].

The Finsler metric in (1.1) satisfies

$$F(Ax, Ay) = F(x, y)$$

for all $A \in O(n)$, i.e., the orthogonal group acts as isometry group of F [18], [13]. Such metrics are called *spherically symmetric Finsler metrics* (or *orthogonally invariant Finsler metrics* in an alternative terminology, see [6]).

Recently, the study of spherically symmetric Finsler metrics has attracted a lot of attention. The classification of projective spherically symmetric Finsler metrics with constant flag curvature has recently been completed by L. Zhou and Mo–Zhu [13], [18], [8]. By establishing equations that characterize spherically symmetric Finsler metrics of scalar curvature, Huang–Mo constructed infinitely many non-projectively flat Finsler metric of scalar curvature [5]. A Finsler metric is said to be of *scalar (flag) curvature* if the flag curvature $K(x, y, P)$ at a point x is independent of the tangent plane $P \subset T_xM$. In particular, F is said to be of *constant flag curvature* if K is constant.

In [19], Zhou has found an expression of spherically symmetric Finsler metric F on $\mathbb{B}^n(\mu) := \{x \in \mathbb{R}^n;\ |x| < \mu\}$, rewriting F as

$$F = |y|\phi\left(|x|, \frac{\langle x, y\rangle}{|y|}\right)$$

where $(x, y) \in T\mathbb{B}^n(\mu)\backslash\{0\}$. Recently, Huang–Mo obtained a second-order PDE for ϕ expressing that the spherically symmetric Finsler metric $F = |y|\phi\big(|x|, \frac{\langle x,y\rangle}{|y|}\big)$ on $\mathbb{B}^n(\mu)$ is projectively flat [6].

In this paper, we find a PDE that characterize pointwise projectively related spherically symmetric Finsler metrics. More precisely, we show the following:

Theorem 1.1. *Let $F = |y|\phi(r, s)$ and $\tilde{F} = |y|\tilde{\phi}(r, s)$ be two spherically symmetric Finsler metrics on $\mathbb{B}^n(\mu)$, where $r := |x|$, $s := \frac{\langle x,y\rangle}{|y|}$. Then $\tilde{F}$ is pointwise projectively related to F if and only if ϕ and $\tilde{\phi}$ satisfy*

$$\frac{r\phi_{ss} - \phi_r + s\phi_{rs}}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}} = \frac{r\tilde{\phi}_{ss} - \tilde{\phi}_r + s\tilde{\phi}_{rs}}{\tilde{\phi} - s\tilde{\phi}_s + (r^2 - s^2)\tilde{\phi}_{ss}}.$$

Our theorem generalizes a result previously known only when $\tilde{F}$ is the standard Euclidean metric [5], [6]. At the same time, Theorem 1.1 tells us the following interesting fact: *the pointwise projective relatedness for two spherically symmetric Finsler metrics on $\mathbb{B}^n(\mu)$ is independent of the dimension n.*

In Finsler geometry, it is a natural problem to determine all Finsler metrics which are pointwise projectively related to the given one [16]. As an application of Theorem 1.1, we describe all spherically symmetric Finsler metrics which are pointwise projectively related to Huang–Mo's spherically symmetric Finsler metrics of scalar curvature (see Theorem 3.2 below). In particular, we give a necessary and sufficient condition for the projective equivalence of two Huang–Mo's spherically symmetric Finsler metrics of scalar curvature, see Theorem 2.4 below.

There are two classes of Finsler metrics with important Weyl curvature properties. The first one is the class of Weyl metrics [5], [10], [14], the second one is the class of W-quadratic metrics [2], [9]. Note that every Weyl metric must be W-quadratic. In [9], Li–Shen find equations that characterize W-quadratic Randers metrics. It seems that W-quadratic Randers metrics form a broader class than Weyl Randers metrics although there is no example supporting this. As an important application of Theorem 1.1, in this paper we find a lot of Randers metrics of quadratic Weyl curvature which are non-trivial in the sense that they are not of Weyl type.

Theorem 1.2. *Let $\tilde{F}(x,y)$ be a Finsler metric on $B^n(\mu)$ defined by*

$$\tilde{F}(x,y) = \frac{\sqrt{\kappa^2\langle x,y\rangle^2 + \varepsilon|y|^2(1+\zeta|x|^2)}}{1+\zeta|x|^2} + \frac{\kappa\langle x,y\rangle}{1+\zeta|x|^2}.$$

where κ, ζ and ε are arbitrary constants such that $\varepsilon > 0$; $\mu = 1/\sqrt{-\zeta}$ if $\zeta < 0$ and $\mu = +\infty$ if $\zeta \geq 0$. Then the Weyl curvature of $\tilde{F}$ is given by

$$\begin{aligned} W^i{}_j = -\frac{(\zeta\varepsilon+\kappa^2)^2}{[(\zeta\varepsilon+\kappa^2)r^2+\varepsilon]^2}\bigg[&\frac{1}{n-1}\left(|x|^2|y|^2 - \langle x,y\rangle^2\right)\delta^i_j - |y|^2x^ix^j \\ &+ \langle x,y\rangle x^iy^j + \frac{\langle x,y\rangle}{n-1}x^jy^i - \frac{|x|^2}{n-1}y^iy^j\bigg], \end{aligned} \tag{1.2}$$

with $r = |x|$, therefore $\tilde{F}$ is a non-trivial W-quadratic Randers metric when $\zeta\varepsilon + \kappa^2 \neq 0$.

For a proof of Theorem 1.2, see Section 4 below. In particular, we show that Chern–Shen's Randers metrics are non-trivial W-quadratic Finsler metrics with isotropic S-curvature, see Corollary 4.3 below.

Recall that a *Randers metric* F on a manifold M is a Finsler metric in the form $F = \alpha+\beta$, where α is a Riemannian metric and β is a 1-form with $\|\beta\|_x < 1$ at any point x of M. Recently, a signification progress has been made in studying Randers metrics [2], [11], [7].

2. Projectively related spherically symmetric metrics

Let F be a Finsler metric on $\mathbb{B}^n(\mu) := \{x \in \mathbb{R}^n;\ |x| < \mu\}$. F is said to be *spherically symmetric* if it satisfies $F(Ax, Ay) = F(x, y)$ for all $x \in \mathbb{B}^n(\mu)$, $y \in T_x\mathbb{B}^n(\mu)$ and $A \in O(n)$.

Let $F = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$ be a spherically symmetric Finsler metric on $\mathbb{B}^n(\mu)$. Let

$$Q := \frac{1}{2r}\frac{r\phi_{ss} - \phi_r + s\phi_{rs}}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}}, \quad r := |x|, \quad s := \frac{\langle x, y\rangle}{|y|}. \tag{2.1}$$

Let $x^1, \ldots, x^n$ be coordinates on $\mathbb{R}^n$ and let $y = \sum y^i \partial/\partial x^i$. Then the geodesic coefficients of F are given by

$$G^i = |y|Py^i + |y|^2 Qx^i, \tag{2.2}$$

where

$$P := \frac{r\phi_s + s\phi_r}{2r\phi} - \frac{Q}{\phi}\left[s\phi + (r^2 - s^2)\phi_s\right]. \tag{2.3}$$

Let $\tilde{F} = |y|\tilde{\phi}\big(|x|, \frac{\langle x,y\rangle}{|y|}\big)$ be another spherically symmetric Finsler metric on $\mathbb{B}^n(\mu)$. Then we have

$$\tilde{G}^i = |y|\tilde{P}y^i + |y|^2\tilde{Q}x^i \tag{2.4}$$

where we denote the corresponding objects with respect to $\tilde{F}$ by adding a tilde $\tilde{}$.

Proof of Theorem 1.1. Assume that $\tilde{F}$ is pointwise projectively related to F. Then

$$\tilde{G}^i = G^i + Ry^i \tag{2.5}$$

where R is positively homogeneous of degree one [17], [15], [1], [4], [16]. Plugging (2.4) and (2.2) into (2.5) yields $|y|^2(\tilde{Q} - Q)x^i + [|y|(\tilde{P} - P) - R]y^i = 0$. It follows that

$$\tilde{Q} = Q, \qquad |y|\tilde{P} = |y|P + R. \tag{2.6}$$

Conversely, suppose that the first equation of (2.6) holds for two spherically symmetric Finsler metrics $F = |y|\phi\big(|x|, \frac{\langle x,y\rangle}{|y|}\big)$ and $\tilde{F} = |y|\tilde{\phi}\big(|x|, \frac{\langle x,y\rangle}{|y|}\big)$. By using (2.2) and (2.4) we have

$$\tilde{G}^i - G^i = |y|\tilde{P}y^i + |y|^2\tilde{Q}x^i - |y|Py^i - |y|^2Qx^i = Ry^i,$$

where $R := |y|(\tilde{P} - P)$. Together with (2.3), we obtain that R is positively homogeneous of degree one. According to Theorem 2.1 in [4] or (2.2) in [15], $\tilde{F}$ must be pointwise projectively related to F. The above arguments and (2.1) complete the proof of Theorem 1.1. $\square$

As a consequence of Theorem 1.1, by taking the standard Euclidean metric $\tilde{F}$, we obtain the following result obtained by Huang and the first author (see [6, Theorem 1.1]):

Corollary 2.1. *Let $F = |y|\phi\big(|x|, \frac{\langle x,y\rangle}{|y|}\big)$ be a spherically symmetric Finsler metric on $\mathbb{B}^n(\mu)$. Then F is projectively flat if and only if ϕ satisfies $r\phi_{ss} - \phi_r + s\phi_{rs} = 0$.*

Corollary 2.2. *The pointwise projective relatedness for two spherically symmetric Finsler metrics is independent of the dimension of the base space.*

Proposition 2.3. *Let $F_1 = |y|\phi_1\big(|x|, \frac{\langle x,y\rangle}{|y|}\big)$ and $F_2 = |y|\phi_2\big(|x|, \frac{\langle x,y\rangle}{|y|}\big)$ be two spherically symmetric Finsler metrics on $\mathbb{B}^n(\mu)$ with*

$$\begin{aligned}\phi_j(r,s) = s\cdot h_j(r) &+ \frac{m_j!}{(2m_j-1)!!}\frac{(2r^2)^{m_j}}{(a_j+b_jr^2)^{(2m_j+1)\lambda_j}}\\ &- \frac{(r^2-s^2)^{m_j}}{(2m_j-1)(a_j+b_jr^2)^{(2m_j+1)\lambda_j}}\\ &- \frac{1}{(a_j+b_jr^2)^{(2m_j+1)\lambda_j}}\sum_{i=2}^{m_j}\frac{m_j!(2m_j-2i-1)!!}{(2m_j-1)!!(m_j-i+1)!}(2r^2)^{i-1}(r^2-s^2)^{m_j-i+1}\\ &+ \frac{\epsilon_j}{(a_j+b_jr^2)^{\lambda_j}} \qquad (2.7)\end{aligned}$$

where $m_j \in \{1,2,3,\dots\}$, ϵ_j, a_j, b_j and λ_j are constants satisfying $\epsilon_j > 0$ and $a_j+b_jr^2 > 0$ and h_j are differentiable functions. Then F_1 is pointwise projectively related to F_2 if and only if a_j, b_j and λ_j $(j = 1,2)$ satisfy

$$b_1b_2(\lambda_1-\lambda_2) = 0, \quad \lambda_1b_1a_2 = \lambda_2b_2a_1. \tag{2.8}$$

Furthermore, F_1 and F_2 are projectively flat if $b_1b_2 = 0$ or $b_1b_2 \neq 0, \lambda_1 = \lambda_2 = 0$; F_1 and F_2 are of scalar curvature if $b_1b_2 = 0$ or $b_1b_2 \neq 0$, λ_1 or $\lambda_2 \in \{0,1\}$; F_1 and F_2 are non-trivial W-quadratic metrics if $\lambda_1(\lambda_1-1)b_1 \neq 0$ or $\lambda_2(\lambda_2-1)b_2 \neq 0$.

Proof. According to Theorem 1.1, F_1 is pointwise projectively related to F_2 if and only if

$$Q_1 = Q_2 \tag{2.9}$$

where

$$Q_j = \frac{1}{2r}\frac{r(\phi_j)_{ss} - (\phi_j)_r + s(\phi_j)_{rs}}{(\phi_j) - s(\phi_j)_s + (r^2-s^2)(\phi_j)_{ss}}, \quad j = 1,2. \tag{2.10}$$

On the other hand, (2.7) gives solutions of (2.10) with

$$Q_j := \frac{\lambda_j b_j}{a_j + b_j r^2}, \quad j = 1, 2 \tag{2.11}$$

where a_j, b_j and λ_j are constants satisfying $a_j + b_j r^2 > 0$ [10]. Thus F_1 is pointwise projectively related to F_2 if and only if

$$\frac{\lambda_1 b_1}{a_1 + b_1 r^2} = \frac{\lambda_2 b_2}{a_2 + b_2 r^2}. \tag{2.12}$$

It is easy to see that (2.12) holds if and only if

$$b_1 b_2(\lambda_1 - \lambda_2) r^2 + (\lambda_1 b_1 a_2 - \lambda_2 b_2 a_1) = 0.$$

Together with the second equation of (2.1) we have (2.8). "Furthermore ..." is an immediate consequence of Theorem 1.1 in [10]. □

Theorem 2.4. *Let $F_1 = |y|\phi_1\big(|x|, \frac{\langle x,y\rangle}{|y|}\big)$ and $F_2 = |y|\phi_2\big(|x|, \frac{\langle x,y\rangle}{|y|}\big)$ be two spherically symmetric Finsler metrics on $\mathbb{B}^n(\mu)$ with*

$$\begin{aligned}\phi_j(r,s) = s \cdot h_j(r) &+ \frac{m_j!}{(2m_j-1)!!}\frac{(2r^2)^{m_j}}{(a_j+b_jr^2)^{2m_j+1}} - \frac{(r^2-s^2)^{m_j}}{(2m_j-1)(a_j+b_jr^2)^{2m_j+1}} \\ &- \frac{1}{(a_j+b_jr^2)^{2m_j+1}}\sum_{i=2}^{m_j}\frac{m_j!(2m_j-2i-1)!!}{(2m_j-1)!!(m_j-i+1)!}(2r^2)^{i-1}(r^2-s^2)^{m_j-i+1} \\ &+ \frac{\epsilon_j}{a_j+b_jr^2}\end{aligned} \tag{2.13}$$

where $m_j \in \{1,2,3,\dots\}$, ϵ_j, a_j and b_j are constants such that $\epsilon_j > 0$, $a_j + b_j r^2 > 0$ and h_j are differentiable functions. Then F_1 is pointwise projectively related to F_2 if and only if a_1, a_2, b_1 and b_2 satisfy $b_1 a_2 = b_2 a_1$.

Proof. The result follows from Proposition 2.3 by taking $\lambda_1 = \lambda_2 = 1$. □

Theorem 2.4 tells us that the pointwise projective relatedness of two Huang–Mo's spherically symmetric Weyl metrics is independent of the functions h_j, the natural numbers m_j and the constants ϵ_j, $j = 1, 2$.

3. Projectively related Weyl quadratic metrics

Given a Finsler metric on a manifold M, a natural problem is to determine all Finsler metrics which are pointwise projectively related to the given metric [16].

In this section, we study the following problem: given a Weyl quadratic spherically symmetric Finsler metric, describe all spherically symmetric Finsler metrics which are pointwise projectively related to the given one.

Proposition 3.1. *Let ϕ be a function defined by*

$$\begin{aligned}\phi(r,s) = s\cdot h(r) &+ \frac{m!}{(2m-1)!!}\frac{(2r^2)^m}{(a+br^2)^{(2m+1)\lambda}} - \frac{(r^2-s^2)^m}{(2m-1)(a+br^2)^{(2m+1)\lambda}} \\ &- \frac{1}{(a+br^2)^{(2m+1)\lambda}}\sum_{i=2}^{m}\frac{m!(2m-2i-1)!!}{(2m-1)!!(m-i+1)!}(2r^2)^{i-1}(r^2-s^2)^{m-i+1} \\ &+ \frac{\epsilon}{(a+br^2)^{\lambda}} \end{aligned}\tag{3.1}$$

where $m\in\{1,2,3,\dots\}$; ϵ, a, b and λ are constants such that $\epsilon>0$ and $a+br^2>0$ and h are differentiable functions. Then any spherically symmetric Finsler metric which is pointwise projectively related to $F=|y|\phi\big(|x|,\frac{\langle x,y\rangle}{|y|}\big)$ is given by

$$\tilde{F} = |y|\tilde{\phi}\left(|x|,\frac{\langle x,y\rangle}{|y|}\right)$$

where

$$\tilde{\phi}(r,s) = s\left(f(r) - \int\frac{\eta(\varphi(r,s))}{s^2\sqrt{r^2-s^2}}ds\right),\tag{3.2}$$

with φ given by

$$\varphi(r,s) = -\frac{r^2-s^2}{(a+br^2)^{2\lambda}};\tag{3.3}$$

f and η are arbitrary differentiable real functions of r and φ respectively, and

$$\frac{-\sqrt{r^2-s^2}}{s}\frac{\partial\eta}{\partial s}>0,\quad \textit{when } n\geq 2,$$

with the additional inequality

$$\frac{\eta}{\sqrt{r^2-s^2}}>0,\quad \textit{when } n\geq 3,$$

where $r^2-s^2>0$ and $s\neq 0$.

Proof. According to the proof of Theorem 1.1 in [10], we have

$$\frac{1}{2r}\frac{r\phi_{ss}-\phi_r+s\phi_{rs}}{\phi-s\phi_s+(r^2-s^2)\phi_{ss}} = Q = \frac{\lambda b}{a+br^2}\tag{3.4}$$

where we have used the first equation of (2.1). Together with Theorem 1.1 we obtain

$$\left[(r^2-s^2)\frac{2\lambda b}{a+br^2}-1\right]r\tilde{\phi}_{ss}+\tilde{\phi}_r-s\tilde{\phi}_{rs}+\frac{2\lambda br}{a+br^2}(\tilde{\phi}-s\tilde{\phi}_s)=0.\tag{3.5}$$

By Lemma 4.1 in [12] and (3.4), F has vanishing Douglas curvature. Then Proposition 3.1 follows from Theorem 1.2 in [12]. $\Box$

As a consequence of Proposition 3.1, by taking $\lambda = 1$, we obtain the following:

Theorem 3.2. *Let ϕ be a function defined by*

$$\phi(r,s) = s\cdot h(r) + \frac{m!}{(2m-1)!!}\frac{(2r^2)^m}{(a+br^2)^{2m+1}} - \frac{(r^2-s^2)^m}{(2m-1)(a+br^2)^{2m+1}}$$
$$-\frac{1}{(a+br^2)^{2m+1}}\sum_{i=2}^{m}\frac{m!(2m-2i-1)!!}{(2m-1)!!(m-i+1)!}(2r^2)^{i-1}(r^2-s^2)^{m-i+1} + \frac{\epsilon}{a+br^2}$$

where $m \in \{1,2,3,\dots\}$, ϵ, a and b are constants such that $\epsilon > 0$ and $a+br^2 > 0$ and h is a differentiable function. Then any spherically symmetric Finsler metric which is pointwise projectively related to $F = |y|\phi\big(|x|, \frac{\langle x,y\rangle}{|y|}\big)$ is given by

$$\tilde{F} = |y|\tilde{\phi}\left(|x|, \frac{\langle x,y\rangle}{|y|}\right)$$

where

$$\tilde{\phi}(r,s) = s\left(f(r) - \int \frac{\eta(\varphi(r,s))}{s^2\sqrt{r^2-s^2}}ds\right),$$

with φ given by

$$\varphi(r,s) = -\frac{r^2-s^2}{(a+br^2)^2};$$

f and η are arbitrary differentiable real functions of r and φ respectively, and

$$\frac{-\sqrt{r^2-s^2}}{s}\frac{\partial\eta}{\partial s} > 0, \text{ when } n \geq 2,$$

with the additional inequality

$$\frac{\eta}{\sqrt{r^2-s^2}} > 0, \text{ when } n \geq 3,$$

where $r^2 - s^2 > 0$ and $s \neq 0$.

4. Non-trivial W-quadratic Randers metrics

In this section, we are going to find Randers metrics of quadratic Weyl curvature which are non-trivial in the sense that they are not of Weyl type. Recall that a Finsler metric is called a *Weyl metric* if it has vanishing Weyl curvature [5], [10], [14] and Finsler metric is said to be *W-quadratic* if it has quadratic Weyl curvature [2], [9]. According to M. Matsumoto's result, a Finsler metric is of Weyl type if and only if it is of scalar curvature.

As a consequence of Proposition 3.1, for $\lambda = \frac{1}{2}$, $b = \zeta\varepsilon + \kappa^2$, $a = \varepsilon$ and hence $\varphi(r,s) = \frac{-(r^2-s^2)}{(\zeta\varepsilon+\kappa^2)r^2+\varepsilon}$, with the choice $\eta(\varphi) = \varepsilon\sqrt{-\big(\frac{1}{\varphi}+\kappa^2\big)^{-1}}$, we get the following result:

Corollary 4.1. *Let $\phi(r,s)$ be a function defined by*

$$\begin{aligned}\phi(r,s) = s\cdot h(r) &+ \frac{m!}{(2m-1)!!}\frac{(2r^2)^m}{[(\zeta\varepsilon+\kappa^2)r^2+\varepsilon]^{\frac{2m+1}{2}}}\\ &- \frac{(r^2-s^2)^m}{(2m-1)\,[(\zeta\varepsilon+\kappa^2)r^2+\varepsilon]^{\frac{2m+1}{2}}}\\ &- \frac{1}{[(\zeta\varepsilon+\kappa^2)r^2+\varepsilon]^{\frac{2m+1}{2}}}\sum_{i=2}^{m}\frac{m!(2m-2i-1)!!}{(2m-1)!!(m-i+1)!}(2r^2)^{i-1}(r^2-s^2)^{m-i+1}\\ &+ \frac{\epsilon}{[(\zeta\varepsilon+\kappa^2)r^2+\varepsilon]^{\frac{1}{2}}}\end{aligned} \tag{4.1}$$

where $m \in \{1,2,3,\dots\}$, ϵ, ε, ζ and κ are constants such that $\epsilon > 0$ and $(\zeta\varepsilon + \kappa^2)r^2 + \varepsilon > 0$ and h is a differentiable function. Then the spherical symmetric Finsler metric

$$F = |y|\phi\left(|x|, \frac{\langle x,y\rangle}{|y|}\right) \tag{4.2}$$

is pointwise projectively related to $\tilde{F}(x,y) = |y|\tilde{\phi}\big(|x|, \frac{\langle x,y\rangle}{|y|}\big)$ where

$$\tilde{\phi}(r,s) = sf(r) + \frac{\sqrt{\zeta\varepsilon r^2 + \kappa^2 s^2 + \varepsilon}}{\zeta r^2 + 1}$$

with any real function f such that $\tilde{\phi}(r,s)$ is positive.

In particular, when

$$f(r) = \frac{\kappa}{1+\zeta r^2},$$

we have the following:

Corollary 4.2. *Let ϕ be a function defined by* (4.1). *Then the spherical symmetric metric*

$$F = |y|\phi\left(|x|, \frac{\langle x,y\rangle}{|y|}\right)$$

is pointwise projectively related to the Randers metric

$$\tilde{F}(x,y) = \frac{\sqrt{\kappa^2\langle x,y\rangle^2 + \varepsilon|y|^2(1+\zeta|x|^2)}}{1+\zeta|x|^2} + \frac{\kappa\langle x,y\rangle}{1+\zeta|x|^2}. \tag{4.3}$$

PROOF OF THEOREM 1.2. According to Corollary 4.2, $\tilde{F}(x,y)$ is pointwise projectively related to $F = |y|\phi\big(|x|, \frac{\langle x,y\rangle}{|y|}\big)$ where ϕ is defined by (4.1). Since the Weyl curvature is a projective invariant, the Weyl curvature formula of MO–LIU [10, Theorem 1.1] gives (1.2). $\square$

When $0 < \zeta = \varepsilon$ and $\kappa^2 = 1 - \varepsilon^2$ we get the following:

Corollary 4.3. *Let F be a Finsler metric defined by*

$$F(x,y) := \frac{\sqrt{(1-\varepsilon^2)\langle x,y\rangle^2 + \varepsilon|y|^2(1+\varepsilon|x|^2)}}{1+\varepsilon|x|^2} + \frac{\sqrt{1-\varepsilon^2}\langle x,y\rangle}{1+\varepsilon|x|^2}$$

with $0 < \varepsilon \leq 1$. Then F is a non-trivial W-quadratic Randers metric of isotropic S-curvature. Moreover, the Weyl curvature of F is given by

$$\begin{aligned} W^i{}_j = -\frac{1}{[\varepsilon + |x|^2]^2}\bigg[&\frac{1}{n-1}\left(|x|^2|y|^2 - \langle x,y\rangle^2\right)\delta^i_j - |y|^2x^ix^j \\ &+ \langle x,y\rangle x^iy^j + \frac{\langle x,y\rangle}{n-1}x^jy^i - \frac{|x|^2}{n-1}y^iy^j\bigg]. \end{aligned}$$

Recall that the S-curvature is one of most important non-Riemannian quantities in Finsler geometry. It interacts with the Riemann curvature in a delicate way [2], [7], [9].

ACKNOWLEDGMENTS. This paper has benefitted from our discussions with L. HUANG. The authors are grateful to the referee for his careful reading of the manuscript and very helpful suggestions.

References

[1] G. CHEN and X. CHENG, A class of Finsler metrics projectively related to a Randers metric, *Publ. Math. Debrecen* **81** (2012), 351–363.

[2] X. CHENG and Z. SHEN, Finsler Geometry, An Approach via Randers Space, *Science Press, Beijing*, 2012.

[3] S. S. CHERN and Z. SHEN, Riemann–Finsler geometry, *World Scientific Publishing Co., Pte. Ltd., Hackensack, NJ*, 2005.

[4] N. CUI and Y. SHEN, Projective change between two classes of (α,β)-metrics, *Diff. Geom. Appl.* **27** (2009), 566–573.

[5] L. HUANG and X. MO, On spherically symmetric Finsler metrics of scalar curvature, *J. Geom. Phys.* **62** (2012), 2279–2287.

[6] L. HUANG and X. MO, Projectively flat Finsler metrics with orthogonal invariance, *Ann. Polon. Math.* **107** (2013), 259–270.

[7] L. Huang and X. Mo, On conformal fields of a Randers metric with isotropic S-curvature, *Illinois J. Math.* **57** (2013), 685–696.

[8] B. Li, On the classification of projectively flat Finsler metrics with constant flag curvature, *Adv. Math.* **257** (2014), 266–284.

[9] B. Li and Z. Shen, On Randers metrics of quadratic Riemann curvature, *Int. J. Math.* **20** (2009), 369–376.

[10] H. Liu and X. Mo, Examples of Finsler metrics with special curvature properties, *Math. Nachr.* **288** (2015), 1527–1537.

[11] X. Mo and L. Huang, On characterizations of Randers norms in a Minkowski space, *Int. J. Math.* **21** (2010), 523–535.

[12] X. Mo, N. M. Solorzano and K. Tenenblat, On spherically symmetric Finsler metrics with vanishing Douglas curvature, *Diff. Geom. Appl.* **31** (2013), 746–758.

[13] X. Mo and H. Zhu, On a class of projectively flat Finsler metrics of negative constant flag curvature, *Int. J. Math.* **23** (2012), 1250084, 14 pp.

[14] B. Najafi, Z. Shen and A. Tayebi, On a projective class of Finsler metrics, *Publ. Math. Debrecen* **70** (2007), 211–219.

[15] Y. Shen and Y. Yu, On projectively related Randers metrics, *Int. J. Math.* **19** (2008), 503–520.

[16] Z. Shen, On projectively related Einstein metrics in Riemann-Finsler geometry, *Math. Ann.* **320** (2001), 625–647.

[17] Y. Yu and Y. You, Projective equivalence between an (α, β)-metric and a Randers metric, *Publ. Math. Debrecen* **82** (2013), 155–162.

[18] L. Zhou, Projective spherically symmetric Finsler metrics with constant flag curvature in R^n, *Geom. Dedicata* **158** (2012), 353–364.

[19] L. Zhou, Spherically symmetric Finsler metrics in R^n, *Publ. Math. Debrecen* **80** (2012), 67–77.

[20] M. Zohrehvand and M.M. Rezaii, On projectively related of two special classes of (α, β)-metrics, *Diff. Geom. Appl.* **29** (2011), 660–669.

XIAOHUAN MO
KEY LABORATORY OF PURE
AND APPLIED MATHEMATICS
SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING 100871
CHINA

E-mail: moxh@pku.edu.cn

XIAOYANG WANG
SCHOOL OF MATHEMATICS AND STATISTICS
BEIJING INSTITUTE OF TECHNOLOGY
BEIJING 100081
CHINA

E-mail: wxy314159@126.com

(Received July 31, 2015; revised August 25, 2015)