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Topologies and orders on function spaces

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Introduction

By Y and Z we denote two fixed topological spaces and by $C(Y,Z)$ we denote the set of all continuous maps of Y into Z. If τ is a topology on the set $C(Y,Z)$, then the corresponding topological space is denoted by $C_\tau(Y,Z)$.

Let X be a space and $F : X \times Y \to Z$ be a continuous map. By F_x, where $x \in X$, we denote the continuous map of Y into Z, for which $F_x(y) = F(x,y)$, for every $y \in Y$. By $\widehat{F}$ we denote the map of X into the set $C(Y,Z)$, for which $\widehat{F}(x) = F_x$, for every $x \in X$.

Let G be a map of the space X into the set $C(Y,Z)$. By $\widetilde{G}$ we denote the map of the space $X \times Y$ into the space Z, for which $\widetilde{G}(x,y) = G(x)(y)$, for every $(x,y) \in X \times Y$. It is easy to verify that $\widehat{\widetilde{G}} = G$ and $\widetilde{\widehat{F}} = F$.

A topology τ on $C(Y,Z)$ is called *splitting* (respectively, *jointly continuous*) (see [1]) if and only if for every space X, the continuity of a map $F : X \times Y \to Z$ (respectively, a map $G : X \to C_\tau(Y,Z)$) implies that of the map $\widehat{F} : X \to C_\tau(Y,Z)$ (respectively, of the map $\widetilde{G} : X \times Y \to Z$). If this condition is satisfied for the elements of the family $\mathcal{A}$ of spaces, then the topology is called $\mathcal{A}$*–splitting* (respectively, $\mathcal{A}$*–jointly continuous*). (See [2]). If $\mathcal{A} = \{X\}$, then instead of "$\mathcal{A}$–splitting" and "$\mathcal{A}$–jointly continuous" we write "X–splitting" and "X–jointly continuous". The greatest $\mathcal{A}$–splitting topology, which always exists, is denoted by $\tau(\mathcal{A})$.

We recall some notions. (See, for example, [3]). For every space X with a topology τ we define a preorder "$\overset{\tau}{\leq}$" and an equivalence relation "$\overset{\tau}{\sim}$" on X as follows: if $x, y \in X$, then we write $x \overset{\tau}{\leq} y$ (respectively, $x \overset{\tau}{\sim} y$) if and only if $x \in \mathrm{Cl}_X(\{y\})$ (respectively, $x \in \mathrm{Cl}_X(\{y\})$ and $y \in \mathrm{Cl}_X(\{x\})$).

(By $\mathrm{Cl}_X(Q)$ we denote the closure of the set Q in the space X). It is easy to see that the preorder "$\overset{\tau}{\leq}$" on X is a partial order if and only if X is a T_0–space. Also, the points x and y are $\overset{\tau}{\sim}$–equivalent if and only if for every open subset U of X either $x, y \in U$ or $x, y \notin U$.

On the set $C(Y,Z)$ we define a preorder "$\leq$" and an equivalence relation "$\sim$" as follows: if $g, f \in C(Y,Z)$, then we write $g \leq f$ (respectively, $g \sim f$) if and only if $g(y) \overset{\tau}{\leq} f(y)$ (respectively, $g(y) \overset{\tau}{\sim} f(y)$), for every $y \in Y$, where τ is the topology of the space Z. Obviously, if Z is a T_0–space, then the preorder "$\leq$" on $C(Y,Z)$ is a partial order. Also, $g \sim f$ if and only if $g \leq f$ and $f \leq g$.

If X is a set equipped with a preorder "$\leq$", then we set $[y, \rightarrow)_{\leq} = \{x \in X : y \leq x\}$ and $(\leftarrow, y]_{\leq} = \{x \in X : x \leq y\}$.

Let $\mathcal{U}$ ba a quasi-uniformity on the space Z. (See, for example, [8]). This quasi-uniformity defines on the set $C(Y,Z)$ a quasi-uniformity $\mathcal{Q} \equiv \mathcal{Q}(\mathcal{U})$ as follows (see [5]): the set of all subsets of $C(Y,Z)$ of the form

$$(Y,U) \equiv \{(f,g) \in C(Y,Z) \times C(Y,Z) : (f(y), g(y)) \in U, \text{ for every } y \in Y\},$$

where $U \in \mathcal{U}$, is a basis for the quasi-uniformity $\mathcal{Q}$. We denote by $\tau_{\mathcal{Q}}$ the topology on $C(Y,Z)$, which is defined by the quasi-uniformity $\mathcal{Q}$ and we say that $\tau_{\mathcal{Q}}$ is *generated* by the quasi-uniformity $\mathcal{U}$ on the space Z.

By $\boldsymbol{S}$ we denote the Sierpinski space, that is, the set $\{0,1\}$ equipped with the topology $\tau(\boldsymbol{S}) \equiv \{\emptyset, \{0,1\}, \{1\}\}$, and by $\boldsymbol{D}$ the set $\{0,1\}$ with the trivial topology.

In the present paper we study the connections of the natural preorder "$\leq$" and equivalence relation "$\sim$" on the set $C(Y,Z)$ with the notions of X–splitting and X–jointly continuous topologies on this set, where X is either the space $\boldsymbol{S}$ or the space $\boldsymbol{D}$.

1. Theorem. *A topology τ on $C(Y,Z)$ is $\boldsymbol{S}$–splitting if and only if from the condition $g \leq f$ it follows that $g \overset{\tau}{\leq} f$.*

Proof. Let τ be an $\boldsymbol{S}$–splitting topology on $C(Y,Z)$ and let $g \leq f$, where $g, f \in C(Y,Z)$. We prove that $g \overset{\tau}{\leq} f$.

Let $F : \boldsymbol{S} \times Y \rightarrow Z$ be a map for which $F(1,y) = f(y)$ and $F(0,y) = g(y)$, $y \in Y$. We prove that F is continuous. Let $F(1,y) = f(y)$ and let U be an open neighbourhood of $f(y)$ in Z. Since f is continuous, the set $f^{-1}(U)$ is open in Y. The set $V = \{1\} \times f^{-1}(U)$ is an open neighbourhood of $(1,y)$ in $\boldsymbol{S} \times Y$ and $F(V) \subseteq U$, which means that F is continuous at the point $(1,y) \in \boldsymbol{S} \times Y$.

Let $F(0,y) = g(y)$ and U be an open neighbourhood of $g(y)$ in Z. The set $V = \boldsymbol{S} \times g^{-1}(U)$ is an open neighbourhood of $(0,y)$ in $\boldsymbol{S} \times Y$. We prove that $F(V) \subseteq U$. Indeed, if $(0,y_1) \in V$, then $F(0,y_1) = g(y_1) \in U$.

If $(1, y) \in V$, then $F(1, y_1) = f(y_1)$. Since $g \leq f$ and $g(y_1) \in U$ we have that $f(y_1) \in U$. Hence, $F(V) \subseteq U$ and, therefore, F is continuous.

Since τ is $\boldsymbol{S}$–splitting the map $\widehat{F} : \boldsymbol{S} \to C_\tau(Y, Z)$ is continuous. We have that $\widehat{F}(1) = f$ and $\widehat{F}(0) = g$. Let W be an open neighbourhood of g in $C_\tau(Y, Z)$. Then, $\widehat{F}^{-1}(W)$ is an open neighbourhood of $0 \in \boldsymbol{S}$. Since $0 \in \mathrm{Cl}_{\boldsymbol{S}}(\{1\})$ we have that $1 \in \widehat{F}^{-1}(W)$, that is, $\widehat{F}(1) = f \in W$. This means that $g \in \mathrm{Cl}_{C_\tau(Y,Z)}(\{f\})$. Hence $g \overset{\tau}{\leq} f$.

Conversely, let τ be a topology on $C(Y, Z)$ such that from the condition $g \leq f$ it follows that $g \overset{\tau}{\leq} f$. We prove that τ is $\boldsymbol{S}$–splitting. Let $F : \boldsymbol{S} \times Y \to Z$ be a continuous map. Consider the map $\widehat{F} : \boldsymbol{S} \to C_\tau(Y, Z)$. Let $\widehat{F}(1) = f$ and $\widehat{F}(0) = g$. We prove that $g \leq f$. Indeed, let $y \in Y$ and let U be an open neighbourhood of $g(y)$ in Z. Since F is continuous, the set $F^{-1}(U)$ is an open neighbourhood of $(0, y)$ in $\boldsymbol{S} \times Y$. It is easy to see that every open neighborhood of $(0, y)$ contains the point $(1, y)$. Hence, $(1, y) \in F^{-1}(U)$, that is, $f(y) \in U$. Therefore, $g \leq f$. By the assumption, $g \overset{\tau}{\leq} f$.

Let U be an open subset of $C_\tau(Y, Z)$. If $f, g \in U$ or $f, g \notin U$, then $\widehat{F}^{-1}(U)$ is open in $\boldsymbol{S}$. If $f \in U$ and $g \notin U$, then $\widehat{F}^{-1}U = \{1\} \in \tau(\boldsymbol{S})$. Let $g \in U$. Since $g \overset{\tau}{\leq} f$ we have that $f \in U$ and, hence, the set $\widehat{F}^{-1}(U)$ is open in $\boldsymbol{S}$. Thus, $\widehat{F}$ is continuous and, therefore, the topology τ is $\boldsymbol{S}$–splitting.

2. Corollary. *The discrete topology and, hence, every topology on $C(Y, Z)$ is $\boldsymbol{S}$–splitting if and only if the space Z is a T_1–space.*

Proof. If Z is a T_1–space, then by the condition $g \leq f$, where $g, f \in C(Y, Z)$, it follows that $g = f$. Hence, $g \overset{\tau}{\leq} f$, for every topology τ on $C(Y, Z)$. Therefore, by Theorem 1, every topology on $C(Y, Z)$ is $\boldsymbol{S}$–splitting.

Conversely, suppose that every topology on $C(Y, Z)$ is $\boldsymbol{S}$–splitting. If Z is not a T_1–space, then there exist points $x, y \in Z$, $x \neq y$, such that $x \leq y$. Let $f, g \in C(Y, Z)$ such that $g(Y) = \{x\}$ and $f(Y) = \{y\}$. Then, $g \leq f$ and $g \neq f$. By Theorem 1, $g \overset{\tau}{\leq} f$, for every topology τ on $C(Y, Z)$. If τ is the discrete topology, then $g = f$, which is a contradiction. Hence, Z is a T_1–space.

3. Theorem. *A topology τ on $C(Y, Z)$ is $\boldsymbol{S}$–jointly continuous if and only if from the condition $g \overset{\tau}{\leq} f$ it follows that $g \leq f$.*

Proof. Let τ be an $\boldsymbol{S}$–jointly continuous topology on $C(Y, Z)$ and let $g \overset{\tau}{\leq} f$, where $g, f \in C(Y, Z)$. We prove that $g \leq f$.

Let $G : \boldsymbol{S} \to C(Y,Z)$ be a map for which $G(0) = f$ and $G(1) = g$. We prove that G is continuous. Let U be an open subset of $C(Y,Z)$. If $g \in U$, then since $g \overset{\tau}{\leq} f$ we have that $f \in U$ and, hence, the set $G^{-1}(U) = \boldsymbol{S}$ is open. Also, if $g \notin U$ and $f \notin U$, then the set $G^{-1}(U) = \emptyset$ is open. Hence, the map G is continuous. Since τ is $\boldsymbol{S}$–jointly continuous, the map $\widetilde{G} : \boldsymbol{S} \times Y \to Z$ is also continuous.

Let $y \in Y$ and let W be an open neighbourhood of $g(y)$ is Z. Then, $\widetilde{G}^{-1}(W)$ is an open subset of $\boldsymbol{S} \times Y$ containing the point $(0,y)$. There exist an open neighbourhood V_1 of 0 in $\boldsymbol{S}$ and an open neighbourhood V_2 of y in Y such that $V_1 \times V_2 \subseteq \widetilde{G}^{-1}(W)$. Since $1 \in V_1$ we have that $(1,y) \in \widetilde{G}^{-1}(W)$, which means that $F(1,y) = f(y) \in W$. Thus, $g(y) \in \mathrm{Cl}_Z(\{f(y)\})$. Hence, $g \leq f$.

Conversely, let τ be a topology on $C(Y,Z)$ such that from the condition $g \overset{\tau}{\leq} f$ it follows that $g \leq f$. We prove that τ is $\boldsymbol{S}$–jointly continuous. Let $G : \boldsymbol{S} \to C(Y,Z)$ be a continuous map and let $G(1) = f$ and $G(0) = g$. We prove that $g \overset{\tau}{\leq} f$. Indeed, let U be an open neighbourhood of g in $C(Y,Z)$. Since G is continuous, the set $G^{-1}(U)$ is an open subset of $\boldsymbol{S}$ containing the point 0. Hence, $1 \in G^{-1}(U)$ and, therefore, $G(1) = f \in U$, which means that $g \overset{\tau}{\leq} f$.

Consider the map $\widetilde{G} : \boldsymbol{S} \times Y \to Z$. Let W be an open subset of Z and let $(1,y) \in \widetilde{G}^{-1}(W)$. Then, the set $\{1\} \times f^{-1}(W)$ is an open subset of $\boldsymbol{S} \times Y$ containing the point $(1,y)$ such that $\widetilde{G}(\{1\} \times f^{-1}(W)) \subseteq W$.

Let $(0,y) \in \widetilde{G}^{-1}(W)$. Consider the open set $\boldsymbol{S} \times g^{-1}(W)$ of $\boldsymbol{S} \times Y$. Obviously, $(0,y) \in \boldsymbol{S} \times g^{-1}(W)$. We prove that $\widetilde{G}(\boldsymbol{S} \times g^{-1}(W)) \subseteq W$. Let $(0,y_1) \in \boldsymbol{S} \times g^{-1}(W)$. Then $\widetilde{G}(0,y_1) = g(y_1) \in W$. Let $(1,y_2) \in \boldsymbol{S} \times g^{-1}(W)$. Then, $y_2 \in g^{-1}(W)$, that is, $g(y_2) \in W$. By the assumption, $f(y_2) \in W$ and, hence, $\widetilde{G}(1,y_2) = f(y_2) \in W$. Thus, the map $\widetilde{G}$ is continuous. Hence, τ is an $\boldsymbol{S}$–jointly continuous topology.

4. Corollary. *The trivial topology and, hence, every topology on the set $C(Y,Z)$ is $\boldsymbol{S}$–jointly continuous if and only if the topology of Z is trivial.*

Proof. Suppose that the topology of Z is trivial. Then, $f \leq g$ for every $f, g \in C(Y,Z)$. By Theorem 3, it follows that every topology τ on $C(Y,Z)$ is $\boldsymbol{S}$–jointly continuous.

Conversely, suppose that every topology on $C(Y,Z)$ is $\boldsymbol{S}$–jointly continuous. Let τ be the trivial topology on $C(Y,Z)$. Then, for every $f, g \in C(Y,Z)$ we have $g \overset{\tau}{\leq} f$. If the topology of Z is not trivial, then there exist points $x, y \in Z$, $x \neq y$, such that $y \notin \mathrm{Cl}_Z(\{x\})$. Let $f, g \in C(Y,Z)$

such that $f(Y) = \{x\}$ and $g(Y) = \{y\}$. Then $g \not\leq f$, which by Theorem 3 is a contradiction.

5. Corollary. *A topology τ on $C(Y,Z)$ is simultaneously $\boldsymbol{S}$–splitting and $\boldsymbol{S}$–jointly continuous if and only if the preorders "$\overset{\tau}{\leq}$" and "$\leq$" coincide.*

6. Theorem. *The subsets of $C(Y,Z)$ of the form $[g,\rightarrow)_{\leq}$, $g \in C(Y,Z)$, compose a basis for open sets in the space $C_{\tau(\{\boldsymbol{S}\})}(Y,Z)$.*

Proof. Let τ be a topology on $C(Y,Z)$, for which the sets of the form $[g,\rightarrow)_{\leq}$, $g \in C(Y,Z)$, compose a subbasis for open sets. Obviously, if $f \in [g,\rightarrow)_{\leq}$, then $[f,\rightarrow)_{\leq} \subseteq [g,\rightarrow)_{\leq}$. Hence, if $g \in U \in \tau$, then $[g,\rightarrow)_{\leq} \subseteq U$. Thus, if $g \leq f$, then $g \overset{\tau}{\leq} f$. Hence, by Theorem 1, τ is an $\boldsymbol{S}$–splitting topology. Therefore, $\tau \subseteq \tau(\{\boldsymbol{S}\})$.

Let $U \in (\{\boldsymbol{S}\})$ and $g \in U$. By Theorem 1, $[g,\rightarrow)_{\leq} \subseteq U$ and, hence, $U \in \tau$. This means that $\tau(\{\boldsymbol{S}\}) \subseteq \tau$ and the sets of the form $[g,\rightarrow)_{\leq}$, $g \in C(Y,Z)$, compose a basis for the topology $\tau(\{\boldsymbol{S}\})$.

7. Corollary. *The set of all subsets of $C(Y,Z)$ of the form $\{f \in C(Y,Z) : g^{-1}(U) \subseteq f^{-1}(U)\}$, for every $U \in \mathcal{O}(Z)\}$, $g \in C(Y,Z)$, is a basis for the greatest $\boldsymbol{S}$–splitting topology on $C(Y,Z)$.*

The proof of this corollary follows by the relation $[g,\rightarrow)_{\leq} = \{f \in C(Y,Z) : g^{-1}(U) \subseteq f^{-1}(U)\}$, for every $U \in \mathcal{O}(Z)\}$.

8. Corollary. *The greatest $\boldsymbol{S}$–splitting topology $\tau(\{\boldsymbol{S}\})$ on $C(Y,Z)$ has the following property: the intersection of any family of open sets is open, that is, every element f of $C(Y,Z)$ has a smallest open neighbourhood in the space $C_{\tau(\{\boldsymbol{S}\})}(Y,Z)$.*

9. Theorem. *The greatest $\boldsymbol{S}$–splitting topology is $\boldsymbol{S}$–jointly continuous.*

Proof. Let $f, g \in C(Y,Z)$ and $g \overset{\tau(\{\boldsymbol{S}\})}{\leq} f$. Since $[g,\rightarrow)_{\leq}$ is an open neighbourhood of g in $C_{\tau(\{\boldsymbol{S}\})}(Y,Z)$ (see Theorem 6) we have that $f \in [g,\rightarrow)_{\leq}$, that is, $g \leq f$. By Theorem 3, $\tau(\{\boldsymbol{S}\})$ is $\boldsymbol{S}$–jointly continuous.

10. Theorem. *In the set of all simultaneously $\boldsymbol{S}$–splitting and $\boldsymbol{S}$–jointly continuous topologies on $C(Y,Z)$ there exists a smallest topology denoted by $\tau_{\min}(\{\boldsymbol{S}\})$. Moreover, the set $C(Y,Z)$ and the subsets of $C(Y,Z)$ of the form $C(Y,Z) \setminus (\leftarrow, g]_{\leq}$, $g \in C(Y,Z)$, compose a subbasis for this topology.*

Proof. Let τ be a topology on the set $C(Y,Z)$, for which the set $C(Y,Z)$ and the subsets of $C(Y,Z)$ of the form $C(Y,Z) \setminus (\leftarrow, g]_{\leq}$, $g \in C(Y,Z)$, compose a subbasis. We prove that τ is $\boldsymbol{S}$–splitting. Indeed, let

$g \leq f$, that is, $f \in [g, \rightarrow)_{\leq}$ and let $g \in U \in \tau$. If $U = C(Y,Z)$, then $f \in U$. If $U \neq C(Y,Z)$, then there exist elements $g_1, \ldots, g_k \in C(Y,Z)$ such that

$$g \in \bigcap\{C(Y,Z) \setminus (\leftarrow, g_i]_{\leq} : i \in \{1,\ldots,k\}\} \subseteq U\,.$$

Hence,

$$g \notin \bigcup\{(\leftarrow, g_i]_{\leq} : i \in \{1,\ldots,k\}\}\,.$$

Therefore,

$$\begin{aligned}[g,\rightarrow)_{\leq} &\subseteq C(Y,Z) \setminus (\bigcup\{(\leftarrow, g_i]_{\leq} : i \in \{1,\ldots,k\}\}) \\ &= \bigcap\{C(Y,Z) \setminus (\leftarrow, g_i]_{\leq} : i \in \{1,\ldots,k\}\} \subseteq U\,.\end{aligned}$$

Therefore, $f \in U$, that is $g \overset{\tau}{\leq} f$. By Theorem 1, τ is $\boldsymbol{S}$–splitting.

We prove that τ is $\boldsymbol{S}$–jointly continuous. Let $g \overset{\tau}{\leq} f$. By Theorem 3 it is sufficient to prove that $g \leq f$, that is, $f \in [g,\rightarrow)_{\leq}$. If $f \notin [g,\rightarrow)_{\leq}$, then the set $C(Y,Z) \setminus (\leftarrow, f]_{\leq}$ is an open neighbourhood of g, which does not contain the element f. Since $g \overset{\tau}{\leq} f$, this is a contradiction. Hence, $g \leq f$.

Now we prove that τ is the smallest $\boldsymbol{S}$–splitting and $\boldsymbol{S}$–jointly continuous topology on $C(Y,Z)$, that is $\tau = \tau_{\min}(\{\boldsymbol{S}\})$. Let τ' be an $\boldsymbol{S}$–splitting and $\boldsymbol{S}$–jointly continuous toplogy on $C(Y,Z)$. We prove that $\tau \subseteq \tau'$. Let $g, f \in C(Y,Z)$ and $f \in C(Y,Z) \setminus (\leftarrow, g]_{\leq}$. It is sufficient to prove that there exists an element $V \in \tau'$ such that

$$f \in V \subseteq C(Y,Z) \setminus (\leftarrow, g]_{\leq}\,.$$

We have that $f \notin (\leftarrow, g]_{\leq}$, that is, $f \not\leq g$. Since τ' is $\boldsymbol{S}$–jointly continuous, by Theorem 3 it follows that $f \overset{\tau'}{\not\leq} g$. Therefore, there exists an element $V \in \tau'$ such that $f \in V$ and $g \notin V$. Since τ' is $\boldsymbol{S}$–splitting, by Theorem 1 we have that $(\leftarrow, g]_{\leq} \cap V = \emptyset$. Hence,

$$f \in V \subseteq C(Y,Z) \setminus (\leftarrow, g]_{\leq}\,.$$

11. Theorem. *The pointwise topology on $C(Y,Z)$ is $\boldsymbol{S}$–jointly continuous.*

Proof. Let τ_p be the pointwise topology on $C(Y,Z)$. Then, the sets of the form

$$(y,U) = \{f \in C(Y,Z) : f(y) \in U\}\,,$$

where $y \in Y$ and U is an open subset of Z, compose a subbasis for τ_p.

Let $f \overset{\tau_p}{\leq} g$, $y \in Y$ and U be an open neighbourhood of $f(y)$ in Z. Then, $f \in (y,U)$. Since $f \overset{\tau_p}{\leq} g$, we have that $g \in (y,U)$, that is, $g(y) \in U$.

This means that $f(y) \in \mathrm{Cl}_Z(\{g(y)\})$ for every $y \in Y$, that is, $f \leq g$. By Theorem 3, τ_p is $\boldsymbol{S}$–jointly continuous.

12. Corollary. *The compact-open topology and the Isbell topology (see, for example,* [4]*) on $C(Y,Z)$ $\boldsymbol{S}$–jointly continuous.*

13. Corollary. *For the compact-open topology and Isbell topology the preorders "$\overset{\tau}{\leq}$" and "$\leq$" in $C(Y,Z)$ coincide.*

14. Remark. Propositions 3.6 of [6] and 3.2 of [7] follow immediately by Corollary 13.

15. Theorem. *For every $f \in C(Y,Z)$, the intersection of all neighbourhoods of f in the space $C_{\tau_{\min(\{\boldsymbol{S}\})}}(Y,Z)$ is the smallest open neighbourhood of f in the space $C_{\tau(\{\boldsymbol{S}\})}(Y,Z)$.*

Proof. Let $f \in U \in \tau_{\min(\{\boldsymbol{S}\})}$. We prove that $[f,\rightarrow)_{\leq} \subseteq U$. It is sufficient to suppose that $U = C(Y,Z) \setminus (\leftarrow, g]_{\leq}$, for some $g \in C(Y,Z)$. Let $h \in [f,\rightarrow)_{\leq}$. Since $f \in U$, we have that $f \notin (\leftarrow, g]_{\leq}$. Hence, $h \notin (\leftarrow, g]_{\leq}$ and, therefore $h \in U$.

For the proof of the theorem, it is sufficient to prove that if $h \notin [f,\rightarrow)_{\leq}$, then there exists an element V of $\tau_{\min(\{\boldsymbol{S}\})}$ such that $f \in V$ and $h \notin V$. Obviously, the set $V = C(Y,Z) \setminus (\leftarrow, h]_{\leq}$ is the required open set.

16. Theorem. *Every topology on the set $C(Y,Z)$, which is generated by a quasi-uniformity on the space Z is $\boldsymbol{S}$–splitting and $\boldsymbol{S}$–jointly continuous.*

Proof. Let $\mathcal{U}$ be a quasi-uniformity on the space Z and let $\tau_{\mathcal{Q}}$ be the corresponding topology on the set $C(Y,Z)$. Since $\tau_{\mathcal{Q}}$ is jointly continuous (see [5]), this topology is also $\boldsymbol{S}$–jointly continuous.

We prove that $\tau_{\mathcal{Q}}$ is $\boldsymbol{S}$–splitting. Let $g, f \in C(Y,Z)$ and $g \leq f$. Let H be a neighbourhood of g in the space $C_{\tau_{\mathcal{Q}}}(Y,Z)$. We can suppose that

$$H \equiv (Y,U)_{(g)} \equiv \{h \in C(Y,Z) : (g,h) \in (Y,U)\},$$

where U is an element of $\mathcal{U}$. We prove that $f \in H$, that is, $(g,f) \in (Y,U)$ or $(g(y), f(y)) \in U$, for every $y \in Y$. Let $y \in Y$. Since $g \leq f$ we have that $g(y) \in \mathrm{Cl}_Z(\{f(y)\})$, that is, the point $f(y)$ belongs to any neighbourhood of $g(y)$. Hence, $f(y)$ belongs to the set

$$U_{(g(y))} \equiv \{z \in Z : (g(y)), z) \in U\}.$$

Thus, $(g(y), f(y)) \in U$. Therefore, $g \overset{\tau_{\mathcal{Q}}}{\leq} f$. By Theorem 1, the topology $\tau_{\mathcal{Q}}$ is $\boldsymbol{S}$–splitting.

17. Example. Let Z be a set equipped with a preorder "$\leq$". By $\tau(\leq)$ we denote the topology on Z, for which the sets of the form $[z, \rightarrow)_{\leq}$, $z \in Z$, compose a subbasis.

Let Y be a space, for which every continuous map of Y into Z is constant and let a be a fixed point of Y. Obviously, $f \leq g$, where $f, g \in C(Y,Z)$, if and only if $f(a) \leq g(a)$. Identifying every element f of $C(Y,Z)$ with the element $f(a)$ of Z, every topology on the set Z can be considered as a topology on $C(Y,Z)$. In particular, on the set $C(Y,Z)$ we can consider the topology $\tau(\leq)$. By Theorem 6, the topology $\tau(\leq)$ on $C(Y,Z)$ is the greatest $\boldsymbol{S}$–splitting topology.

Let $\tau_{\min}(\leq)$ be a topology on Z, for which the set Z and the subsets of Z of the form $Z \setminus (\leftarrow, z]_{\leq}$, $z \in Z$, compose a subbasis. By Theorem 10, it follows that the topology $\tau_{\min}(\leq)$ on $C(Y,Z)$ is the smallest $\boldsymbol{S}$–splitting and $\boldsymbol{S}$–jointly continuous topology on $C(Y,Z)$.

18. Example. Let Z be the set of real numbers with the usual order "$\leq$". Then the sets of the form $[a, \infty)$, $a \in Z$ compose a basis of the topology $\tau(\leq)$ and the sets of the form (a, ∞), $a \in Z$, compose a basis of the topology $\tau_{\min}(\leq)$. It is easy to see that $\tau_{\min}(\leq) \subseteq \tau(\leq)$ and $\tau_{\min}(\leq) \neq \tau(\leq)$.

Let τ be a topology on the set Z for which the sets of the form $Z \setminus [a, b]$, where $a, b \in Z$, $a \leq b$, compose a subbasis. It is easy to see that $\tau \not\subseteq \tau(\leq)$, that is, the topology τ on $C(Y,Z)$ is not $\boldsymbol{S}$–splitting.

On the other hand, the topology τ on $C(Y,Z)$ satisfies the following condition: if $f \overset{\tau}{\leq} g$, then $f = g$. By Theorem 3 it follows that τ is $\boldsymbol{S}$–jointly continuous. Obviously, $\tau_{\min}(\leq) \not\subseteq \tau$, which means that on the set $C(Y,Z)$ there is no smallest $\boldsymbol{S}$–jointly continuous topology.

19. Example. Let $Z = \boldsymbol{S}$. Consider the set $\mathcal{O}(Y)$ of all open subsets of Y, the set $\mathcal{K}(Y)$ of all closed subsets of Y and the set $C(Y, \boldsymbol{S})$. If we identify every element U of $\mathcal{O}(Y)$ with the element $Y \setminus U$ of $\mathcal{K}(Y)$ and with the element f of $C(Y, \boldsymbol{S})$, for which $f(U) \subseteq \{1\}$ and $f(Y \setminus U) \subseteq \{0\}$, then for every topology on one of the above sets we can consider the corresponding topology on the other sets. In particular, on the sets $\mathcal{O}(Y)$, $\mathcal{K}(Y)$ and $C(Y, \boldsymbol{S})$ we can consider the Scott topology (see [3]), the Vietoris topology (see, for example, [M]), an $\mathcal{A}$–splitting topology, for some family $\mathcal{A}$ of spaces, etc. Also, the preorder "$\leq$", which is defined on the set $C(Y, \boldsymbol{S})$, can be considered on the sets $\mathcal{K}(Y)$ and $\mathcal{O}(Y)$. It is easy to prove that if $K, F \in \mathcal{K}(Y)$, then $K \leq F$ if and only if $F \subseteq K$. Thus, by Theorems 6 and 10 we have the following corollaries:

20. Corollary. *The sets of the from*

$$\mathcal{K}(F) \equiv \{K \in \mathcal{K}(Y) : K \subseteq F\}\,,$$

$F \in \mathcal{K}(Y)$, *compose a basis for the greatest* $\boldsymbol{S}$*–splitting topology on* $\mathcal{K}(Y)$.

21. Corollary. *The set* $\mathcal{K}(Y)$ *and the sets of the form*

$$\mathcal{K}_{\min}(F) \equiv \{K \in \mathcal{K}(Y) : F \nsubseteq K\}\,,$$

$F \in \mathcal{K}(Y)$, *compose a subbasis of open sets for the smallest* $\boldsymbol{S}$*–splitting and* $\boldsymbol{S}$*–jointly continuous topology on* $\mathcal{K}(Y)$.

22. Remark. We observe that the sets of the form

$$\mathcal{K}(Y) \setminus \mathcal{K}(F), \quad F \in \mathcal{K}(Y)\,,$$

and the sets of the form

$$\mathcal{K}_0(F) \equiv \{K \in \mathcal{K}(Y) : K \cap F = \emptyset\}\,,$$

$F \in \mathcal{K}(Y)$, compose a subbasis of the Vietoris topology on $\mathcal{K}(Y)$. Since the element $\emptyset \in \mathcal{K}(Y)$ belongs to any open set of the greatest $\boldsymbol{S}$–splitting topology on $\mathcal{K}(Y)$ while this element is an isolated element of $\mathcal{K}(Y)$ with Vietoris topology, we have that if $Y \neq \emptyset$, then the Vietoris topology is not $\boldsymbol{S}$–splitting.

The proofs of the following results concerning the space $\boldsymbol{D}$ are similar to the corresponding results concerning the space $\boldsymbol{S}$ and we omit them.

23. Theorem. *The following are true:*

(1) A topology τ *on* $C(Y,Z)$ *is* $\boldsymbol{D}$*–splitting if and only if by the relation* $f \sim g$, *where* $f, g \in C(Y,Z)$, *it follows that* $f \overset{\tau}{\sim} g$.

(2) A topology τ *on* $C(Y,Z)$ *is* $\boldsymbol{D}$*–jointly continuous if and only if by the relation* $f \overset{\tau}{\sim} g$, *where* $f, g \in C(Y,Z)$, *it follows that* $f \sim g$.

(3) The set whose elements are the equivalence classes of the relation "$\sim$*" on the set* $C(Y,Z)$, *composes a basis for the greatest* $\boldsymbol{D}$*–splitting topology* $\tau(\{\boldsymbol{D}\})$ *on* $C(Y,Z)$.

(4) The greatest $\boldsymbol{D}$*–splitting topology is* $\boldsymbol{D}$*–splitting continuous.*

(5) In the set of all simaltaneously $\boldsymbol{D}$*–splitting and* $\boldsymbol{D}$*–jointly continuous topologies on* $C(Y,Z)$ *there exists a smallest topology denoted by* $\tau_{\min}(\{\boldsymbol{D}\})$. *Moreover, the set* $C(Y,Z)$ *and the subsets of* $C(Y,Z)$ *of the form* $C(Y,Z) \setminus E$, *where* E *is an equivalence class of the relation "*$\sim$*" on the set* $C(Y,Z)$, *compose a subbasis for this topology.*

(6) Every $\boldsymbol{S}$*–splitting (respectively,* $\boldsymbol{S}$*–jointly continuous) topology on* $C(Y,Z)$ *is* $\boldsymbol{D}$*–splitting (respectively,* $\boldsymbol{D}$*–jointly continuous).*

24. Corollary. *(1) The discrete topology and, hence, every topology on the set $C(Y,Z)$ is $\boldsymbol{D}$–splitting if and only if the space Z is a T_0–space.*

(2) The trivial topology and, hence, every topology on the set $C(Y,Z)$ is $\boldsymbol{D}$–jointly continuous if and only if the topology of Z is trivial.

(3) A topology τ on $C(Y,Z)$ is simaltaneously $\boldsymbol{D}$–splitting and $\boldsymbol{D}$–jointly continuous if and only if the relations "$\overset{\tau}{\sim}$" and "$\sim$" coincide.

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