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On a result of Shallit and Pethő

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In this paper, by using matrix methods we give a general form of some results given by SHALLIT [2], [3] and PETHŐ [1].

1. Introduction

Let $B(u,\infty) = \sum\limits_{u=0}^{\infty} \frac{1}{u^{2^k}}$, $u \geq 3$, $u \in \mathbb{Z}$ then it is well-known that $B(u,\infty)$ is a transcendental number (see [5], p. 35). In 1979 SHALLIT [2] proved that if $B(u,v) = \sum\limits_{u=0}^{v} \frac{1}{u^{2^k}}$, $u \geq 3$, $u \in \mathbb{Z}$ then $B(u,0) = [0; u]$, $B(u,1) = [0; u-1, u+1]$ and if $B[u,v] = [a_0; a_1, \ldots, a_n] = p_n/q_n$ then $B[u, v+1) = [a_0; a_1, a_2, \ldots, a_{n-1}, a_n + 1, a_n - 1, a_{n-1}, \ldots, a_1]$ for $v \geq 2$.

Moreover in 1982 he examined [3] the numbers of the form: $S(u,\infty) = \sum\limits_{k=0}^{\infty} u^{-c(k)}$, where $\{c(k)\}_{k=0}^{\infty}$ is a sequence of positive integers such that $c(v+1) \geq 2c(v)$ for all $v \geq v'$, where v' is a non-negative integer. From this result follows some continued fraction expansion for the Liouville transcendental numbers $L(m) = \sum\limits_{k=0}^{\infty} m^{-(k+1)!}$, $m \geq 2$, $m \in \mathbb{Z}$.

In 1982 PETHŐ [1] considered the series: $\sum\limits_{i=1}^{\infty} \frac{d_i}{Q_i}$, where $d_1 = 1, d_i = \pm 1$ for $i = 1, 2, 3, \ldots$, and $Q_i = a_{i-1} Q_{i-1}^k$, $k \geq 2$ for $i = 2, 3, \ldots$ and $a_1 \geq 2$, $Q_1 = 1$. For $C_k(a,u) = \sum\limits_{i=0}^{u} \frac{d_i}{Q_i}$ he proved a general theorem (see [1], Thm p. 235) concerning continued fraction expansion. From this theorem follows in the case $k = 2$ a result of KMOŠEK reported by SCHINZEL in 1979 at the Oberwolfach meeting: "Diophantische Approximationen". In 1986

Trung Wu [4] applying matrix methods gave new proofs of the results given by Shallit.

In the present paper by application of matrix methods we give a general form of the results given by Shallit and Pethő. Namely we prove the following

Theorem. *Let* $S(\infty) = \sum_{i=0}^{\infty} \frac{m_i}{n_i} < \infty$, *where* $\langle m_i, n_i \rangle \in \mathbb{Z}^2$ *and let* $S(u) = \sum_{i=0}^{u} \frac{m_i}{n_i} = \frac{M_u}{N_u}$. *If* $S(u) = [A_0; A_1, \dots A_n] = p_n/q_n$ *and* $M_u = p_n$, $N_u = q_n = n_u$, $n_{u+1} = sn_u^2$, $m_{u+1} = (-1)^u$ *for some* $s \geq 1$, *then*

$$S(u+1) = [A_0, A_1, \dots, A_n, s-1, A_{n-1}+1, A_{n-1}, \dots, A_1], \quad \text{if } A_n = 1,\ s \neq 1, \tag{1}$$

$$S(u+1) = [A_0, A_1, \dots, A_n+1, \dots, A_1], \quad \text{if } A_n = 1,\ s = 1, \tag{2}$$

$$S(u+1) = [A_0, A_1, \dots, A_{n-2}, A_{n-1}A_{n-1}+2, A_{n-2}, \dots, A_1], \quad \text{if } A_n = s = 1, \tag{3}$$

$$S(u+1) = [A_0, A_1, \dots, A_n, s-1, 1, A_{n-1}, \dots, A_1], \quad \text{if } A_n \neq 1,\ s \neq 1\,. \tag{4}$$

2. Basic lemmas

Lemma 1. *Let* $M_2(K)$ *denote the set of all* 2×2 *matrices with elements in* K. *Then for* $A_i \in M_2(K)$

$$(A_1 \cdot A_2 \cdot \ldots \cdot A_r)^T = A_r^T \cdot A_{r-1}^T \cdot \ldots \cdot A_1^T \tag{5}$$

where A^T *denotes the transpose to* A.

Lemma 2. *If* $p_n/q_n = [a_0; a_1, \dots, a_n]$ *then* $[a_n; a_{n-1}, \dots, a_1] = q_n/q_{n-1}$.

Proof. We use the following well-known identity:

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \ldots \cdot \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \tag{6}$$

where $p_n/q_n = [a_0; a_1, \dots, a_n]$. Since

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -a_0 \end{bmatrix} \qquad \text{and} \tag{7}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & -a_0 \end{bmatrix} \cdot \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} q_n & q_{n-1} \\ p_n - a_0 q_n & p_{n-1} - a_0 q_{n-1} \end{bmatrix}, \tag{8}$$

by (6), (7) and (8) we obtain

$$\begin{bmatrix} q_n & q_{n-1} \\ p_n - a_0 q_n & p_{n-1} - a_0 q_{n-1} \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \ldots \cdot \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}. \tag{9}$$

From (9) and Lemma 1 we obtain

$$\begin{bmatrix} q_n & p_n - a_0 q_n \\ q_{n-1} & p_{n-1} - a_0 q_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}^T \cdot \ldots \cdot \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}^T. \tag{10}$$

Since $\begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$, by (10) it follows that $q_n/q_{n-1} = [a_n; a_{n-1}, \dots, a_1]$ and the proof is finished.

Lemma 3. *Let $p_n/q_n = [a_0; a_1, \dots a_n]$ and $r_m/s_m = [b_0; b_1, \dots, b_m]$, then*

$$\frac{p_{n-1} \cdot s_m + p_n \cdot r_m}{q_{n-1} \cdot s_m + q_n \cdot r_m} = [a_0; a_1, \dots, a_n, b_0, b_1, \dots, b_m].$$

Proof. From the assumptions we have

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \ldots \cdot \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} r_m & r_{m-1} \\ s_m & s_{m-1} \end{bmatrix} = \begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \ldots \cdot \begin{bmatrix} b_{m-1} & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} b_m & 1 \\ 1 & 0 \end{bmatrix}. \tag{12}$$

Since

$$\begin{aligned} \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} \cdot \begin{bmatrix} r_m & r_{m-1} \\ s_m & s_{m-1} \end{bmatrix} &= \\ &= \begin{bmatrix} p_n r_m + p_{n-1} s_m & p_n r_{m-1} + p_{n-1} s_{m-1} \\ q_n r_m + q_{n-1} s_m & q_n r_{m-1} + q_{n-1} s_{m-1} \end{bmatrix}, \end{aligned} \tag{13}$$

by (11), (12) and (13) Lemma 3 follows.

Lemma 4. (see [1], Lemma, p. 234). *Let* $s \geq 1$ *be an integer and* $p_n/q_n = [b_0; b_1, \ldots, b_n]$, *then*

$$[b_0; b_1, \ldots, b_n, s-1, 1, b_n - 1, b_{n-1}, \ldots, b_1] = \frac{sp_nq_n + (-1)^n}{s \cdot q_n^2}\,. \tag{14}$$

Proof. Let $r_m/s_m = [b_0; b_1, \ldots, b_n, s-1, 1, b_n-1, b_{n-1}, \ldots, b_1]$ then we have

$$\begin{bmatrix} r_m & r_{m-1} \\ s_m & s_{m-1} \end{bmatrix} = \begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \ldots \cdot \begin{bmatrix} b_n & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} s-1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \\ \cdot \begin{bmatrix} s & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} b_n - 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \ldots \cdot \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{15}$$

Since $p_n/q_n = [b_0; b_1, \ldots, b_n]$,

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \ldots \cdot \begin{bmatrix} b_n & 1 \\ 1 & 0 \end{bmatrix}. \tag{16}$$

From (15) and (16) we get

$$\begin{bmatrix} r_m & r_{m-1} \\ s_m & s_{m-1} \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} \begin{bmatrix} s-1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \\ \cdot \begin{bmatrix} b_n - 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \ldots \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{17}$$

Since

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_n - 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b_n & 1 \\ b_{n-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_n & 1 \\ 1 & 0 \end{bmatrix} \tag{18}$$

and

$$\begin{bmatrix} b_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_n - 1 & 1 \\ 1 & 0 \end{bmatrix} \ldots \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} q_n & p_n - b_0q_n \\ q_{n-1} & p_{n-1} - b_0q_{n-1} \end{bmatrix}, \tag{19}$$

by (17), (18) and (19) we obtain

$$\begin{bmatrix} r_m & r_{m-1} \\ s_m & s_{m-1} \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} \begin{bmatrix} s-1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \\ \cdot \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} q_n & p_n - b_0q_n \\ q_{n-1} & p_{n-1} - b_0q_{n-1} \end{bmatrix}. \tag{20}$$

On the other hand we have

$$\begin{bmatrix} s-1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 1 & 0 \end{bmatrix} \tag{21}$$

and

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} \begin{bmatrix} s & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} sp_n + p_{n-1} & -p_n \\ sq_n + q_{n-1} & -q_n \end{bmatrix}. \tag{22}$$

From (20, (21) and (22) we obtain

$$\begin{bmatrix} r_m & r_{m-1} \\ s_m & s_{m-1} \end{bmatrix} = \begin{bmatrix} sp_n + p_{n-1} & -p_n \\ sq_n + q_{n-1} & -q_n \end{bmatrix} \begin{bmatrix} q_n & p_n - b_0 q_n \\ q_{n-1} & p_{n-1} - b_0 q_{n-1} \end{bmatrix}. \tag{23}$$

From (23) we obtain $r_m = q_n(sp_n + p_{n-1}) - p_n q_{n-1} = sp_n q_n + p_{n-1} q_n - p_n q_{n-1}$ and since $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$, we have

$$r_m = sp_n q_n + (-1)^n. \tag{24}$$

Moreover, we have by (23)

$$s_m = q_n(sq_n + q_{n-1}) - q_n q_{n-1} = sq_n^2. \tag{25}$$

By (24) and (25) it follows that $\dfrac{r_m}{s_m} = \dfrac{sp_n q_n + (-1)^n}{s \cdot q_n^2}$ and the proof of Lemma 4 is complete.

Remark 1. In a simple continued fraction 0 is not allowed to be a partial quotient except as q_0. In many cases, however, it is convenient to allow this. For such continued fractions the given properties are true and one can transform them using the following property:

$$[a_0; a_1, \dots, a_i, 0, a_{i+1}, \dots, a_n] = [a_0; a_1, \dots, a_i + a_{i+1}, \dots, a_n]. \tag{$*$}$$

This property $(*)$ we can deduce by the following matrix identity: Let $p_n/q_n = [a_0; a_1, \dots, a_n]$, then

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{a_0} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{a_1} \cdot \ldots \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{a_n} \quad \text{if } n = 2\ell - 1, \tag{26}$$

and

$$\begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{a_0} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{a_1} \cdot \ldots \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{a_n} \quad \text{if } n = 2\ell. \tag{27}$$

Let i and n be odd. As $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, by (26) we obtain

$$\begin{aligned} \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{a_0} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{a_1} \cdot \ldots \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{a_i} \cdot \\ &\cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{0} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{a_{i+1}} \cdot \ldots \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{a_n} = \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{a_0} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{a_1} \cdot \ldots \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{a_i + a_{i+1}} \cdot \ldots \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{a_n} \end{aligned}$$

and $(*)$ follows. In a similar way we obtain $(*)$ in the other cases.

3. Proof of the Theorem

Let $s \geq 1$. Then by Lemma 4 and the assumtion of the Theorem we have

$$\text{(28)} \quad \begin{cases} [A_0; A_1, \ldots, A_n, s-1, A_{n-1}+1, A_{n-2}, \ldots, A_1] = \\ = \dfrac{sp_n q_n + (-1)^n}{sq_n^2} = \dfrac{p_n}{q_n} + \dfrac{(-1)^n}{sq_n^2} = S(u) + \dfrac{m_{u+1}}{n_{u+1}} = S(u+1)\,. \end{cases}$$

If $s \neq 1$, $A_n \neq 1$ then by (28) the assertion (4) follows. If $A_n = 1$ and $s \neq 1$ then by (28) and $(*)$ we obtain

$$S(u+1) = [A_0; A_1, \ldots, A_n, s-1, A_{n-1}+1, A_{n-2}, \ldots, A_1]$$

and we get the assertion (1) of the Theorem. If $s = 1$ and $A_n \neq 1$ then by (28) and $(*)$ we obtain

$$\begin{aligned} S(u+1) &= [A_0; A_1, \ldots, A_n, 0, 1, A_n - 1, A_{n-1}, \ldots, A_1] = \\ &= [A_0; A_1, \ldots, A_n + 1, A_n - 1, A_{n-1}, \ldots, A_1] \end{aligned}$$

and (2) follows.

In a similar way in the case $s = A_n = 1$ we obtain

$$\begin{aligned} S(u+1) &= [A_0; A_1, \ldots, A_n, 0, 1, A_n - 1, A_{n-1}, \ldots, A_1] = \\ &= [A_0; A_1, \ldots, A_n + 1, A_n - 1, A_{n-1}, \ldots, A_1] = \\ &= [A_0; A_1, \ldots, A_n + 1, 0, A_{n-1}, \ldots, A_1] = \\ &= [A_0; A_1, \ldots, A_{n-1}, A_{n-1} + 2, A_{n-2}, \ldots, A_1] \end{aligned}$$

and statement (3) of the Theorem is proved.

The proof of the Theorem is complete.

Remark 2. For the case considered by Pethő, we have from the assumptions of the Theorem $s = a_n Q_n^{k-2}$, $k \geq 2$. In the Theorem given by Shallit we have $s = 1$, if $B(u,v) = \sum_{k=0}^{v} \frac{1}{u^{2^k}}$ and $s = u^{d(v)}$, if $s(u,v) = \sum_{k=0}^{v} u^{-c(k)}$, where $d(v) = c(v+1) - 2c(v)$; $v \geq v'$.

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