Publ. Math. Debrecen
88/3-4 (2016), 305–317
DOI: 10.5486/PMD.2016.7308

Multivalued F-contractive mappings with a graph and some fixed point results

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Abstract. The main goal of this paper is to introduce a new type contraction, that is, multivalued F-G-contraction, on a metric space with a graph. In terms of this new contraction, we establish some fixed point results. At the end, we give an illustrative example, which shows the importance of graph on the contractive condition.

1. Introduction

Combining fixed point theory and graph theory, Echenique [12] gave a proof of Tarski fixed point theorem by using graphs. In 2006, Espinola and Kirk [13] applied fixed point results in graph theory. Recently, two fundamental results have appeared for fixed point theory with a graph. The first result was given by J. Jachymski [14] for single valued mappings and subsequently Beg, Butt and Radojević [6] extended Jachymski's result for set valued mappings. After then, in [23] Sultana and Vetrivel proved a fixed point theorem for Mizoguchi–Takahashi contraction and in [22] Sistani and Kazemipour gave some theorems for α-φ-contractions in these directions. One can consult to papers [9] and [11] for more details. In this paper, we prove some fixed point theorems for multivalued mappings on a metric space involving a graph using F-G-contractions.

Mathematics Subject Classification: 54H25, 47H10.
Key words and phrases: fixed point, multivalued maps, directed graph, F-contraction, complete metric space.

2. Preliminaries

2.1. Graph theory. Let X be a nonempty set and Δ denotes the diagonal of Cartesian product $X \times X$. A graph on X is an object $G = (V(G), E(G))$, where $V(G)$ is vertex set, whose elements are called vertices and $E(G)$ is edge set. We assume that G has no parallel edges and $\Delta \subset E(G)$.

If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}_{n\in\{0,1,2,\dots,k\}}$ of vertices such that

$$x_0 = x, x_k = y \text{ and } (x_{i-1}, x_i) \in E(G) \text{ for } i \in \{1, 2, \dots, k\}.$$

Notice that a graph G is connected if there is a path between any two vertices and it is weakly connected if $\tilde{G}$ is connected, where $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges.

Denote by $G^{-1}$the graph obtained from G by reversing the direction of edges. Thus

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Since it is more convenient to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention, we have that

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

We call (V', E') a subgraph of G if $V' \subseteq V(G)$ and $E' \subseteq E(G)$ and for any edge $(x, y) \in E'$, $x, y \in V'$.

If G is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation $\mathcal{R}$ defined on $V(G)$ by the rule:

$$y\mathcal{R}z \text{ if there is path in } G \text{ from } y \text{ to } z.$$

We can find more information about graph theory in [15].

Definition 2.1. Let (X, d) be a metric space, $G = (V(G), E(G))$ be a graph such that $V(G) = X$ and let $T : X \to CB(X)$. Then T is said to be graph-preserving if

$$(x, y) \in E(G) \Rightarrow (u, v) \in E(G) \text{ for all } u \in Tx \text{ and } v \in Ty.$$

2.2. Pompeiu–Hausdorff metric and some known results. Let (X,d) be a metric space. $P(X)$ denotes the family of all nonempty subsets of X, $CB(X)$ denotes the family of all nonempty, closed and bounded subsets of X and $K(X)$ denotes the family of all nonempty compact subsets of X. It is clear that, $K(X) \subseteq CB(X) \subseteq P(X)$. For $A, B \in CB(X)$, let

$$H(A,B) = \max\left\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(y,A)\right\},$$

where $D(x,B) = \inf\{d(x,y) : y \in B\}$. Then H is a metric on $CB(X)$, which is called Pompeiu–Hausdorff metric induced by d. We can find detailed information about Pompeiu–Hausdorff metric in [3], [8].

Using Pompeiu–Hausdorff metric, Nadler [18] introduced the concept of multivalued contraction mapping and show that such mapping has a fixed point on complete metric space. Then many authors focused on this direction [7], [10], [17], [20], [21], [24]. In the present paper, we use the recent technique, which was given by Wardowski [26].

Let $F : (0,\infty) \to \mathbb{R}$ be a function. For the sake of completeness, we will consider the following conditions:

$(F1)$ F is strictly increasing, i.e., for all $\alpha, \beta \in (0,\infty)$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,

$(F2)$ for each sequence $\{a_n\}$ of positive numbers,

$$\lim_{n\to\infty} a_n = 0 \text{ if and only if } \lim_{n\to\infty} F(a_n) = -\infty,$$

$(F3)$ there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

$(F4)$ $F(\inf A) = \inf F(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$.

We consider by $\mathcal{F}$ and $\mathcal{F}_*$ be the set of all functions F satisfying $(F1)$–$(F3)$ and $(F1)$–$(F4)$, respectively. It is clear that $\mathcal{F}_* \subset \mathcal{F}$.

Some examples of the functions belonging $\mathcal{F}_*$ are $F_1(\alpha) = \ln\alpha$, $F_2(\alpha) = \alpha + \ln\alpha$, $F_3(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln(\alpha^2+\alpha)$. If we define $F_5(\alpha) = \ln\alpha$ for $\alpha \le 1$ and $F_5(\alpha) = 2\alpha$ for $\alpha > 1$, then $F_5 \in \mathcal{F}\diagdown\mathcal{F}_*$.

Definition 2.2 ([26]). Let (X,d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be an F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\forall x,y \in X[d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \le F(d(x,y))].$$

Also, Wardowski concluded that if $F_1, F_2 \in \mathcal{F}$ with $F_1(\alpha) \leq F_2(\alpha)$ for all $\alpha > 0$ and $G = F_2 - F_1$ is nondecreasing, then every F_1-contraction T is an F_2-contraction.

He noted that for the mappings $F_1(\alpha) = \ln \alpha$ and $F_2(\alpha) = \alpha + \ln \alpha$, $F_1 < F_2$ and a mapping $F_2 - F_1$ is strictly increasing. On the other side, Example 2.5 in [26] shows that the mapping T is not an F_1-contraction (Banach contraction), but still is an F_2-contraction.

Thus, the following theorem, which was given by Wardowski, is a proper generalization of Banach Contraction Principle.

Theorem 2.1 ([26])**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point in X.*

Following Wardowski, Minak *et al.* [16], and Wardowski and Van Dung [27] obtained some fixed point results for single valued maps by generalizing Theorem 2.1. Then considering the Pompeiu–Hausdorff metric H, the multivalued version of these mentioned results are obtained in [4] and [2]. (see also [5], [19]). After then, Acar and Altun [1] proved a fixed point theorem for multivalued F-contraction using δ-distance.

Lemma 2.2. *Let (X, d) be a metric space and $T : X \to P(X)$ be an upper semicontinuous mapping such that Tx is closed for all $x \in X$. If $x_n \to x_0$, $y_n \to y_0$ and $y_n \in Tx_n$, then $y_0 \in Tx_0$.*

3. Main results

In this section, we start with introducing weakly graph-preserving property of a multivalued map on a metric space. We also define multivalued F-G-contraction, which is a new type contraction, then prove some theorems for multivalued mappings considering these two new concepts. Actually, there are lots of theorems which are proved by using graph preserving property (see for example [25]), but the following property is weaker than this one.

Definition 3.1. Let (X, d) be a metric space, $G = (V(G), E(G))$ be a graph such that $V(G) = X$ and let $T : X \to CB(X)$. Then we say that T has weakly graph-preserving property whenever for each $x \in X$ and $y \in Tx$ with $(x, y) \in E(G)$ implies $(y, z) \in E(G)$ for all $z \in Ty$.

Remark 3.1. It is clear that every graph preserving map is a weakly graph preserving. But the converge may not be true. For example, let $X = [-1, 1]$ with the usual metric. Consider a graph given by $V(G) = X$ and $E(G) = X \times X \setminus \Delta$. Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \{-x\}, & x \notin \{-1, 0\} \\ \{0, 1\}, & x = -1 \\ \{1\}, & x = 0\,. \end{cases}$$

Then it can be seen that T is weakly graph-preserving map but not graph-preserving.

Definition 3.2. Let (X, d) be a metric space, G be a directed graph on X and $T : X \to CB(X)$ be a mapping. Define a set

$$T_G = \{(x, y) \in E(G) : H(Tx, Ty) > 0\}\,.$$

Given $F \in \mathcal{F}$ we say that T is a multivalued F-G-contraction if there exists $\tau > 0$ such that

$$\tau + F(H(Tx, Ty)) \leq F(M(x, y)) \tag{3.1}$$

for $x, y \in X$ with $(x, y) \in T_G$, where

$$M(x, y) = \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}\,[D(x, Ty) + D(y, Tx)]\right\}.$$

We shall present our main result for the mapping $T : X \to K(X)$ which makes sense since $K(X) \subseteq CB(X)$.

Theorem 3.1. *Let (X, d) be a complete metric space, G be a directed graph on X and $T : X \to K(X)$ be a multivalued F-G-contraction. Assume that T is upper semicontinuous and weakly graph-preserving map and the set*

$$X_T = \{x \in X : (x, u) \in E(G) \text{ for some } u \in Tx\}$$

is nonempty. Then T has a fixed point.

Proof. Suppose that T has no fixed point. Then for all $x \in X$, $D(x, Tx) > 0$. Let $x_0 \in X_T$. Then $(x_0, x_1) \in E(G)$ for some $x_1 \in Tx_0$. So we get

$$0 < D(x_1, Tx_1) \leq H(Tx_0, Tx_1).$$

Thus $(x_0, x_1) \in T_G$ and so we can use the condition (3.1) for x_0 and x_1. Then we have

$$\begin{aligned}
F(D(x_1,Tx_1)) &\leq F(H(Tx_0,Tx_1)) \leq F(M(x_0,x_1)) - \tau, \\
&= F(\max\left\{\begin{array}{c} d(x_0,x_1), D(x_0,Tx_0), D(x_1,Tx_1), \\ \frac{1}{2}\left[D(x_0,Tx_1)+D(x_1,Tx_0)\right] \end{array}\right\}) - \tau \\
&\leq F(\max\{d(x_0,x_1), D(x_1,Tx_1)) - \tau, \qquad (3.2)
\end{aligned}$$

since

$$\begin{aligned}
M(x_0,x_1) &= \max\left\{\begin{array}{c} d(x_0,x_1), D(x_0,Tx_0), D(x_1,Tx_1), \\ \frac{1}{2}\left[D(x_0,Tx_1)+D(x_1,Tx_0)\right] \end{array}\right\} \\
&= \max\left\{d(x_0,x_1), D(x_1,Tx_1), \frac{1}{2}D(x_0,Tx_1)\right\} \\
&\leq \max\left\{d(x_0,x_1), D(x_1,Tx_1), \frac{1}{2}\left[d(x_0,x_1)+D(x_1,Tx_1)\right]\right\} \\
&\leq \max\left\{d(x_0,x_1), D(x_1,Tx_1), \max\{d(x_0,x_1), D(x_1,Tx_1)\}\right\} \\
&= \max\{d(x_0,x_1), D(x_1,Tx_1)\}.
\end{aligned}$$

Now if $d(x_0,x_1) \leq D(x_1,Tx_1)$, then from (3.2), we have

$$F(D(x_1,Tx_1)) \leq F(D(x_1,Tx_1)) - \tau,$$

which is a contradiction. Thus $D(x_1,Tx_1) < d(x_0,x_1)$ and so from (3.2), we have

$$F(D(x_1,Tx_1)) \leq F(d(x_0,x_1)) - \tau \qquad (3.3)$$

Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that $d(x_1,x_2) = D(x_1,Tx_1)$. From (3.3),

$$F(d(x_1,x_2)) \leq F(d(x_0,x_1)) - \tau.$$

Since $(x_0,x_1) \in E(G)$, $x_1 \in Tx_0$ and $x_2 \in Tx_1$, using weakly graph-preserving property we can write $(x_1,x_2) \in E(G)$. Because of $0 < D(x_2,Tx_2) \leq H(Tx_1,Tx_2)$, we have $(x_1,x_2) \in T_G$. Then

$$F(D(x_2,Tx_2)) \leq F(H(Tx_1,Tx_2)) \leq F(M(x_1,x_2)) - \tau. \qquad (3.4)$$

By considering the same way, we can get

$$M(x_1,x_2) \leq \max\{d(x_1,x_2), D(x_2,Tx_2)\}.$$

Thus from (3.4)

$$F(D(x_2, Tx_2)) \leq F(d(x_1, x_2)) - \tau. \tag{3.5}$$

Since Tx_2 is compact, there exists $x_3 \in Tx_2$ such that $d(x_2, x_3) = D(x_2, Tx_2)$. Therefore we have

$$F(d(x_2, x_3)) \leq F(d(x_2, x_1)) - \tau.$$

By induction, we can find a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, $(x_n, x_{n+1}) \in T_G$ and

$$F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau \tag{3.6}$$

for all $n \in \mathbb{N}$. Denote $a_n = d(x_n, x_{n+1})$ for $n \in \mathbb{N}$, then $a_n > 0$ and from (3.6), $\{a_n\}$ is decreasing. Therefore there exists $\delta \geq 0$ such that $\lim_{n\to\infty} a_n = \delta$. Now let $\delta > 0$. Using (3.6), the following holds:

$$\begin{aligned} F(a_n) \leq & F(a_{n-1}) - \tau \\ \leq & F(a_{n-2}) - 2\tau \\ & \vdots \\ \leq & F(a_0) - n\tau. \end{aligned} \tag{3.7}$$

From (3.7), we get $\lim_{n\to\infty} F(a_n) = -\infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} a_n = 0.$$

From (F3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} a_n^k F(a_n) = 0.$$

By (3.7), the following holds for all $n \in \mathbb{N}$

$$a_n^k F(a_n) - a_n^k F(a_0) \leq -a_n^k n\tau \leq 0. \tag{3.8}$$

Letting $n \to \infty$ in (3.8), we obtain that

$$\lim_{n\to\infty} n a_n^k = 0. \tag{3.9}$$

From (3.9), there exits $n_1 \in \mathbb{N}$ such that $n a_n^k \leq 1$ for all $n \geq n_1$. So we have

$$a_n \leq \frac{1}{n^{1/k}}. \tag{3.10}$$

for all $n \geq n_1$. In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (3.10), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= a_n + a_{n+1} + \cdots + a_{m-1} \\ &= \sum_{i=n}^{m-1} a_i \leq \sum_{i=n}^{\infty} a_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

By the convergence of the series $\sum\limits_{i=1}^{\infty} \frac{1}{i^{1/k}}$, we get $d(x_n, x_m) \to 0$ as $n \to \infty$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$. Since T is upper semicontinuous, then by Lemma 2.2 we have $z \in Tz$, which is a contradict to the assumption being T has no fixed point. □

Remark 3.2. By adding the condition (F4) on F, we can consider $CB(X)$ instead of $K(X)$.

Theorem 3.2. *Let (X, d) be a complete metric space, G be a directed graph on X and $T : X \to CB(X)$ be a multivalued F-G-contraction with $F \in \mathcal{F}_*$. Assume that T is upper semicontinuous and weakly graph-preserving map and the set X_T is nonempty. Then T has a fixed point.*

Proof. We begin as in the proof of Theorem 3.1. By taking into account the condition (F4), we get

$$F(D(x_1, Tx_1)) = F(\inf\{d(x_1, y) : y \in Tx_1\}) = \inf\{F(d(x_1, y) : y \in Tx_1)\}$$

and so from (3.3) we have

$$\inf\{F(d(x_1, y) : y \in Tx_1)\} < F(d(x_0, x_1)) - \frac{\tau}{2}.$$

Thus there exists $x_2 \in Tx_1$ such that

$$F(d(x_1, x_2)) \leq F(d(x_0, x_1)) - \frac{\tau}{2}.$$

The rest of the proof can be completed as in the proof of Theorem 3.1. □

Remark 3.3. If we consider the following condition (3.11) on X instead of the upper semicontinuity of T, we can get the following result. But we need to use the continuity of F.

Theorem 3.3. *Let (X, d) be a complete metric space and G be a directed graph such that the following property hold:*

$$\begin{aligned} &\text{for any } \{x_n\} \text{ in } X, \text{ if } x_n \to x \text{ and } (x_n, x_{n+1}) \in E(G), \\ &\text{then there is a subsequence } \{x_{n_k}\} \text{ with } (x_{n_k}, x) \in E(G). \end{aligned} \tag{3.11}$$

Let $T : X \to K(X)$ be a multivalued F-G-contraction (resp. $T : X \to CB(X)$ be a multivalued F-G-contraction with $F \in \mathcal{F}_*$). Assume that T is weakly graph-preserving map and X_T is nonempty. If F is continuos, then T has a fixed point.

Proof. Suppose that T has no fixed point. By similar way of proof of Theorem 3.1 (resp. Theorem 3.2), we can construct a sequence $\{x_n\}$ such that $x_n \to z$ for some $z \in X$. By the property (3.11), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, z) \in E(G)$ for each $k \in \mathbb{N}$. Since $\lim_{n\to\infty} x_n = z$ and $D(z, Tz) > 0$, then there exists $n_0 \in \mathbb{N}$ such that $D(x_{n_k+1}, Tz) > 0$ for all $n_k \geq n_0$. Therefore for all $n_k \geq n_0$

$$H(Tx_{n_k}, Tz) > 0,$$

thus $(x_{n_k}, z) \in T_G$ for all $n_k \geq n_0$. From (3.1) and (F1), we have

$$\begin{aligned} F(D(x_{n_k+1}, Tz)) &\leq F(H(Tx_{n_k}, Tz)) - \tau \\ &\leq F(M(x_{n_k}, z)) - \tau \\ &\leq F(\max\{d(x_{n_k}, z), d(x_{n_k}, x_{n_k+1}), D(z, Tz), \\ &\quad \frac{1}{2}[D(x_{n_k}, Tz) + d(z, x_{n_k+1})]\}) - \tau \end{aligned}$$

for all $n_k \geq n_0$. Taking the limit $k \to \infty$ and using the continuity of F, we have $\tau + F(D(z, Tz)) \leq F(D(z, Tz))$, which is a contradiction. Thus T has a fixed point. □

Corollary 3.4. *Let (X, d) be a complete metric space, G be a directed graph on X and $T : X \to K(X)$ be a mapping. Assume that there exists $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$\tau + F(H(Tx, Ty)) \leq F(d(x, y)) \tag{3.12}$$

for $x, y \in X$ with $(x, y) \in T_G$. Suppose that T is upper semicontinuous and weakly graph-preserving map and the set X_T is nonempty. Then T has a fixed point.

Corollary 3.5. *Let (X, d) be a complete metric space, G be a directed graph on X and $T : X \to CB(X)$ be a mapping. Assume that there exists $F \in \mathcal{F}_*$ and $\tau > 0$ such that*

$$\tau + F(H(Tx, Ty)) \leq F(d(x, y))$$

for $x, y \in X$ with $(x, y) \in T_G$. Suppose that T is upper semicontinuous and weakly graph-preserving map and the set X_T is nonempty. Then T has a fixed point.

Remark 3.4. As in Remark 3.3, we can say that the condition (3.11) can be considered instead of the upper semicontinuity of T in Corollary 3.4 and in Corollary 3.5. But in this case we don't need to continuity of F.

Corollary 3.6. *Let (X, d) be a complete metric space, G be a directed graph such that the* (3.11) *property hold and $T : X \to K(X)$ (resp. $T : X \to CB(X)$) be a mapping. Assume that there exists $F \in \mathcal{F}$ (resp. $F \in \mathcal{F}_*$) and $\tau > 0$ satisfying the inequality* (3.12) *for $x, y \in X$ with $(x, y) \in T_G$. If T is weakly graph-preserving map and X_T is nonempty, then T has a fixed point.*

Proof. Suppose that T has no fixed point. By similar way of proof of Theorem 3.1 (resp. Theorem 3.2), we can construct a sequence $\{x_n\}$ such that $x_n \to z$ for some $z \in X$. By the property (3.11), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, z) \in E(G)$ for each $k \in \mathbb{N}$. Since $\lim_{n\to\infty} x_n = z$ and $D(z, Tz) > 0$, then there exists $n_0 \in \mathbb{N}$ such that $D(x_{n_k+1}, Tz) > 0$ for all $n_k \geq n_0$. Therefore for all $n_k \geq n_0$

$$H(Tx_{n_k}, Tz) > 0,$$

thus $(x_{n_k}, z) \in T_G$ for all $n_k \geq n_0$. Again, since $\lim_{n\to\infty} x_n = z$ and $D(z, Tz) > 0$, then there exists $n_1 \in \mathbb{N}$ such that $d(x_{n_k}, z) < \frac{D(z,Tz)}{2}$ for all $n_k \geq n_1$. From (3.12) and (F1), we have

$$\begin{aligned} F(D(x_{n_k+1}, Tz)) &\leq F(H(Tx_{n_k}, Tz)) - \tau \\ &\leq F(d(x_{n_k}, z)) - \tau \\ &\leq F(\frac{D(z, Tz)}{2}) - \tau \end{aligned}$$

for all $n_k \geq \max\{n_0, n_1\}$. Therefore from (F1) we have

$$D(x_{n_k+1}, Tz) \leq \frac{D(z, Tz)}{2}$$

for all $n_k \geq \max\{n_0, n_1\}$. Taking the limit $k \to \infty$ we get $0 < D(z, Tz) \leq \frac{D(z,Tz)}{2}$, which is a contradiction. Thus T has a fixed point. □

Now we give an example, which shows the importance of graph on the contractive condition.

Example 3.7. Let

$$X = \{0, 1, 2, 3, \dots\} \quad \text{and} \quad d(x, y) = \begin{cases} 0, & x = y \\ x + y, & x \neq y \,. \end{cases}$$

Then (X, d) is a complete metric space. Consider a graph given by $V(G) = X$ and $E(G) = X \times X \backslash \{(0, 1), (1, 0)\}$. Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \{x\}, & x = 0, x = 1 \\ \{0, 1, 2, \dots, x - 1\}, & x \geq 2 \,. \end{cases}$$

Then T is upper semicontinuous because τ_d is discrete topology. Also T is weakly graph-preserving map. Now, we claim that T is multivalued F-G-contraction with $F(\alpha) = \alpha + \ln \alpha$ and $\tau = 1$. Note that $T_G = E(G) \backslash \Delta$. Therefore we have to consider the following cases for contractive condition.

Case 1. For $y = 0$ and $x > 1$, we have

$$\begin{aligned} \frac{H(Tx, Ty)}{M(x, y)} e^{H(Tx,Ty) - M(x,y)} &= \frac{x - 1}{x} e^{x-1-x} \\ &= \frac{x - 1}{x} e^{-1} < e^{-1}. \end{aligned}$$

Case 2. For $y = 1$ and $x > 1$, we have

$$\begin{aligned} \frac{H(Tx, Ty)}{M(x, y)} e^{H(Tx,Ty) - M(x,y)} &= \frac{x}{x + 1} e^{x-x-1} \\ &= \frac{x}{x + 1} e^{-1} < e^{-1}. \end{aligned}$$

Case 3. For $x > y > 1$, we have

$$\begin{aligned} \frac{H(Tx, Ty)}{M(x, y)} e^{H(Tx,Ty) - M(x,y)} &= \frac{x - 1}{x + y} e^{x-1-x-y} \\ &= \frac{x - 1}{x + y} e^{-1-y} < e^{-1}. \end{aligned}$$

Thus all conditions of Theorem 3.1 (or Theorem 3.2) are satisfied. Therefore, T has a fixed point.

On the other hand, if we don't consider the graph on X, the contractive condition is not satisfied. Indeed, let $x = 0$ and $y = 1$, then $H(Tx, Ty) = 1$ and $d(x, y) = 1$, so for all $F \in \mathcal{F}$ and $\tau > 0$ we get

$$\tau + F(H(Tx, Ty)) > F(d(x, y)).$$

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(Received March 5, 2015)