

## The influence of weakly $\mathfrak{3}$ -permutable subgroups on the structure of finite groups

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**Abstract.** Let  $\mathfrak{3}$  be a complete set of Sylow subgroups of a finite group  $G$ , that is, for each prime  $p$  dividing the order of  $G$ ,  $\mathfrak{3}$  contains exactly one and only one Sylow  $p$ -subgroup of  $G$ . A subgroup  $H$  of  $G$  is said to be  $\mathfrak{3}$ -permutable of  $G$  if  $H$  permutes with every member of  $\mathfrak{3}$ . A subgroup  $H$  of  $G$  is said to be a weakly  $\mathfrak{3}$ -permutable subgroup of  $G$  if there exists a subnormal subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \leq H_3$ , where  $H_3$  is the subgroup of  $H$  generated by all those subgroups of  $H$  which are  $\mathfrak{3}$ -permutable subgroups of  $G$ . We investigate the structure of the finite group  $G$  under the assumption that every cyclic subgroup of prime order  $p$  or of order 4 (if  $p = 2$ ) is a weakly  $\mathfrak{3}$ -permutable subgroup of  $G$ . Our results extend and generalize several results in the literature.

### 1. Introduction and statement of results

Throughout this paper, all groups are assumed to be finite. Two subgroups  $H$  and  $K$  of a group  $G$  are said to be permutable if  $HK = KH$ , that is,  $HK$  is a subgroup of  $G$ . A subgroup  $H$  of  $G$  is said to be  $S$ -permutable if it permutes with every Sylow subgroup of  $G$ . This concept was introduced in 1962 by KEGEL [9] who called these subgroups  $S$ -quasinormal, and has been studied extensively by many authors. Recently, in 2003, ASAAD and HELIEL [3] generalized  $S$ -permutability property by requiring permutability only with the members of a complete set of Sylow subgroups. Let  $\mathfrak{3}$  be a complete set of Sylow subgroups of

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a group  $G$ , that is, a set composed of a Sylow  $p$ -subgroup of  $G$  for each prime  $p$  dividing the order of  $G$ . A subgroup  $H$  of  $G$  is said to be  $\mathfrak{Z}$ -permutable if  $H$  permutes with all members of  $\mathfrak{Z}$ . In 1996, WANG [17] introduced the concept of  $c$ -normality as follows: A subgroup  $H$  of a group  $G$  is said to be  $c$ -normal in  $G$  if there exists a normal subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \leq H_G$ , where  $H_G = \text{Core}_G(H)$  is the largest normal subgroup of  $G$  contained in  $H$ . All the above mentioned concepts are nontrivial generalization of normality. In 2015, HELIEL *et al.* [5] introduced a new subgroup embedding property, namely, the weakly  $\mathfrak{Z}$ -permutability which unified and generalized both of  $\mathfrak{Z}$ -permutability and  $c$ -normality concepts as follows:

*Definition.* Let  $\mathfrak{Z}$  be a complete set of Sylow subgroups of a group  $G$ . A subgroup  $H$  of  $G$  is said to be a weakly  $\mathfrak{Z}$ -permutable subgroup of  $G$  if there exists a subnormal subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \leq H_{\mathfrak{Z}}$ , where  $H_{\mathfrak{Z}}$  is the subgroup of  $H$  generated by all those subgroups of  $H$  which are  $\mathfrak{Z}$ -permutable subgroups of  $G$ .

It is clear that every  $\mathfrak{Z}$ -permutable subgroup and every  $c$ -normal subgroup of a group  $G$  are weakly  $\mathfrak{Z}$ -permutable subgroups of  $G$ . Examples 1.2, 1.3 and 1.4 in HELIEL *et al.* [5] show that the converse does not hold in general and there is no inclusion relationship between the notions of  $\mathfrak{Z}$ -permutability and  $c$ -normality.

In recent years, there has been a considerable interest to study the structure of a finite group  $G$  under the assumption that all minimal subgroups of  $G$  and the cyclic subgroups of order 4 are well-situated in  $G$  (a minimal subgroup of  $G$  is a subgroup of prime order). In 1970, BUCKLEY [4] proved that if  $G$  is a group of odd order whose minimal subgroups are normal, then  $G$  is supersolvable. Later on, in 1990, SHAALAN [15] proved that if  $G$  is a group such that all of its cyclic subgroups of prime order  $p$  or of order 4 (if  $p = 2$ ) are  $S$ -permutable, then  $G$  is supersolvable. Working within the framework of formation theory, in 2004, HELIEL *et al.* [6] extended and generalized the above mentioned results as follows: Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$ , and let  $\mathfrak{Z}$  be a complete set of Sylow subgroups of a group  $G$ . Then  $G \in \mathfrak{F}$  if and only if  $G$  has a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and the cyclic subgroups of  $G_p \cap F^*(H)$  of prime order or of order 4 (if  $p = 2$ ) are  $\mathfrak{Z}$ -permutable subgroups of  $G$ , for all  $G_p \in \mathfrak{Z}$ , where  $F^*(H)$  is the generalized Fitting subgroup of  $H$ . Using  $c$ -normality concept, in 2003, WEI *et al.* [20] obtained the  $c$ -normal version of the previous result as follows: Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$ . Suppose that  $G$  is a group with a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$ . If all the cyclic subgroups of  $F^*(H)$  of prime order  $p$  or

of order 4 (if  $p = 2$ ) are  $c$ -normal in  $G$ , then  $G \in \mathfrak{F}$ . For more results in this direction, see [1], [2], [4], [6], [10]–[15] and [17]–[20].

The main object of this paper is to improve and extend the above mentioned results and all the related results in the literature that dealing with the structure of finite groups under the assumption that the cyclic subgroups of prime order  $p$  or of order 4 (if  $p = 2$ ) are  $\mathfrak{3}$ -permutable or  $c$ -normal by using the new concept weakly  $\mathfrak{3}$ -permutable. Our main results are as follows:

**Theorem 1.1.** *Let  $\mathfrak{3}$  be a complete set of Sylow subgroups of a group  $G$  and let  $p$  be the smallest prime dividing the order of  $G$ . If the cyclic subgroups of  $G_p \in \mathfrak{3}$  of order  $p$  or of order 4 (if  $p = 2$ ) are weakly  $\mathfrak{3}$ -permutable subgroups of  $G$ , then  $G$  is  $p$ -nilpotent.*

**Theorem 1.2.** *Let  $\mathfrak{3}$  be a complete set of Sylow subgroups of a group  $G$ . If the cyclic subgroups of  $G_p$  of prime order or of order 4 (if  $p = 2$ ) are weakly  $\mathfrak{3}$ -permutable subgroups of  $G$ , for all  $G_p \in \mathfrak{3}$ , then  $G$  is supersolvable.*

**Theorem 1.3.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $\mathfrak{3}$  be a complete set of Sylow subgroups of a group  $G$ . Then the following are equivalent:*

- (a)  $G \in \mathfrak{F}$ .
- (b) *There is a normal subgroup  $H$  of  $G$  such that  $G/H \in \mathfrak{F}$  and the cyclic subgroups of  $G_p \cap F^*(H)$  of prime order or of order 4 (if  $p = 2$ ) are weakly  $\mathfrak{3}$ -permutable subgroups of  $G$ , for all  $G_p \in \mathfrak{3}$ .*

## 2. Basic definitions and preliminaries

Let  $\mathfrak{F}$  be a class of groups. We call  $\mathfrak{F}$  a formation provided that:

- (a) If  $G \in \mathfrak{F}$  and  $H$  is a normal subgroup of  $G$ , then  $G/H \in \mathfrak{F}$ , and
- (b) If  $M$  and  $N$  are normal subgroups of  $G$  such that  $G/M \in \mathfrak{F}$  and  $G/N \in \mathfrak{F}$ , then  $G/(M \cap N) \in \mathfrak{F}$ .

A formation  $\mathfrak{F}$  is said to be saturated if  $G/\Phi(G) \in \mathfrak{F}$  implies that  $G \in \mathfrak{F}$ . In this paper,  $\mathfrak{U}$  denotes the class of all supersolvable groups. It is known that  $\mathfrak{U}$  is a saturated formation, see [7, p. 713, Satz 8.6].

For any group  $G$ , the generalized Fitting subgroup  $F^*(G)$  is the unique maximal normal quasinilpotent subgroup of  $G$ . This subgroup,  $F^*(G)$ , is an important characteristic subgroup of  $G$  and it is a natural generalization of  $F(G)$ , the Fitting subgroup of  $G$ . The basic properties of  $F^*(G)$  can be found in [8, X 13].

For the basic properties of  $\mathfrak{Z}$ -permutable and weakly  $\mathfrak{Z}$ -permutable, the reader is referred to [3] and [5].

**Lemma 2.1.** *Let  $\mathfrak{Z}$  be a complete set of Sylow subgroups of a group  $G$  and let  $p$  be the smallest prime dividing the order of  $G$ . If the cyclic subgroups of  $G_p \in \mathfrak{Z}$  of order  $p$  or of order 4 (if  $p = 2$ ) are  $\mathfrak{Z}$ -permutable subgroups of  $G$ , then  $G$  is  $p$ -nilpotent.*

PROOF. See [12, Theorem 3.1]. □

**Lemma 2.2.** *Let  $H$  and  $N$  be subgroups of a group  $G$  such that  $N$  is normal in  $G$  and let  $\mathfrak{Z}$  be a complete set of Sylow subgroups of  $G$ . Then:*

- (a) *If  $H \leq N$  and  $H$  is a weakly  $\mathfrak{Z}$ -permutable subgroup of  $G$ , then  $H$  is a weakly  $\mathfrak{Z} \cap N$ -permutable subgroup of  $N$ .*
- (b) *If  $(|H|, |N|) = 1$  and  $H$  is a weakly  $\mathfrak{Z}$ -permutable subgroup of  $G$ , then  $HN/N$  is a weakly  $\mathfrak{Z}N/N$ -permutable subgroup of  $G/N$ .*

PROOF. See [5, Lemma 2.3]. □

**Lemma 2.3.** *Let  $G$  be a group. If  $F^*(G)$  is solvable, then  $F^*(G) = F(G)$ .*

PROOF. See [8, X 13]. □

**Lemma 2.4.** *Let  $H$  be a subgroup of a group  $G$ . If  $H$  is  $S$ -permutable in  $G$  and  $x \in G$ , then  $H^x$  is  $S$ -permutable in  $G$ .*

PROOF. Let  $\text{Syl}(G)$  denotes the set of all Sylow subgroups of  $G$ . By hypothesis,  $HP = PH$ , for all  $P \in \text{Syl}(G)$ . This implies that  $H^x P^x = P^x H^x$ , for all  $P^x \in (\text{Syl}(G))^x$ . Since  $(\text{Syl}(G))^x = \text{Syl}(G)$ , then  $H^x$  permute with every Sylow subgroup of  $G$ . Thus  $H^x$  is  $S$ -permutable in  $G$ . □

Following SKIBA [16], we say that a subgroup  $H$  of a group  $G$  is a weakly  $S$ -permutable subgroup in  $G$  if there exists a subnormal subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \leq H_{SG}$ , where  $H_{SG}$  is the subgroup of  $H$  generated by all those subgroups of  $H$  which are  $S$ -permutable in  $G$ .

**Lemma 2.5.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $G$  be a group with a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$ . If the cyclic subgroups of  $F^*(H)$  of prime order  $p$  or of order 4 (if  $p = 2$ ) are weakly  $S$ -permutable subgroups in  $G$ , then  $G \in \mathfrak{F}$ .*

PROOF. Immediate consequence of [16, Theorem 1.3]. □

### 3. Proofs

PROOF OF THEOREM 1.1. Assume that the result is false and let  $G$  be a counterexample of minimal order. If every cyclic subgroup of  $G_p \in \mathfrak{3}$  of order  $p$  or of order 4 (if  $p = 2$ ) is  $\mathfrak{3}$ -permutable subgroup of  $G$ , then  $G$  is  $p$ -nilpotent by Lemma 2.1. Thus, there exists a cyclic subgroup  $H$  of  $G_p$  of order  $p$  or 4 (if  $p = 2$ ) such that  $H$  is not  $\mathfrak{3}$ -permutable subgroup of  $G$ . By hypothesis,  $H$  is a weakly  $\mathfrak{3}$ -permutable subgroup of  $G$ . So, there exists a subnormal subgroup  $K$  of  $G$  with  $K \neq G$  such that  $G = HK$  and  $H \cap K \leq H_3$ . Let  $N$  be a proper normal subgroup of  $G$  containing  $K$ . Since  $G = HN$ , we have  $G_p = G_p \cap HN = H(G_p \cap N)$ . By Lemma 2.2(a), the cyclic subgroups of  $G_p \cap N \in \mathfrak{3} \cap N$  of order  $p$  or of order 4 (if  $p = 2$ ) are weakly  $\mathfrak{3} \cap N$ -permutable subgroups of  $N$ . The minimal choice of  $G$  implies that  $N$  is  $p$ -nilpotent. Therefore  $N = (G_p \cap N)R$ , where  $R$  is a normal Hall  $p'$ -subgroup of  $N$ . Since  $R \text{ char } N$  and  $N$  is normal in  $G$ , it follows that  $R$  is a normal subgroup of  $G$ . So,  $G = HN = H(G_p \cap N)R = G_p R$  and  $G_p \cap R = 1$ . Thus  $G$  is  $p$ -nilpotent, a contradiction completing the proof of the theorem.  $\square$

We need the following lemma:

**Lemma 3.1.** *Let  $\mathfrak{3}$  be a complete set of Sylow subgroups of a group  $G$  and let  $P$  be a normal  $p$ -subgroup of  $G$ . If every cyclic subgroup of  $P$  of order  $p$  or of order 4 (if  $p = 2$ ) is a weakly  $\mathfrak{3}$ -permutable subgroup of  $G$ , then every cyclic subgroup of  $P$  of order  $p$  or of order 4 (if  $p = 2$ ) is a weakly  $S$ -permutable subgroup in  $G$ .*

PROOF. We distinguish two cases:

*Case 1.  $p \neq 2$ .*

Let  $H$  be a subgroup of  $P$  of order  $p$ . By hypothesis,  $H$  is a weakly  $\mathfrak{3}$ -permutable subgroup of  $G$ , that is, there exists a subnormal subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K \leq H_3$ . If  $H \cap K = 1$ , then  $H$  is a weakly  $S$ -permutable subgroup in  $G$  and we are done. So, we may assume that  $H \cap K = H$  and hence  $H$  is  $\mathfrak{3}$ -permutable subgroup of  $G$ . Clearly, every conjugate of  $H$  is contained in  $P$  as  $P$  is normal in  $G$ . Suppose that every conjugate of  $H$  is  $\mathfrak{3}$ -permutable subgroup of  $G$ . Then  $H^{x^{-1}}G_q \leq G$ , for all  $x \in G$ , and for all  $G_q \in \mathfrak{3}$ . This implies that  $HG_q^x = (H^{x^{-1}}G_q)^x \leq G$ , for all  $x \in G$ , and for all  $G_q \in \mathfrak{3}$ . Since the Sylow subgroups of  $G$  are conjugates, we have that  $H$  is  $S$ -permutable in  $G$  as desired. So, we may assume that  $G$  has an element  $y$  such that  $H^y$  is not  $\mathfrak{3}$ -permutable subgroup of  $G$ . By hypothesis,  $H^y$  is a weakly  $\mathfrak{3}$ -permutable subgroup of  $G$ . Then there exists a subnormal subgroup  $N$  of  $G$  such that  $G = H^y N$  and  $H^y \cap N = 1$ . This implies that  $N^{y^{-1}}$  is a subnormal subgroup of  $G$  with  $G = HN^{y^{-1}}$  and

$H \cap N^{y^{-1}} = 1$ . Thus  $H$  is a weakly  $S$ -permutable subgroup in  $G$  which completes the proof of this case.

*Case 2.  $p = 2$ .*

By using the same arguments as in Case 1, we have that every subgroup of  $P$  of order 2 is a weakly  $S$ -permutable subgroup in  $G$ . Let  $a \in P$  such that  $|a| = 4$ . By hypothesis,  $\langle a \rangle$  is a weakly  $\mathfrak{3}$ -permutable subgroup of  $G$ . So, there exists a subnormal subgroup  $K$  of  $G$  such that  $G = \langle a \rangle K$  and  $\langle a \rangle \cap K \leq \langle a \rangle_{\mathfrak{3}}$ . Similarly, as in Case 1,  $\langle a \rangle \cap K \neq 1$ . Suppose that  $\langle a \rangle \cap K = \langle a^2 \rangle$ . Since  $\langle a^2 \rangle$  is a weakly  $S$ -permutable subgroup in  $G$  and  $\langle a^2 \rangle$  cannot be complemented in  $G$  (if  $N$  is a complement of  $\langle a^2 \rangle$  in  $G$ , then  $G = \langle a^2 \rangle N = \langle a \rangle N$  and  $\langle a \rangle \cap N = 1$ . But  $|\langle a^2 \rangle N| = |\langle a^2 \rangle| |N| < |\langle a \rangle| |N| = |\langle a \rangle N|$ , a contradiction), it follows that  $\langle a^2 \rangle$  is  $S$ -permutable in  $G$ . Thus  $\langle a \rangle$  is a weakly  $S$ -permutable subgroup in  $G$ . So, we may assume that  $\langle a \rangle \cap K = \langle a \rangle$  and hence  $\langle a \rangle$  is  $\mathfrak{3}$ -permutable subgroup of  $G$ . By using the same arguments as in Case 1, we have an element  $y \in G$  such that  $\langle a \rangle^y$  is not  $\mathfrak{3}$ -permutable subgroup of  $G$ . By hypothesis, there exists a subnormal subgroup  $R$  of  $G$  such that  $G = \langle a \rangle^y R$  and  $\langle a \rangle^y \cap R = 1$  or  $\langle a^2 \rangle^y$ . Clearly,  $R^{y^{-1}}$  is a subnormal subgroup of  $G$ . If  $\langle a \rangle^y \cap R = 1$ , then  $G = \langle a \rangle R^{y^{-1}}$  and  $\langle a \rangle \cap R^{y^{-1}} = 1$ . Consequently,  $\langle a \rangle$  is a weakly  $S$ -permutable subgroup in  $G$ . So, we may assume that  $\langle a \rangle^y \cap R = \langle a^2 \rangle^y$ . By using the same arguments as above,  $\langle a^2 \rangle^y$  is  $S$ -permutable in  $G$ . Therefore  $\langle a^2 \rangle$  is  $S$ -permutable in  $G$  by Lemma 2.4. This implies that  $G = \langle a \rangle R^{y^{-1}}$  and  $\langle a \rangle \cap R^{y^{-1}} = \langle a^2 \rangle$  is  $S$ -permutable in  $G$ . Thus  $\langle a \rangle$  is a weakly  $S$ -permutable subgroup in  $G$  which completes the proof of the case.  $\square$

**PROOF OF THEOREM 1.2.** Assume that the result is false and let  $G$  be a counterexample of minimal order. By Lemma 2.2(a) and repeated applications of Theorem 1.1,  $G$  has a Sylow tower of supersolvable type. Let  $q$  be the largest prime dividing the order of  $G$ . Clearly,  $G_q$  is a normal subgroup of  $G$  and  $q \neq 2$ . Let  $H/G_q$  be a cyclic subgroup of  $G_p G_q / G_q \in \mathfrak{3}G_q / G_q$  of order  $p$  or of order 4 (if  $p = 2$ ), where  $p \neq q$ . Then  $H/G_q = \langle xG_q \rangle = \langle x \rangle G_q / G_q \cong \langle x \rangle$  and so  $|x| = p$  or 4 (if  $p = 2$ ). Since  $(|\langle x \rangle|, |G_q|) = 1$  and  $\langle x \rangle$  is a weakly  $\mathfrak{3}$ -permutable subgroup of  $G$ , it follows, by Lemma 2.2(b), that  $H/G_q = \langle x \rangle G_q / G_q$  is a weakly  $\mathfrak{3}G_q / G_q$ -permutable subgroup of  $G/G_q$ . So, the cyclic subgroups of  $G_p G_q / G_q$  of prime order or of order 4 (if  $p = 2$ ) are weakly  $\mathfrak{3}G_q / G_q$ -permutable subgroups of  $G/G_q$ , for all  $G_p G_q / G_q \in \mathfrak{3}G_q / G_q$ . Therefore  $G/G_q$  satisfies the hypothesis of the theorem and hence  $G/G_q$  is supersolvable by the minimal choice of  $G$ . By hypothesis and Lemma 3.1, every subgroup of  $G_q$  of order  $q$  is a weakly  $S$ -permutable subgroup in  $G$ . Applying Lemma 2.5 yields  $G$  is supersolvable, a

contradiction completing the proof of the theorem. □

PROOF OF THEOREM 1.3. (a)  $\Rightarrow$  (b). If  $G \in \mathfrak{F}$ , then (b) is true with  $H = 1$ .

(b)  $\Rightarrow$  (a). By Lemma 2.2(a), the cyclic subgroups of  $G_p \cap F^*(H)$  of prime order or of order 4 (if  $p = 2$ ) are weakly  $\mathfrak{3} \cap F^*(H)$ -permutable subgroups of  $F^*(H)$ , for all  $G_p \cap F^*(H) \in \mathfrak{3} \cap F^*(H)$ . Thus  $F^*(H)$  is supersolvable by Theorem 1.2 and hence  $F^*(H) = F(H)$  by Lemma 2.3. Consequently, the cyclic subgroups of the Sylow subgroups of  $F^*(H) = F(H)$  of prime order or of order 4 (if  $p = 2$ ) are weakly  $S$ -permutable subgroups in  $G$  by hypothesis and Lemma 3.1. Applying Lemma 2.5 yields  $G \in \mathfrak{F}$ . This completes the proof of the theorem. □

The following corollaries are immediate consequences of Theorem 1.3:

**Corollary 3.2.** *Let  $\mathfrak{U}$  be the class of supersolvable groups and let  $\mathfrak{3}$  be a complete set of Sylow subgroups of a group  $G$ . Then the following are equivalent:*

- (a)  $G \in \mathfrak{U}$ .
- (b) *There is a normal subgroup  $H$  of  $G$  such that  $G/H \in \mathfrak{U}$  and the cyclic subgroups of  $G_p \cap F^*(H)$  of prime order  $p$  or of order 4 (if  $p = 2$ ) are weakly  $\mathfrak{3}$ -permutable subgroups of  $G$ , for all  $G_p \in \mathfrak{3}$ .*

**Corollary 3.3.** *Let  $G$  be a group, and let  $\mathfrak{3}$  be a complete set of Sylow subgroups of  $G$ . If the subgroup generated by every element of prime order or of order 4 of  $F^*(G)$  is a weakly  $\mathfrak{3}$ -permutable in  $G$ , then  $G$  is supersolvable.*

**Corollary 3.4.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $\mathfrak{3}$  be a complete set of Sylow subgroups of a group  $G$ . Then the following are equivalent:*

- (a)  $G \in \mathfrak{F}$ .
- (b) *There is a solvable normal subgroup  $H$  of  $G$  such that  $G/H \in \mathfrak{F}$  and the cyclic subgroups of the Sylow subgroups of  $F(H)$  of prime order or of order 4 (if  $p = 2$ ) are weakly  $\mathfrak{3}$ -permutable subgroups of  $G$ , for all  $G_p \in \mathfrak{3}$ .*

**Corollary 3.5.** *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $\mathfrak{3}$  be a complete set of Sylow subgroups of a group  $G$ . Then the following are equivalent:*

- (a)  $G \in \mathfrak{F}$ .
- (b) *There is a normal subgroup  $H$  of  $G$  such that  $G/H \in \mathfrak{F}$  and the cyclic subgroups of  $G_p \cap H$  of prime order or of order 4 (if  $p = 2$ ) are weakly  $\mathfrak{3}$ -permutable subgroups of  $G$ , for all  $G_p \in \mathfrak{3}$ .*

#### 4. Some applications

In this section, we list some well-known results from the literature which may be considered as special cases of our main theorems.

**Corollary 4.1** ([15, Theorem 3.2(iii)]). *Let  $p$  be the smallest prime dividing the order of a group  $G$  and let  $G_p$  be a Sylow  $p$ -subgroup of  $G$ . If the cyclic subgroups of  $G_p$  of order  $p$  or of order 4 (if  $p = 2$ ) are  $S$ -permutable in  $G$ , then  $G$  is  $p$ -nilpotent.*

**Corollary 4.2** ([15, Theorem 3.4(iii)]). *Let  $G$  be a group with a normal subgroup  $H$  such that  $G/H$  is supersolvable. If the cyclic subgroups of  $H$  of prime order  $p$  or of order 4 (if  $p = 2$ ) are  $S$ -permutable in  $G$ , then  $G$  is supersolvable.*

**Corollary 4.3** ([13, Theorem 3.1]). *Let  $G$  be a group with a normal subgroup  $H$  such that  $G/H$  is supersolvable. If the cyclic subgroups of  $F^*(H)$  of prime order  $p$  or of order 4 (if  $p = 2$ ) are  $S$ -permutable in  $G$ , then  $G$  is supersolvable.*

**Corollary 4.4** ([1, Theorem 1]). *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$ . Suppose that  $G$  is a group with a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and the cyclic subgroups of  $H$  of prime order  $p$  or of order 4 (if  $p = 2$ ) are  $S$ -permutable in  $G$ , then  $G \in \mathfrak{F}$ .*

**Corollary 4.5** ([2, Theorem]). *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a solvable normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and the cyclic subgroups of  $F(H)$  of prime order  $p$  or of order 4 (if  $p = 2$ ) are  $S$ -permutable in  $G$ .*

**Corollary 4.6** ([13, Theorem 3.3]). *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and the cyclic subgroups of  $F^*(H)$  of prime order  $p$  or of order 4 (if  $p = 2$ ) are  $S$ -permutable in  $G$ .*

**Corollary 4.7** ([6, Theorem 3.2]). *Let  $\mathfrak{Z}$  be a complete set of Sylow subgroups of a group  $G$ . If the cyclic subgroups of  $G_p$  of prime order or of order 4 (if  $p = 2$ ) are  $\mathfrak{Z}$ -permutable subgroups of  $G$ , for all  $G_p \in \mathfrak{Z}$ , then  $G$  is supersolvable.*

**Corollary 4.8** ([6, Theorem 3.3]). *Let  $\mathfrak{Z}$  be a complete set of Sylow subgroups of a group  $G$ . Then  $G$  is supersolvable if and only if  $G$  has a normal subgroup  $H$  such that  $G/H$  is supersolvable and the cyclic subgroups of  $G_p \cap F^*(H)$  of prime order or of order 4 (if  $p = 2$ ) are  $\mathfrak{Z}$ -permutable subgroups of  $G$ , for all  $G_p \in \mathfrak{Z}$ .*



**Corollary 4.9** ([6, Main Theorem]). *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $\mathfrak{Z}$  be a complete set of Sylow subgroups of a group  $G$ . Then  $G \in \mathfrak{F}$  if and only if  $G$  has a normal subgroup  $H$  of  $G$  such that  $G/H \in \mathfrak{F}$  and the cyclic subgroups of  $G_p \cap F^*(H)$  of prime order or of order 4 (if  $p = 2$ ) are  $\mathfrak{Z}$ -permutable subgroups of  $G$ , for all  $G_p \in \mathfrak{Z}$ .*

**Corollary 4.10** ([14, Lemma 3.8]). *Let  $p$  be the smallest prime dividing the order of a group  $G$  and let  $G_p$  be a Sylow  $p$ -subgroup of  $G$ . If the cyclic subgroups of  $G_p$  of order  $p$  or of order 4 (if  $p = 2$ ) are  $c$ -normal in  $G$ , then  $G$  is  $p$ -nilpotent.*

**Corollary 4.11** ([17, Theorem 4.2]). *Let  $G$  be a group such that all of its cyclic subgroups of prime order  $p$  or of order 4 (if  $p = 2$ ) are  $c$ -normal in  $G$ , then  $G$  is supersolvable.*

**Corollary 4.12** ([10, Theorem 3.4]). *Let  $G$  be a group with a normal subgroup  $H$  such that  $G/H$  is supersolvable. If the cyclic subgroups of  $H$  of prime order  $p$  or of order 4 (if  $p = 2$ ) are  $c$ -normal in  $G$ , then  $G$  is supersolvable.*

**Corollary 4.13** ([11, Theorem 3]). *Let  $G$  be a solvable group of odd order with a normal subgroup  $H$  such that  $G/H$  is supersolvable. If the subgroups of  $F(H)$  of prime order are  $c$ -normal in  $G$ , then  $G$  is supersolvable.*

**Corollary 4.14** ([18, Theorem 3.2]). *Let  $G$  be a group with a normal subgroup  $H$  such that  $G/H$  is supersolvable. If the cyclic subgroups of  $F^*(H)$  of prime order  $p$  or of order 4 (if  $p = 2$ ) are  $c$ -normal in  $G$ , then  $G$  is supersolvable.*

**Corollary 4.15** ([14, Theorem 3.9]). *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$  and let  $G$  be a group. Then  $G \in \mathfrak{F}$  if and only if  $G$  has a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$  and the cyclic subgroups of  $H$  of prime order  $p$  or of order 4 (if  $p = 2$ ) are  $c$ -normal in  $G$ .*

**Corollary 4.16** ([19, Theorem 2]). *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$ . Suppose that  $G$  is a group with a solvable normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$ . If the cyclic subgroups of  $F(H)$  of prime order  $p$  or of order 4 (if  $p = 2$ ) are  $c$ -normal in  $G$ , then  $G \in \mathfrak{F}$ .*

**Corollary 4.17** ([20, Theorem 3.2]). *Let  $\mathfrak{F}$  be a saturated formation containing the class of supersolvable groups  $\mathfrak{U}$ . Suppose that  $G$  is a group with a normal subgroup  $H$  such that  $G/H \in \mathfrak{F}$ . If the cyclic subgroups of  $F^*(H)$  of prime order  $p$  or of order 4 (if  $p = 2$ ) are  $c$ -normal in  $G$ , then  $G \in \mathfrak{F}$ .*

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