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Incomplete poly-Bernoulli numbers associated with incomplete Stirling numbers

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Abstract. By using the associated and restricted Stirling numbers of the second kind, we give some generalizations of the poly-Bernoulli numbers. We also study their analytic and combinatorial properties. As an application, at the end of the paper we present a new infinite series representation of the Riemann zeta function via the Lambert W.

1. Introduction

Let $\mu \geq 1$ be an integer in the whole text. Our goal is to generalize the following relation for the poly-Bernoulli numbers $B_n^{(\mu)}$ ([Kan, Theorem 1]):

$$B_n^{(\mu)} = \sum_{k=0}^{n} (-1)^{n-k} \frac{k!}{(k+1)^\mu} \left\{ {n \atop k} \right\} \quad (n \geq 0,\ \mu \geq 1)\,,$$

where $\left\{ {n \atop k} \right\}$ are the Stirling numbers of the second kind, determined by

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^i \binom{k}{j} (k-j)^n$$

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(see e.g., [Jor]). When $\mu = 1$, $B_n^{(1)}$ are the classical Bernoulli numbers, defined by the generating function

$$\frac{x}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(1)} \frac{x^n}{n!} . \tag{1}$$

Notice that the classical Bernoulli numbers B_n are also defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} ,$$

satisfying $B_n^{(1)} = B_n$ $(n \neq 1)$ with $B_1^{(1)} = 1/2 = -B_1$.

The generating function of the poly-Bernoulli numbers $B_n^{(\mu)}$ is given by

$$\frac{\mathrm{Li}_\mu\bigl(1-e^{-x}\bigr)}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(\mu)} \frac{x^n}{n!} , \tag{2}$$

where

$$\mathrm{Li}_\mu(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^\mu}$$

is the μ-th polylogarithm function ([Kan, (1)]). The generating function of the poly-Bernoulli numbers can also be written in terms of iterated integrals ([Kan, (2)]):

$$e^x \cdot \underbrace{\frac{1}{e^x-1}\int_0^x \frac{1}{e^x-1}\int_0^x \cdots \frac{1}{e^x-1}\int_0^x}_{\mu-1} \frac{x}{e^x-1} \underbrace{dxdx\ldots dx}_{\mu-1} = \sum_{n=0}^{\infty} B_n^{(\mu)} \frac{x^n}{n!} . \tag{3}$$

Several generalizations of the poly-Bernoulli numbers have been considered ([BayHam1], [BayHam2], [CopCan], [Jol], [Sas]). However, most kinds of generalizations are based upon the generating functions of (1) and/or (2). On the contrary, our generalizations are based upon the explicit formula in terms of the Stirling numbers. In [KomMezSza], a similar approach is used to generalize the Cauchy numbers c_n, defined by $x/\log(1+x) = \sum_{n=0}^{\infty} c_n x^n/n!$. In this paper, by using the associated and restricted Stirling numbers of the second kind, we give substantial generalizations of the poly-Bernoulli numbers. One of the main results is to generalize the formula in (2) as

$$\sum_{n=0}^{\infty} B_{n,\le m}^{(\mu)} \frac{t^n}{n!} = \frac{\mathrm{Li}_\mu\bigl(1-E_m(-t)\bigr)}{1-E_m(-t)}$$

and

$$\sum_{n=0}^\infty B_{n,\ge m}^{(\mu)}\frac{t^n}{n!}=\frac{\mathrm{Li}_\mu\bigl(E_{m-1}(-t)-e^{-t}\bigr)}{E_{m-1}(-t)-e^{-t}}\,,$$

where $E_m(t)=\sum_{k=0}^m\frac{t^k}{k!}$. See Theorem 1 below.

2. Incomplete Stirling numbers of the second kind

In place of the classical Stirling numbers of the second kind $\left\{ n \atop k \right\}$ we substitute the restricted Stirling numbers and the associated Stirling numbers

$$\left\{ n \atop k \right\}_{\le m}\quad\text{and}\quad\left\{ n \atop k \right\}_{\ge m}\,,$$

respectively. Some combinatorial and modular properties of these numbers can be found in [Mez], and other properties can be found in the cited papers of [Mez]. The generating functions of these numbers are given by

$$\sum_{n=k}^{mk}\left\{ n \atop k \right\}_{\le m}\frac{x^n}{n!}=\frac{1}{k!}(E_m(x)-1)^k \tag{4}$$

and

$$\sum_{n=mk}^{\infty}\left\{ n \atop k \right\}_{\ge m}\frac{x^n}{n!}=\frac{1}{k!}\bigl(e^x-E_{m-1}(x)\bigr)^k \tag{5}$$

respectively, where

$$E_m(t)=\sum_{k=0}^m\frac{t^k}{k!}$$

is the mth partial sum of the exponential function sum. These give the number of the k-partitions of an n-element set, such that each block contains at most or at least m elements, respectively. Since the generating function of $\left\{ n \atop k \right\}$ is given by

$$\sum_{n=k}^{\infty}\left\{ n \atop k \right\}\frac{x^n}{n!}=\frac{(e^x-1)^k}{k!}$$

(see e.g., [Jor]), by $E_\infty(x)=e^x$ and $E_0(x)=1$, we have

$$\left\{ n \atop k \right\}_{\le\infty}=\left\{ n \atop k \right\}_{\ge 1}=\left\{ n \atop k \right\}.$$

These give the number of the k-partitions of an n-element set, such that each block contains at most or at least m elements, respectively. Notice that these numbers where $m = 2$ have been considered by several authors (e.g., [Com], [How], [Rio], [Zha]).

It is well-known that the Stirling numbers of the second kind satisfy the recurrence relation:

$$\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\} \tag{6}$$

for $k > 0$, with the initial conditions

$$\left\{ {0 \atop 0} \right\} = 1 \quad \text{and} \quad \left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 0$$

for $n > 0$. The restricted and associated Stirling numbers of the second kind satisfy the similar relations. It is easy to see the initial conditions

$$\left\{ {0 \atop 0} \right\}_{\le m} = 1 \quad \text{and} \quad \left\{ {n \atop 0} \right\}_{\le m} = \left\{ {0 \atop n} \right\}_{\le m} = 0\,,$$
$$\left\{ {0 \atop 0} \right\}_{\ge m} = 1 \quad \text{and} \quad \left\{ {n \atop 0} \right\}_{\ge m} = \left\{ {0 \atop n} \right\}_{\ge m} = 0$$

for $n > 0$.

Proposition 1. *For $k > 0$ we have*

$$\left\{ {n+1 \atop k} \right\}_{\le m} = \sum_{i=0}^{m-1} \binom{n}{i} \left\{ {n-i \atop k-1} \right\}_{\le m} \tag{7}$$

$$= k \left\{ {n \atop k} \right\}_{\le m} + \left\{ {n \atop k-1} \right\}_{\le m} - \binom{n}{m} \left\{ {n-m \atop k-1} \right\}_{\le m}, \tag{8}$$

$$\left\{ {n+1 \atop k} \right\}_{\ge m} = \sum_{i=m-1}^{n} \binom{n}{i} \left\{ {n-i \atop k-1} \right\}_{\ge m} \tag{9}$$

$$= k \left\{ {n \atop k} \right\}_{\ge m} + \binom{n}{m-1} \left\{ {n-m+1 \atop k-1} \right\}_{\ge m}. \tag{10}$$

Remark. The fourth relation (10) appeared in a different form in [How]. Since

$$\sum_{i=1}^{n-k+1} \binom{n}{i} \left\{ {n-i \atop k-1} \right\} = k \left\{ {n \atop k} \right\},$$

the relation (7) and the relation (9) are both reduced to the relation (6), if $m \ge n - k + 2$ and if $m = 1$, respectively. It is trivial to see that the relations (8) and (10) are also reduced to the original relation (6), if $m > n$ and if $m = 1$, respectively.

Proof of Proposition 1. The combinatorial proofs of the previous theorem are given as follows. We shall give combinatorial proofs. First, identity (7). To construct a partition with k blocks on $n+1$ element we can do the following. The last element in its block can have i elements by side, where $i = 0, 1, \dots, m-1$. We have to choose these i elements from n. This can be done in $\binom{n}{i}$ ways. The rest of the elements go into $k-1$ blocks in $\left\{ {n-i \atop k-1} \right\}_{\le m}$ ways. Summing over the possible values of i we are done.

The proof of (8). The above construction can be described in another way: the last element we put into a singleton and the other n elements must form a partition with $k-1$ blocks: $\left\{ {n \atop k-1} \right\}_{\le m}$ possibilities. Or we put this element into one existing block after constructing a partition of n elements into k blocks. This offers us $k \left\{ {n \atop k} \right\}_{\le m}$ possibilities, but we must subtract the possibilities when we exceed the block size limit m. This happens if we put the last element into a block of m elements. There are $\binom{n}{m} \left\{ {n-m \atop k-1} \right\}_{\le m}$ such partitions in total. The proof is done.

The proof of (9) and (10) is similar. □

Note that the classical Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ satisfy the identities:

$$\left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = 1, \quad \left\{ {n \atop n-1} \right\} = \binom{n}{2},$$
$$\left\{ {n \atop n-2} \right\} = \frac{3n-5}{4}\binom{n}{3}, \quad \left\{ {n \atop n-3} \right\} = \binom{n}{4}\binom{n-2}{2},$$
$$\left\{ {n \atop 2} \right\} = 2^{n-1} - 1, \quad \left\{ {n \atop 3} \right\} = \frac{3^{n-1}}{2} - 2^{n-1} + \frac{1}{2},$$
$$\left\{ {n \atop 4} \right\} = \frac{4^{n-1}}{6} - \frac{3^{n-1}}{2} + 2^{n-2} - \frac{1}{6}\,.$$

By the definition (4) or Proposition 1 (7), we list several basic properties about the restricted Stirling numbers of the second kind. Some basic properties about the associated Stirling numbers of the second kind can be found in [Com], [How], [Mez], [Zha].

Lemma 1. *For $0 \le n \le k-1$ or $n \ge mk+1$, we have*

$$\left\{ {n \atop k} \right\}_{\le m} = 0\,. \tag{11}$$

For $k \le n \le mk$, we have

$$\left\{ {n \atop k} \right\}_{\le m} = \left\{ {n \atop k} \right\} \quad (k \le n \le m)\,, \tag{12}$$

$$\left\{ {n \atop n} \right\}_{\le m} = 1 \quad (n \ge 0,\ m \ge 1)\,, \tag{13}$$

$$\left\{ {n \atop n-1} \right\}_{\le m} = \binom{n}{2} \quad (n \ge 2,\ m \ge 2)\,, \tag{14}$$

$$\left\{ {n \atop n-2} \right\}_{\le m} = \begin{cases} \frac{3n-5}{4}\binom{n}{3} & (n \ge 4,\ m \ge 3); \\ 3\binom{n}{4} & (n \ge 4,\ m = 2)\,, \end{cases} \tag{15}$$

$$\left\{ {n \atop n-3} \right\}_{\le m} = \begin{cases} \binom{n}{4}\binom{n-2}{2} & (n \ge 4,\ m \ge 4); \\ 15\binom{n}{6} + 10\binom{n}{5} & (n \ge 4,\ m = 3); \\ 15\binom{n}{6} & (n \ge 4,\ m = 2)\,, \end{cases} \tag{16}$$

$$\left\{ {n \atop 1} \right\}_{\le m} = 1 \quad (1 \le n \le m)\,, \tag{17}$$

$$\left\{ {n \atop 2} \right\}_{\le m} = 2^{n-1} - 1 \quad (2 \le n \le m+1)\,. \tag{18}$$

Proof. Some of the above special values are trivial. Some of them can be proven by analyzing the possible block structures.

We take (15) as a concrete example.

Let $m = 2$, and the number of blocks be $k = n - 2$. Then for the block structure we have the only one possibility

$$\underbrace{.\,|\,.\,|\cdots|\,.\,|}_{n-4}\,..\,|\,..$$

That is, there are $n-4$ singletons and two blocks of length 2. There are $\frac{1}{2}\binom{4}{2}\binom{n}{4} = 3\binom{n}{4}$ such partitions: we have to choose those four elements going to the non singleton blocks in $\binom{n}{4}$ ways. Then we put two of four into the first block and the other two goes to the other: $\binom{4}{2} = 6$ cases. Finally, we have to divide by two because the order of the blocks does not matter. The last case of (15) follows.

If $m = 3$ then we have one more possible distribution of blocks sizes apart from the above:

$$\underbrace{.\,|\,.\,|\cdots|\,.\,|}_{n-5}\,...$$

Into the last block we have $\binom{n}{3}$ possible option to put 3 elements. So if $m = 3$ and $k = n - 2$ then we have $\binom{n}{3} + 3\binom{n}{4} = \frac{3n-5}{4}\binom{n}{3}$ cases in total.

The rest of the cases can be treated similarly. □

3. Incomplete poly-Bernoulli numbers

3.1. Generating function and its integral representation. By using two types of incomplete Stirling numbers, define *restricted poly-Bernoulli numbers* $B_{n,\le m}^{(\mu)}$ and *associated poly-Bernoulli numbers* $B_{n,\ge m}^{(\mu)}$ by

$$B_{n,\le m}^{(\mu)} = \sum_{k=0}^{n}(-1)^{n-k}\frac{k!}{(k+1)^\mu}\left\{ {n \atop k} \right\}_{\le m} \quad (n \ge 0), \tag{19}$$

and

$$B_{n,\ge m}^{(\mu)} = \sum_{k=0}^{n}(-1)^{n-k}\frac{k!}{(k+1)^\mu}\left\{ {n \atop k} \right\}_{\ge m} \quad (n \ge 0)\,, \tag{20}$$

respectively. These numbers can be considered as generalizations of the usual poly-Bernoulli numbers $B_n^{(\mu)}$, since

$$B_{n,\le\infty}^{(\mu)} = B_{n,\ge 1}^{(\mu)} = B_n^{(\mu)}\,.$$

We call these numbers as *incomplete poly-Bernoulli numbers*.

One can deduce that these numbers have the generating functions.

Theorem 1. *We have*

$$\sum_{n=0}^{\infty} B_{n,\le m}^{(\mu)}\frac{t^n}{n!} = \frac{\mathrm{Li}_\mu\big(1-E_m(-t)\big)}{1-E_m(-t)} \tag{21}$$

and

$$\sum_{n=0}^{\infty} B_{n,\ge m}^{(\mu)}\frac{t^n}{n!} = \frac{\mathrm{Li}_\mu\big(E_{m-1}(-t)-e^{-t}\big)}{E_{m-1}(-t)-e^{-t}}\,. \tag{22}$$

Remark. In the first formula $m \to \infty$ gives back the poly-Bernoulli numbers (2) since $E_\infty(-t) = e^{-t}$ and $\mathrm{Li}_1(z) = -\log(1-z)$, while in the second we must take $m = 1$ since $E_0(-t) = 1$.

Proof of Theorem 1. By the definition of (20) and using (4), we get

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\le m}^{(\mu)}\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left\{ {n \atop k} \right\}_{\le m}\frac{(-1)^{n-k}k!}{(k+1)^\mu}\frac{t^n}{n!}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k k!}{(k+1)^\mu}\sum_{n=k}^{\infty}\left\{ {n \atop k} \right\}_{\le m}\frac{(-t)^n}{n!}\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(k+1)^\mu} \frac{1}{k!} \left((-t) + \frac{(-t)^2}{2!} + \cdots + \frac{(-t)^m}{m!} \right)^k$$

$$= \sum_{k=0}^{\infty} \frac{\bigl(1 - E_m(-t)\bigr)^k}{(k+1)^\mu} = \frac{\mathrm{Li}_\mu\bigl(1 - E_m(-t)\bigr)}{1 - E_m(-t)} .$$

Similarly, by the definition of (19) and using (5), we get

$$\sum_{n=0}^{\infty} B_{n,\ge m}^{(\mu)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\le m} \frac{(-1)^{n-k} k!}{(k+1)^\mu} \frac{t^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(k+1)^\mu} \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\}_{\ge m} \frac{(-t)^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(k+1)^\mu} \frac{1}{k!} \left(\frac{(-t)^m}{m!} + \frac{(-t)^{m+1}}{(m+1)!} + \cdots \right)^k$$

$$= \sum_{k=0}^{\infty} \frac{\bigl(E_{m-1}(-t) - e^{-t}\bigr)^k}{(k+1)^\mu}$$

$$= \frac{\mathrm{Li}_\mu\bigl(E_{m-1}(-t) - e^{-t}\bigr)}{E_{m-1}(-t) - e^{-t}} . \qquad \square$$

For $\mu \ge 1$, the generating functions can be written in the form of iterated integrals. We set $E_{-1}(-t) = 0$ for convenience.

Theorem 2.

$$\frac{1}{1 - E_m(-t)} \cdot \underbrace{\int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \int_0^t \cdots \frac{E_{m-1}(-t)}{1 - E_m(-t)} \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)}}_{\mu-1}$$

$$\times \bigl(-\log\bigl(E_m(-t)\bigr)\bigr) \underbrace{dt \dots dt}_{\mu-1} = \sum_{n=0}^{\infty} B_{n,\le m}^{(\mu)} \frac{x^n}{n!} , \tag{23}$$

$$\frac{1}{E_{m-1}(-t) - e^{-t}} \cdot \underbrace{\int_0^t \frac{e^{-t} - E_{m-2}(-t)}{E_{m-1}(-t) - e^{-t}} \int_0^t \cdots \frac{e^{-t} - E_{m-2}(-t)}{E_{m-1}(-t) - e^{-t}} \int_0^t \frac{e^{-t} - E_{m-2}(-t)}{E_{m-1}(-t) - e^{-t}}}_{\mu-1}$$

$$\times \bigl(-\log\bigl(1 + e^{-t} - E_{m-1}(-t)\bigr)\bigr) \underbrace{dt \dots dt}_{\mu-1} = \sum_{n=0}^{\infty} B_{n,\ge m}^{(\mu)} \frac{x^n}{n!} . \tag{24}$$

Remark. If $m \to \infty$ in (23), by $E_\infty(-t) = e^{-t}$, and if $m = 1$ in (24), by $E_0(-t) = 1$ and $E_{-1}(-t) = 0$, both of them are reduced to (3).

Proof of Theorem 2. Since for $\mu \ge 1$

$$\frac{d}{dt}\mathrm{Li}_\mu\bigl(1-E_m(-t)\bigr)=\frac{E_{m-1}(-t)}{1-E_m(-t)}\mathrm{Li}_{\mu-1}\bigl(1-E_m(-t)\bigr)\,,$$

we have

$$\begin{aligned}
\mathrm{Li}_\mu\bigl(1-E_m(-t)\bigr)&=\int_0^t\frac{E_{m-1}(-t)}{1-E_m(-t)}\mathrm{Li}_{\mu-1}\bigl(1-E_m(-t)\bigr)dt\\
&=\int_0^t\frac{E_{m-1}(-t)}{1-E_m(-t)}\int_0^t\frac{E_{m-1}(-t)}{1-E_m(-t)}\mathrm{Li}_{\mu-2}\bigl(1-E_m(-t)\bigr)dtdt\\
&=\int_0^t\frac{E_{m-1}(-t)}{1-E_m(-t)}\int_0^t\frac{E_{m-1}(-t)}{1-E_m(-t)}\cdots\int_0^t\frac{E_{m-1}(-t)}{1-E_m(-t)}\mathrm{Li}_1\bigl(1-E_m(-t)\bigr)\underbrace{dt\cdots dt}_{\mu-1}\\
&=\int_0^t\frac{E_{m-1}(-t)}{1-E_m(-t)}\int_0^t\frac{E_{m-1}(-t)}{1-E_m(-t)}\cdots\int_0^t\frac{E_{m-1}(-t)}{1-E_m(-t)}\bigl(-\log\bigl(E_m(-t)\bigr)\bigr)\underbrace{dt\cdots dt}_{\mu-1}\,.
\end{aligned}$$

Therefore, we obtain (23). Similarly, by

$$\frac{d}{dt}\mathrm{Li}_\mu\bigl(E_{m-1}(-t)-e^{-t}\bigr)=\frac{e^{-t}-E_{m-2}(-t)}{E_{m-1}(-t)-e^{-t}}\mathrm{Li}_{\mu-1}\bigl(E_{m-1}(-t)-e^{-t}\bigr)\,,$$

we obtain (24). □

If $\mu=1$ in Theorem 1 or in Theorem 2, the generating functions of the *restricted Bernoulli numbers* $B_{n,\le m}^{(1)}$ and *associated Bernoulli numbers* $B_{n,\ge m}^{(1)}$ are given. Both functions below are reduced to the generating function (1) of the Bernoulli numbers $B_n^{(1)}$ if $m\to\infty$ and $m=1$, respectively.

Corollary 1. *We have*

$$\sum_{n=0}^\infty B_{n,\le m}^{(1)}\frac{t^n}{n!}=\frac{\log E_m(-t)}{E_m(-t)-1}$$

and

$$\sum_{n=0}^\infty B_{n,\ge m}^{(1)}\frac{t^n}{n!}=\frac{\log\bigl(1+e^{-t}-E_{m-1}(-t)\bigr)}{e^{-t}-E_{m-1}(-t)}\,.$$

3.2. Basic divisibility for non-positive μ. In this short subsection we deduce a basic divisibility property for both the restricted and associated poly-Bernoulli numbers.

It is known [GraKnuPat] that

$$\left\{ {p \atop k} \right\} \equiv 0 \pmod p \quad (1 < k < p)$$

for any prime p. The proof of this basic divisibility is the same for the restricted and associated Stirling numbers, so we can state that

$$\left\{ {p \atop k} \right\}_{\le m} \equiv 0 \pmod p \quad (k = 0, 1, \dots),$$

and

$$\left\{ {p \atop k} \right\}_{\ge m} \equiv 0 \pmod p \quad (k \ge 2).$$

(Note that $\left\{ {p \atop 1} \right\}_{\ge m} = 1$.) These immediately lead to the next statement.

Theorem 3. *For any $\mu \le 0$ we have that*

$$\begin{aligned} B_{p,\le m}^{(\mu)} &\equiv 0 \pmod p, \\ B_{p,\ge m}^{(\mu)} &\equiv 2^{|\mu|} \pmod p \end{aligned}$$

hold for any prime p.

4. A new series representation for the Riemann zeta function

To present our result, we need to recall the definition of the Lambert W function. $W(a)$ is the solution of the equation

$$xe^x = a,$$

that is, $W(a)e^{W(a)} = a$. Since this equation, in general, has infinitely many solutions, the W function has infinitely many complex branches denoted by $W_k(a)$ where $k \in \mathbb{Z}$. What we prove is the following:

Theorem 4. *For any $\mu \in \mathbb{C}$ with $\Re(\mu) > 1$ we have that*

$$\zeta(\mu) = \sum_{n=0}^{\infty} B_{n,\ge 2}^{(\mu)} \frac{(W_k(-1))^n}{n!}$$

for $k = 0, -1$, where ζ is the Riemann zeta function.

Proof. Let us recall the generating function of $B^{(\mu)}_{n,\geq m}$ in the particular case when $m = 2$:

$$\sum_{n=0}^{\infty} B^{(\mu)}_{n,\geq 2}\frac{(-t)^n}{n!} = \frac{\mathrm{Li}_\mu(1+t-e^t)}{1+t-e^t}. \tag{25}$$

By a simple transformation it can be seen that the equation $1+t-e^t = 1$ is solvable in terms of the Lambert W function, and that the solution is $-W_k(-1)$ for any branch $k \in \mathbb{Z}$. However, (25) is valid only for t such that $|1+t-e^t| \leq 1$, at least when $\Re(\mu) > 1$. (This comes from the proof of Theorem 1.) Since the absolute value of $-W_k(-1)$ grows with k, the only two branches which belong the convergence domain of (25) is $k = -1, 0$. Hence, substituting one of these in place of t we have that

$$\sum_{n=0}^{\infty} B^{(\mu)}_{n,\geq 2}\frac{(W_k(-1))^n}{n!} = \frac{\mathrm{Li}_\mu(1-W_k(-1)-e^{-W_k(-1)})}{1-W_k(-1)-e^{-W_k(-1)}} = \frac{\mathrm{Li}_\mu(1)}{1} = \zeta(\mu). \qquad \square$$

Note that

$$W_0(-1) = \overline{W_{-1}(-1)} \approx -0.318132 + 1.33724i,$$

so all the terms in the incomplete Bernoulli sum are complex, but the sum itself always converges to the real number $\zeta(\mu)$.

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