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Quadratic Lie triple systems admitting symplectic structures

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Abstract. The aim of this article is to determine quadratic symplectic Lie triple systems, which are Lie triple systems admitting both quadratic and symplectic structures, by T^*-extensions of Lie triple systems.

1. Introduction

The symplectic structures on Lie algebras and quadratic Lie algebras attract the interest in many fields of mathematic and physics. Recently, the symplectic structures on quadratic Lie algebras have been studied by Bajo, Benayadi and Medina in [BBM]. We know that quadratic symplectic Lie algebras are the Lie algebras of Lie Groups which admit a bi-invariant pseudo-Riemannian metric and a left-invariant symplectic form. If the symplectic form on the Lie group is viewed as a solution r of the classical Yang–Baxter equation, then the Poisson–Lie tensor $\pi = r^+ - r^-$ and the geometry of the double Lie groups $D(r)$ can be well described ([DM]).

Similar to the case of Lie algebras, we define symplectic structures on Lie triple systems based on scalar cohomology. Then we study the symplectic structures on quadratic Lie triple systems by means of T^*-extensions, which are studied in [LWD]. It is shown in [LWD] that every nilpotent quadratic Lie triple system over an algebraically closed field of characteristic 0 is either a T^*-extension or a non-degenerate ideal of codimension 1 in a T^*-extension of some Lie triple system.

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In general, a T^*-extension of a Lie triple system need not admit a non-degenerate scalar 2-cocycle, even if the extended system is nilpotent. Thus we need to impose some additional properties to a Lie triple system $\mathfrak{a}$ to ensure that a T^*-extension $T^*_\theta\mathfrak{a}$ might be furnished with a symplectic structure. This will be done in section 3, where we will show that if $\mathbb{K}$ is algebraically closed, then every quadratic symplectic Lie triple system is a T^*-extension of a Lie triple system which admits an invertible derivation and we give necessary and sufficient conditions on $\mathfrak{a}$ to assure that the extended system admits a skew-symmetric derivation, and hence a symplectic structure.

2. Preliminaries

Definition 2.1 ([L]). A *Lie triple system* is a vector space T over a field $\mathbb{K}$ with a trilinear multiplication $[a, b, c]$ satisfying

$$[a, a, b] = 0,$$

$$[a, b, c] + [b, c, a] + [c, a, b] = 0,$$

$$[a, b, [c, d, e]] = [[a, b, c], d, e] + [c, [a, b, d], e] + [c, d, [a, b, e]],$$

for any $a, b, c, d, e \in T$.

Definition 2.2 ([L]). A *derivation* of a Lie triple system T is a linear transformation D of T into T such that

$$D([x, y, z]) = [Dx, y, z] + [x, Dy, z] + [x, y, Dz],$$

for any $x, y, z \in T$. The set $\mathrm{Der}(T)$ of derivations of T is a Lie algebra of linear transformations, we call it the *derivation algebra* of T. Further, if $a_i, b_i, i = 1, 2, \cdots, n$ are arbitrary in T, then $x \mapsto \sum_i L(a_i, b_i)(x) = \sum_i [a_i, b_i, x]$ is a derivation of T. $L(T, T) = \{\sum_i L(a_i, b_i) | a_i, b_i \in T\}$ is a subalgebra of $\mathrm{Der}(T)$, we call it the *inner derivation algebra* of T, its element is called an *inner derivation* of T.

Definition 2.3 ([ZSZ]). A symmetric bilinear form f on a Lie triple system is said to be right-invariant(resp. left-invariant) if $f(R(a, b)x, y) = f(x, R(b, a)y)$ (resp. $f(L(a, b)x, y) = f(x, L(b, a)y)$) for all $x, y, a, b \in T$, where $L(a, b)x := [a, b, x]$ and $R(a, b)x := [x, a, b]$. Furthermore, f is said to be invariant if it is both right-invariant and left-invariant.

Definition 2.4. Let T be a Lie triple system over $\mathbb{K}$.

(1) We say that (T, B) is a *quadratic Lie triple system* if B is a non-degenerate symmetric invariant bilinear form on T. Here B is called a *quadratic structure* on T. A quadratic Lie triple system (T, B) is said to be *reducible (or B-reducible)* if it admits an ideal J such that the restriction of B to $J \times J$ is non-degenerate. Otherwise, we will say (T, B) is *irreducible*.

(2) We say that (T, ω) is a *symplectic Lie triple system* if ω is a non-degenerate skew-symmetric bilinear form on T such that the identity

$$\omega([x, y, z], a) - \omega(x, [a, z, y]) + \omega(y, [a, z, x]) - \omega(z, [y, x, a]) = 0$$

holds for any $x, y, z, a \in T$. The above identity means that ω is a non-degenerate 2-cocycle for the scalar cohomology of T which is defined in [LWD]. Note that in such case, T must be even-dimensional. Here ω is called a *symplectic structure* on T.

(3) We say that (T, B, ω) is a *quadratic symplectic Lie triple system* if (T, B) is quadratic and (T, ω) is symplectic.

Definition 2.5. An ideal of a symplectic Lie triple system (T, ω) is called *lagrangian* if and only if it coincides with its orthogonal complement with respect to ω.

Lemma 2.6 ([SM])**.** *Let T be a Lie triple system, D a derivation of T, $T = T_{\lambda_1}(D) + T_{\lambda_2}(D) + \cdots + T_{\lambda_s}(D)$ the decomposition of T by D. Then*

(1) $[T_{\lambda_i}(D), T_{\lambda_j}(D), T_{\lambda_k}(D)] \subseteq T_{\lambda_i+\lambda_j+\lambda_k}(D)$,

(2) $\mathrm{Der}(T)$ *contains the semisimple and nilpotent parts (in* $\mathrm{End}T$*) of all its elements.*

Lemma 2.7. *Let (T, B) be a quadratic Lie triple system. There is a symplectic structure on T if and only if there is an invertible derivation of T which is skew-symmetric with respect to B.*

Proof. Suppose that D is an invertible derivation of T which is skew-symmetric with respect to B. Define $\omega(x, y) := B(Dx, y)$. Since

$$\omega(x, y) = B(Dx, y) = -B(x, Dy) = -B(Dy, x) = -\omega(y, x),$$

we know that ω is skew-symmetric. Furthermore

$$\omega([x, y, z], a) - \omega(x, [a, z, y]) + \omega(y, [a, z, x]) - \omega(z, [y, x, a])$$

$$= B(D[x,y,z],a) - B([Dx,y,z],a) + B([Dy,x,z],a) - B([x,y,Dz],a = 0.$$

Notice that the first equality follows from the invariant property of the quadratic form of the quadratic Lie triple system. Namely ω is symplectic.

Suppose that ω is a symplectic structure on T. Define D by $\omega(x,y) = B(Dx,y)$ for any $x,y \in T$. Then D is a derivation of T. We have

$$B(Dx,y) = \omega(x,y) = -\omega(y,x) = -B(Dy,x) = -B(x,Dy).$$

That is, D is skew-symmetric with respect to B. Define D' by $\omega(D'x,y) = B(x,y)$ for any $x,y \in T$. Then

$$\omega(D'Dx,y) = B(Dx,y) = \omega(x,y), \quad B(DD'x,y) = \omega(D'x,y) = B(x,y).$$

By the non-degeneracy of ω and B, we get $D'D = DD' =$id. Thus D is invertible. □

Remark 2.8. The skew-symmetric derivation D of (T,B) in the above lemma is also skew-symmetric with respect to the symplectic form ω. In fact, for any $x,y \in T$,

$$\omega(Dx,y) = B(D^2x,y) = -B(Dx,Dy) = -\omega(x,Dy).$$

Example 2.9. Let $T = \mathbb{R}\{x_1,x_2,x_3,x_4\}$ be the 4-dimensional Lie triple system defined by

$$[x_1,x_2,x_4] = x_3.$$

For any symmetric invariant bilinear form B on T, we have

$$B(x_3,x_3) = B(x_3,x_1) = B(x_3,x_2) = B(x_3,x_4) = 0.$$

So there is no quadratic structure on T. The skew-symmetric bilinear form on T defined by

$$\omega(x_1,x_4) = \omega(x_2,x_3) = 1$$

gives a symplectic structure on T and the linear endomorphism of T defined by

$$D(x_1) = 2x_1, D(x_2) = -x_2, D(x_3) = x_3, D(x_4) = -2x_4$$

is an invertible skew-symmetric derivation of (T,ω).

Example 2.10. Let T be a Lie triple system and $n \in \mathbb{N}, n > 1$. Consider the non-unitary associative algebra $A_n = t\mathbb{K}[t]/t^n\mathbb{K}[t]$. Define the bracket on the vector space $T_n = T \otimes A_n$ by

$$[x \otimes \bar{t^p}, y \otimes \bar{t^q}, z \otimes \bar{t^r}] = [x, y, z]_T \otimes \overline{t^{p+q+r}},$$

where $x, y, z \in T, p, q, r \in \mathbb{N}/\{0\}$. Then T_n is a nilpotent Lie triple system. The endomorphism D of T_n defined by $D(x \otimes \bar{t^p}) = p(x \otimes \bar{t^p})$ for any $x \in T$ and $p \in \{1, \ldots, n-1\}$ is an invertible derivation of T_n. Define the bracket on the vector space $\tilde{T_n} = T_n \oplus (T_n)^*$ by

$$[X + f, Y + g, Z + h] = [X, Y, Z]_{T_n} + [f, Y, Z] + [X, g, Z] + [X, Y, h],$$

where, for any $X, Y, Z, W \in T_n, f, g, h \in (T_n)^*$,

$$[f, Y, Z](W) := f([W, Z, Y]), \quad [X, g, Z](W) := g([Z, W, X]),$$
$$[X, Y, h](W) := h([Y, X, W]).$$

Define a bilinear form on $\tilde{T_n}$ by

$$B(X + f, Y + g) = f(Y) + g(X).$$

Then $(\tilde{T_n}, B)$ is a quadratic Lie triple system. Define $\tilde{D}$ on $\tilde{T_n}$ by

$$\tilde{D}(X + f) = D(X) + D^* f, \quad \forall X \in T_n, f \in (T_n)^*,$$

where $D^*(f) = -f \circ D$. Then $\tilde{D}$ is an invertible derivation which is skew-symmetric with respect to B, hence the quadratic Lie triple system $(\tilde{T_n}, B)$ admits a symplectic structure.

3. Quadratic Lie triple systems with symplectic structures

Definition 3.1 ([LWD]). Let $\mathfrak{a}$ be a Lie triple system over a field $\mathbb{K}$ and $\mathfrak{a}^*$ its dual space. Consider a Yamaguti 3-cocycle (cf. [Y]) $\theta : \mathfrak{a} \times \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}^*$ and define the bracket on the vector space $T^*_\theta \mathfrak{a} = \mathfrak{a} \oplus \mathfrak{a}^*$ by

$$[x + f, y + g, z + h] = [x, y, z] + \theta(x, y, z) + [f, y, z] + [x, g, z] + [x, y, h],$$

where

$$[f, y, z](a) := f([a, z, y]), \quad [x, g, z](a) := g([z, a, x]), \quad [x, y, h](a) := h([y, x, a]),$$

for any $x, y, z, a \in \mathfrak{a}$, and $f, g, h \in \mathfrak{a}^*$. The pair $(T^*_\theta \mathfrak{a}, [\cdot, \cdot, \cdot])$ is a Lie triple system, which is called the T^*-*extension* of the Lie triple system $\mathfrak{a}$ by means of θ.

If, in addition, the 3-cocycle θ satisfies $\theta(x,y,z)(a) = \theta(a,z,y)(x)$ for any $x,y,z,a \in \mathfrak{a}$, then the symmetric bilinear form B on $T^*_\theta\mathfrak{a}$, given by $B(x+f,y+g) = f(y)+g(x)$ for any $x,y \in \mathfrak{a}$, $f,g \in \mathfrak{a}^*$, defines a quadratic structure on $T^*_\theta\mathfrak{a}$. Define $\theta^\sharp(a,x,y,z) := \theta(x,y,z)(a)$. Then it is proved in [LWD], p. 2080 that $\theta^\sharp$ is a scalar 4-cocycle of $\mathfrak{a}$ to $\mathbb{K}$.

As in the case of Lie algebras [J1], we have the similar result for Lie triple systems.

Theorem 3.2. *Let T be a finite dimensional Lie triple system over an algebraically closed field of characteristic zero and suppose that there exists a nilpotent subalgebra $\mathcal{D}$ of the algebra of derivations of T such that there are no nonzero $\mathcal{D}$-constants, i.e., $Dc = 0$ for any $D \in \mathcal{D}$ implies that $c = 0$. Then T is nilpotent.*

Proof. We denote the derivation algebra of T by $\mathfrak{D}$ and consider T as a module for $\mathfrak{D}$, hence for $\mathcal{D}$. If R denotes the representation of $\mathcal{D}$ in T and $\mathrm{ad}_{\mathfrak{D}}$ the adjoint representation in $\mathfrak{D}$, then the characteristic roots of D^R and $\mathrm{ad}_{\mathfrak{D}}D$, $D \in \mathcal{D}$, are in the base field $\mathbb{K}$ since $\mathbb{K}$ is algebraically closed, that is to say, $\mathcal{D}^R$ and $\mathrm{ad}_{\mathfrak{D}}\mathcal{D}$ are split Lie algebras of linear transformations. If $D^R \to \rho(D^R)$ is a weight on $\mathcal{D}^R$, then $D \to \rho(D) \equiv \rho(D^R)$ is a weight for $\mathcal{D}$ in the module T. Since $\mathcal{D}$ is nilpotent, the result on weight spaces for a split nilpotent Lie algebra of linear transformations ([J, Theorem 2.7]) implies that T is a direct sum of weight modules T_ρ. Similarly, we have a decomposition of the algebra $\mathfrak{D}$ of derivations into weight modules $\mathfrak{D}_\alpha$. Thus we have

$$T = T_\rho \oplus T_\sigma \oplus \cdots \oplus T_\tau, \quad \mathfrak{D} = \mathfrak{D}_\alpha \oplus \mathfrak{D}_\beta \oplus \cdots \oplus \mathfrak{D}_\gamma$$

where $\rho,\sigma,\cdots,\tau$ are weights of T and $\alpha,\beta,\cdots,\gamma$ are roots of $\mathcal{D}$ in $\mathfrak{D}$. Take $R = \mathrm{id}$. Since there are no nonzero $\mathcal{D}$-constants, 0 is not a weight. By Proposition 3.5 of [J], we have $[\mathfrak{D}_\alpha, T_\rho] \subseteq T_{\alpha+\rho}$ if $\alpha+\rho$ is a weight of T relative to $\mathcal{D}$ otherwise $[\mathfrak{D}_\alpha, T_\rho] = 0$ ($\mathrm{ad}D(x) = [D,x] := Dx$). It follows that $[T_\sigma, T_\tau, T_\rho] = [L(T_\sigma, T_\tau), T_\rho] \subseteq T_{\sigma+\tau+\rho}$, if $\sigma+\tau+\rho$ is a weight of T relative to $\mathcal{D}$ otherwise $[T_\sigma, T_\tau, T_\rho] = 0$. This implies that every $R(x,y)(R(x,y)(z) := [z,x,y])$ for any $x \in T_\tau, y \in T_\rho$ is nilpotent on T if $\tau+\rho \neq 0$. In addition, we can prove that $\mathfrak{B} = \bigcup_{\tau+\rho\neq 0} R(T_\tau, T_\rho)$ is a weakly closed set. By Theorem 2.1 of [J], the enveloping associative algebra of $\mathfrak{B}$ is nilpotent. Hence, $(\mathrm{ad}x)^2 = R(x,x)$ for any $x \in T$ is nilpotent. According to Engel's theorem [H] of Lie triple systems, we know that T is nilpotent. □

Corollary 3.3. *Let (T,B) be a finite dimensional quadratic Lie triple system over an algebraically closed field of characteristic zero. If T admits a symplectic structure, then T is nilpotent.*

Proof. It follows from Lemma 2.7 and Theorem 3.2. □

Proposition 3.4 ([LWD], Theorem 4.3). *Suppose that $\mathbb{K}$ is algebraically closed of characteristic different from 2 and let (T, B) be a quadratic even-dimensional Lie triple system over $\mathbb{K}$. If T is nilpotent, then (T, B) is isometrically isomorphic to a quadratic T^*-extension $(T^*_\theta\mathfrak{a}, B)$, where $\mathfrak{a}$ is isomorphic to the quotient system of T by a completely isotropic ideal.*

Thus we know that every Lie triple system admitting both a quadratic structure and a symplectic structure is isometrically isomorphic to a T^*-extension of some Lie triple system. In particular, we have the following theorem.

Theorem 3.5. *Let (T, B) be a quadratic Lie triple system over an algebraically closed field $\mathbb{K}$ which admits a skew-symmetric invertible derivation $\bar{D}$. Then there exist a Lie triple system $\mathfrak{a}$, an invertible derivation D of $\mathfrak{a}$ and a Yamaguti cocycle $\theta \in Z^3(\mathfrak{a}, \mathfrak{a}^*)$ such that $T = T^*_\theta\mathfrak{a}$. Moreover the map Θ defined by*

$$\Theta(x, y, z, a) = \theta(Dx, y, z)(a) + \theta(x, Dy, z)(a) + \theta(x, y, Dz)(a) + \theta(x, y, z)(Da),$$

for any $x, y, z, a \in \mathfrak{a}$, is a 4-coboundary for the scalar cohomology of $\mathfrak{a}$.

Proof. Consider the semidirect sum of Lie algebras $L = L(T, T) \oplus \mathbb{K}\bar{D}$. Since T is nilpotent, the Lie algebra $L(T, T)$ is nilpotent and L is solvable. Thus, according to Lemma 3.2 in [B], there is a maximal isotropic (with respect to the quadratic form B) ideal J of T which is also stable under the derivation $\bar{D}$. Now, if $\mathfrak{a} = T/J$ then $\mathfrak{a}^*$ is isomorphic to $T^*_\theta\mathfrak{a}$ ([LWD, Theorem 4.3]). Furthermore, $\mathfrak{a}^* = J$ is stable under $\bar{D}$ and hence, there exist linear mappings $D_{11} : \mathfrak{a} \to \mathfrak{a}, D_{21} : \mathfrak{a} \to \mathfrak{a}^*$ and $D_{22} : \mathfrak{a}^* \to \mathfrak{a}^*$ such that $\bar{D}(x + f) = D_{11}x + D_{21}x + D_{22}f$ holds for any $x \in \mathfrak{a}, f \in \mathfrak{a}^*$. Clearly, D_{11} and D_{22} are invertible since $\bar{D}$ is. The skew-symmetry of $\bar{D}$ is equivalent to the conditions $D_{22}f = -f \circ D_{11}$, for any $f \in \mathfrak{a}$ and $B(D_{21}x, y) = -B(D_{21}y, x)$ for any $x, y \in \mathfrak{a}$. Indeed, $B(\bar{D}(x + f), y + g) = B(D_{11}x+D_{21}f, y+g) = g\circ D_{11}x+D_{22}\circ f(y)+B(D_{21}x, y)$ and $B(x+f, \bar{D}(y+g)) = B(x + f, D_{11}(y) + D_{21}(y) + D_{22}g) = f \circ D_{11}(y) + D_{22} \circ g(x) + B(x, D_{21}(y))$. Let $H = -D_{21}$ and $D = D_{11}$. Since $\bar{D}$ is a derivation, we get

$$\begin{aligned}
0 &= [\bar{D}x, y, z] + [x, \bar{D}y, z] + [x, y, \bar{D}z] - \bar{D}[x, y, z] \\
&= [Dx, y, z]_\mathfrak{a} + [x, Dy, z]_\mathfrak{a} + [x, y, Dz]_\mathfrak{a} - D[x, y, z]_\mathfrak{a} \\
&\quad + H[x, y, z]_\mathfrak{a} - [Hx, y, z]_\mathfrak{a} - [x, Hy, z]_\mathfrak{a} - [x, y, Hz]_\mathfrak{a} \\
&\quad + \theta(x, y, z) \circ D + \theta(Dx, y, z) + \theta(x, Dy, z) + \theta(x, y, Dy),
\end{aligned}$$

for any $x, y, z \in \mathfrak{a}$, which shows that D is a derivation of $\mathfrak{a}$ and that if

$$\Theta(x,y,z,a) = \theta(Dx,y,z)(a) + \theta(x,Dy,z)(a) + \theta(x,y,Dz)(a) + \theta(x,y,z)(Da)$$

and F is the skew-symmetric bilinear form on $\mathfrak{a}$ defined by $F(x,y) = -B(Hx,y) = -Hx(y)$, then we have $\Theta(x,y,z,a) - F([x,y,z]_{\mathfrak{a}},a) + F(x,[a,z,y]_{\mathfrak{a}}) + F(y,[z,a,x]_{\mathfrak{a}}) = \Theta(x,y,z,a) - dF(x,y,z,a) = 0$ for any $x,y,z,a \in \mathfrak{a}$, which ends the proof. □

Remark 3.6. (1) It is interesting to point out that if the derivation $\bar{D}$ is semisimple then the derivation D is also semisimple. Actually, if we choose a basis of the completely isotropic ideal I composed of eigenvectors of D, then each element in its dual basis (with respect to B) is an eigenvector of D.

(2) Theorem 3.5 does not hold in general in the case of a non-algebraically closed field. For example, an even-dimensional abelian Lie triple system T over $\mathbb{R}$ with a positive definite bilinear form is obviously a quadratic Lie triple system which admits an invertible skew-symmetric derivation. However, it cannot be a T^*-extension since there are no isotropic subspaces.

Indeed, the inverse problem of Theorem 3.5 also holds.

Theorem 3.7. *Let $\mathfrak{a}$ be a Lie triple system admitting an invertible derivation D. Consider a Yamaguti 3-cocycle $\theta \in Z^3(\mathfrak{a},\mathfrak{a}^*)$ and define*

$$\Theta(x,y,z,a) = \theta(Dx,y,z)(a) + \theta(x,Dy,z)(a) + \theta(x,y,Dz)(a) + \theta(x,y,z)(Da), \quad (1)$$

for any $x,y,z,a \in \mathfrak{a}$. If Θ is a 4-coboundary for the scalar cohomology of $\mathfrak{a}$, then the quadratic Lie triple system $T^*_\theta\mathfrak{a}$ admits a symplectic structure.

Proof. Let B be the quadratic form on $T^*_\theta\mathfrak{a}$ defined in Definition 3.1. By Lemma 2.6, it is enough to prove the existence of an invertible skew-symmetric derivation of $(T^*_\theta\mathfrak{a}, B)$.

Consider $F : \mathfrak{a}\times\mathfrak{a} \to \mathbb{K}$ such that $\Theta = \delta F$ where δ is defined in ([LWD, page 2080]). Explicitly, for any $x,y,z,a \in \mathfrak{a}$,

$$(\delta F)(x,y,z,a) = F([x,a,z],y) - F([x,a,y],z) + F([z,y,x],a) - F(x,[y,z,a]). \quad (2)$$

Let $H : \mathfrak{a} \to \mathfrak{a}^*$ be the mapping defined by $B(Hx,y) = F(x,y)$ for any $x,y \in \mathfrak{a}$. Then according to Definition 3.1, we have

$$B(Hx,y) = Hx(y) = F(x,y), \quad \forall x,y \in \mathfrak{a}. \quad (3)$$

Define $\bar{D} : T^*_\theta \mathfrak{a} \to T^*_\theta \mathfrak{a}$ by

$$\bar{D}(x + f) = Dx - Hx - f \circ D$$

for any $x \in \mathfrak{a}, f \in \mathfrak{a}^*$. Define $\bar{D}' : T^*_\theta \mathfrak{a} \to T^*_\theta \mathfrak{a}$ by

$$\bar{D}'(x + f) = D^{-1}x - HD^{-1}x \circ D^{-1} - f \circ D^{-1}$$

for any $x \in \mathfrak{a}, f \in \mathfrak{a}^*$. Then

$$\begin{aligned}\bar{D}'\bar{D}(x + f) &= \bar{D}'(Dx - Hx - f \circ D)\\ &= D^{-1}Dx - HD^{-1}Dx \circ D^{-1} + Hx \circ D^{-1} + f \circ D \circ D^{-1} = x + f,\end{aligned}$$

and

$$\begin{aligned}\bar{D}\bar{D}'(x + f) &= \bar{D}(D^{-1}x - HD^{-1}x \circ D^{-1} - f \circ D^{-1})\\ &= DD^{-1}x - HD^{-1}x + HD^{-1}x \circ D^{-1} \circ D + f \circ D^{-1} \circ D = x + f,\end{aligned}$$

that is, $\bar{D}$ is invertible and $\bar{D}'$ is the inverse of $\bar{D}$. Since

$$\begin{aligned}B(\bar{D}(x + f), y + g) &= B(Dx - Hx - f \circ D, y + g)\\ &= B(Dx, y) - F(x, y) - f \circ Dy + g \circ Dx\end{aligned}$$

and

$$\begin{aligned}B(x + f, \bar{D}(y + g)) &= B(x + f, Dy - Hy - g \circ D)\\ &= B(x, Dy) + f \circ Dy - F(y, x) - g \circ Dx,\end{aligned}$$

we have $B(\bar{D}(x + f), y + g) = B(x + f, \bar{D}(y + g))$, i.e., $\bar{D}$ is skew-symmetric with respect to B.

Furthermore, since D is a derivation of $\mathfrak{a}$ and

$$\begin{aligned}&(-[f \circ D, y, z] + [Dx, g, z] + [Dx, y, h] - [x, g \circ D, z] + [f, Dy, z] + [x, Dy, h]\\ &\quad - [x, y, h \circ D] + [f, y, Dz] + [x, g, Dz] - [f, y, z] \circ D - [x, g, z] \circ D\\ &\quad - [x, y, h] \circ D)(a)\\ &= -f \circ D([a, z, y]_\mathfrak{a}) + g([z, a, Dx]_\mathfrak{a}) + h([y, Dx, a]) - g \circ D([z, a, x])\\ &\quad + f([a, z, Dy]) + h([Dy, x, a]_\mathfrak{a}) - h \circ D([y, x, a]_\mathfrak{a}) + f([a, Dz, y])\\ &\quad + g([Dz, a, x]) + f([Da, z, y]_\mathfrak{a}) + g([z, Da, x]) + h([y, x, Da]) \quad = 0,\end{aligned}$$

we get

$$\begin{aligned}
&[\bar{D}(x+f), y+g, z+h] + [x+f, \bar{D}(y+g), z+h] + [x+f, y+g, \bar{D}(z+h)] \\
&\quad - \bar{D}([x+f, y+g, z+h]) \\
&= [Dx, y, z]_{\mathfrak{a}} + \theta(Dx, y, z) - [Hx, y, z] - [f \circ D, y, z] + [Dx, g, z] + [Dx, y, h] \\
&\quad + [x, Dy, z]_{\mathfrak{a}} + \theta(x, Dy, z) - [x, Hy, z] - [x, g \circ D, z] + [f, Dy, z] + [x, Dy, h] \\
&\quad + [x, y, Dz]_{\mathfrak{a}} + \theta(x, y, Dz) - [x, y, Hz] - [x, y, h \circ D] + [f, y, Dz] + [x, g, Dz] \\
&\quad - D([x, y, z]_{\mathfrak{a}}) + H([x, y, z]_{\mathfrak{a}}) + [f, y, z] \circ D + [x, g, z] \circ D + [x, y, h] \circ D \\
&\quad + \theta(x, y, z) \circ D \\
&= \theta(Dx, y, z) + \theta(x, Dy, z) + \theta(x, y, Dz) + \theta(x, y, z) \circ D + H([x, y, z]_{\mathfrak{a}}) \\
&\quad - [Hx, y, z] - [x, Hy, z] - [x, y, Hz].
\end{aligned}$$

In particular,

$$\begin{aligned}
&(\theta(Dx, y, z) + \theta(x, Dy, z) + \theta(x, y, Dz) + \theta(x, y, z) \circ D + H([x, y, z]_{\mathfrak{a}}) - [Hx, y, z] \\
&\qquad - [x, Hy, z] - [x, y, Hz])(a) \\
&\overset{\text{Def.3.1}}{=} \theta(Dx, y, z)(a) + \theta(x, Dy, z)(a) + \theta(x, y, Dz)(a) + \theta(x, y, z)(Da) \\
&\qquad + H([x, y, z]_{\mathfrak{a}})(a) - Hx([a, z, y]_{\mathfrak{a}}) - Hy([z, a, x]_{\mathfrak{a}}) - Hz([y, x, a]_{\mathfrak{a}}) \\
&\overset{(1)(3)}{=} \Theta(x, y, z, a) + F([x, y, z]_{\mathfrak{a}}, a) - F(x, [a, z, y]_{\mathfrak{a}}) - F(y, [z, a, x]_{\mathfrak{a}}) \\
&\qquad - F(z, [y, x, a]_{\mathfrak{a}}) \\
&\overset{(2)}{=} \Theta(x, y, z, a) - \delta F(x, y, z, a) = 0.
\end{aligned}$$

Hence, $\bar{D}$ is a derivation. □

By Corollary 3.3, we know that if a Lie triple system admits an invertible derivation, then it must be nilpotent. However, there are many nilpotent Lie triple systems whose derivations are all non-invertible. The following result gives a characterization of Lie triple systems admitting such a derivation. Note that the result is valid for an arbitrary base field of characteristic zero (not necessarily algebraically closed).

Proposition 3.8. *Let $\mathbb{K}$ be a field of characteristic zero and let $\mathfrak{a}$ be a Lie triple system over $\mathbb{K}$. There exists an invertible derivation of $\mathfrak{a}$ if and only if $\mathfrak{a}$ is isomorphic to the quotient Lie triple system T/I of a quadratic symplectic Lie triple system (T, B, ω) by a lagrangian and completely isotropic ideal I.*

Proof. If $\mathfrak{a}$ admits an invertible derivation, then the Lie triple system $T = T_0^*\mathfrak{a}$ obtained by T^*-extension by the null cocycle $\theta = 0$ is, according to Theorem 3.7, a quadratic symplectic Lie triple system and $I = \mathfrak{a}^*$ is a lagrangian, completely isotropic ideal of T. Conversely, suppose that $\mathfrak{a}$ is isomorphic to T/I, where (T, B, ω) is a quadratic symplectic Lie triple system and I is a Lagrangian completely isotropic ideal of T. According to Corollary 3.1 of [LWD], T is isometrically isomorphic to $T_\theta^*(T/I) = T_\theta^*\mathfrak{a}$ since I is completely isotropic. Let $\bar{D} \in Der_{\mathfrak{a}}(T, B)$ be the invertible derivation such that $\omega(X, Y) = B(DX, Y)$ for any $X, Y \in T$. Clearly, $\omega(I, I) = 0$ implies that $\bar{D}(I) \subseteq I^\perp = I$. Now, since I stable by $\bar{D}$, the same arguments used in the proof of Theorem 3.5 prove that the projection of $\bar{D}|_{\mathfrak{a}}$ to $\mathfrak{a}$ provides an invertible derivation of $\mathfrak{a}$. □

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