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An associated graph to a graded ring

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Abstract. This work is mainly devoted to study of the graph $\Gamma_S(R)$ associated to a graded ring R and a multiplicatively closed subset S of R. Recall that the vertices of $\Gamma_S(R)$ are all of the elements of R and two distinct vertices are adjacent if their sum belongs to S.

In fact, we investigate some basic graph-theoretical properties of $\Gamma_S(R)$, where $R = \bigoplus_i R_i$ is a graded ring. Moreover, we deal with the relationship between the graph-theoretical properties of $\Gamma_S(R)$ and $\Gamma_{S\cap R_0}(R_0)$.

1. Introduction

Throughout the paper, $R = \bigoplus_n R_n$ is a commutative graded (or more precisely $\mathbb{Z}$-graded) ring with non-zero identity unless otherwise stated, S is a multiplicatively closed subset of R and $S_0 = S \cap R_0$. Also, we denote the set of all zero-divisors of R, the nilradical and the set of unit elements of R by $Z(R)$, $\mathrm{Nil}(R)$ and $U(R)$, respectively.

The graphs associated to algebraic structures have been extensively studied by various authors. Recently there has been a lot of interest in this subject and various papers were published establishing different properties of these graphs as well as relationships between graphs of various ring extensions (see e.g. [1], [2], [4], [8], [12], [15], [16]). The graded ring R is one of the well-known extensions of R_0 which is a natural generalization of the rings of polynomials and power series. On the other hand, the graph $\Gamma_S(R)$ introduced in [9] is a natural generalization of total graph [5] and unit graph [7]. Also, in [6], the authors studied $\Gamma_S(R)$ for a

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multiplicatively closed subset S of R such that $R\backslash S$ is a saturated multiplicatively closed subset of R. The main theme of this work is the study of the graph-theoretical properties of $\Gamma_S(R)$, where R is a graded ring. Also, we find some relationships between $\Gamma_S(R)$ and $\Gamma_{S_0}(R_0)$ and check the preservation of the graph-theoretic properties of $\Gamma_{S_0}(R_0)$ under this extension of R_0. Also, we generalize or present new versions of some of the results obtained in [3], [9] and [13].

In order to make this paper easier to follow, we recall here the various notions from graph theory which will be used in the sequel.

Let G be a graph. Then the *valency of a vertex* a, denoted by $\deg_G(a)$, is the number of edges of G incident to a. For two distinct vertices a and b in G, the notation $a \sim b$ means that a and b are adjacent. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if every two distinct vertices are adjacent. The *distance* between two vertices a and b of G, denoted by $d_G(a,b)$ or briefly $d(a,b)$, is the length of a shortest path connecting a and b if such a path exists; otherwise we set $d_G(a,b) = \infty$. For a positive integer r, an *r-partite graph* is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. A graph is called *planar* if it can be drawn in the plane so that its edges intersects only at their ends. Also, for a graph G, a subset B of the vertex set of G is called a *dominating set* if every vertex not in B is adjacent to a vertex in B. The reader may refer to [10] for undefined terms and concepts concerning graph theory.

2. Preliminaries

We devote this section to the definition of $\Gamma_S(R)$ and some elementary remarks about graded rings which may be valuable in turn. Recall that a multiplicatively closed subset S of R is called *saturated* if $xy \in S$ implies that $x \in S$ and $y \in S$.

Definition 2.1. (See [9].) Let S be an arbitrary multiplicatively closed subset of R. $\Gamma_S(R)$ is the simple graph whose vertices are all of the elements of R and two distinct vertices a and b are adjacent if and only if $a + b \in S$.

Remark 2.2. If S is a saturated multiplicatively closed subset of R, then for each $x \in R\backslash S$, (x) is disjoint from S, since S is saturated. Expand (x) to an ideal

I maximal with respect to disjointness from S. One can show that I is a prime ideal. Hence $R \setminus S = \bigcup_{j\in A} P_j$, where P_js are prime ideals of R (see [11, Page 2, Theorem 2]). Hereafter, we set $I_S = \bigcap_{j\in A} P_j$. It is easily seen that $\mathrm{Nil}(R) \subseteq I_S$.

Lemma 2.3. *Let S be a saturated multiplicatively closed subset of R and $\sum f_i$ be an element of R such that for each $i \neq 0$, $f_i \in I_S$. Then $\sum_i f_i \in S$ if and only if $f_0 \in S_0$.*

Proof. ($\Rightarrow$) If $f_0 \notin S_0$, then $f_0 \notin S$. Therefore, by Remark 2.2, $f_0 \in P_j$ for some $j \in A$. On the other hand, by our assumption, $\sum_{i\neq 0} f_i \in I_S$ and so $\sum_{i\neq 0} f_i \in P_j$. Hence $\sum_i f_i \in P_j$. Now Remark 2.2 implies that $\sum_i f_i \notin S$.

($\Leftarrow$) Assume that $\sum_i f_i \notin S$. Then Remark 2.2 insures that $\sum_i f_i \in P_j$ for some $j \in A$. By assumption, $\sum_{i\neq 0} f_i \in I_S$ which implies that $f_0 \in P_j$. Using Remark 2.2 again implies that $f_0 \notin S$. This completes the proof. □

Definition 2.4. We say that S is an I_S-graded m.c.s of R if S is a saturated multiplicatively closed subset of R such that for each $\sum f_i$ in S, we have $f_i \in I_S$ for all $i \neq 0$.

Note that if S is a saturated multiplicatively closed subset of R such that $0 \in S$, then $S = R$ and so $\Gamma_S(R)$ is a complete graph. In this case we also have $I_S = R$ and therefore S is an I_S-graded m.c.s of R. Hence hereafter we may assume that $0 \notin S$. If S is an I_S-graded m.c.s of R and R has at least one non-zero homogeneous element x with non-zero degree, then x and 0 are not adjacent by Lemma 2.3, and so $\Gamma_S(R)$ necessarily can't be complete.

When $S = U(R)$, we can state Lemma 2.3 as follows.

Lemma 2.5. (Compare [14, Exercise 3.3]).

(i) *Let R be a positively graded ring such that every minimal prime ideal is homogeneous and $\sum_i f_i$ an element of R. Then $\sum_i f_i$ is a unit element of R if and only if f_0 is a unit element of R_0 and f_i is nilpotent for all $i \neq 0$.*

(ii) *Let R be a graded ring such that every minimal prime ideal is homogeneous and $\sum_i f_i$ an element of R such that f_0 is not in any minimal prime of R. Then $\sum_i f_i$ is a unit element of R if and only if f_0 is a unit element of R_0 and f_i is nilpotent for all $i \neq 0$.*

Proof. (i) , (ii) ($\Leftarrow$) The result follows from the fact that the sum of a nilpotent element and a unit element is a unit.

(i) ($\Rightarrow$) Assume that $\sum_i f_i$ is a unit element of R. Then it has an inverse, say $\sum_i g_i$. Hence, $f_0 g_0 = 1$ which implies that $f_0 \in U(R_0)$. Now, we are going to show that $f_i \in \mathrm{Nil}(R)$ for each $i \neq 0$. Since $\mathrm{Nil}(R) = \bigcap_{P\in \mathrm{Min}(R)} P$, we show that $f_i \in P$ for each $P \in \mathrm{Min}(R)$ and $i \neq 0$. Suppose that $P \in \mathrm{Min}(R)$. Since $(\sum_i f_i)(\sum_i g_i) = 1$, we have $(\sum_i (f_i + P))(\sum_i (g_i + P)) = 1_{R/P}$. Therefore, $\sum_i (f_i + P)$ is a unit element in R/P. By Exercise 1.1 of [14], we know that, all units in a graded domain are homogeneous. Hence $\sum_i (f_i + P)$ is homogeneous. Since f_0 is unit, we conclude that $f_0 \notin P$, and so we have $f_i \in P$ for all $i \neq 0$ as desired.

(ii) ($\Rightarrow$) By applying a method similar to that we used in the proof of (i), we conclude that $\sum_i (f_i + P)$ is homogeneous for each $P \in \mathrm{Min}(R)$. Since $f_0 \notin P$ for any minimal prime P of R, then $f_i \in P$ for all $i \neq 0$ as desired. So, $\sum_{i \neq 0} f_i \in \mathrm{Nil}(R)$. Now since $\sum_i f_i \in U(R)$, we have $f_0 \in U(R)$, and hence $f_0 \in U(R_0)$. This completes the proof. □

Remark 2.6. If $S = U(R)$, then $U(R)$ is a saturated multiplicatively closed subset of R. In this case, we have $I_S = J(R)$. Since $\mathrm{Nil}(R) \subseteq J(R)$, Lemma 2.5 shows that S is an I_S-graded m.c.s of R, if R is a positively graded ring or for each element $\sum_i f_i$ of S, $f_0 \notin P$ for all $P \in \mathrm{Nil}(R)$. Also, note that for an arbitrary element s in an I_S-graded m.c.s S of R, $1 \cdot s \in S$. Hence $1 \in S$ which implies $U(R) \subseteq S$. The next example illustrates that an I_S-graded m.c.s of R is not necessarily $U(R)$.

Example 2.7. Consider the trivial grading on $R = \mathbb{Z}_{12}$. Then $U(R) = \{1, 5, 7, 11\}$. Set $S = \{1, 3, 5, 7, 9, 11\}$. It is easy to see that S is a saturated multiplicatively closed subset of R such that for each element $\sum_i g_i$ in S, we have $0 = g_i \in I_S$ for all $i \neq 0$. Hence, S is an I_S-graded m.c.s of R.

Remarks 2.8. (1) Let S be a saturated multiplicatively closed subset of R. Then S_0 is a saturated multiplicatively closed subset of R_0.

(2) Let R be a graded local ring. Then, for each $i \neq 0$, every element in R_i is nilpotent. (See [14, Exercise 3.4].)

(3) Consider the graded subring $\mathcal{A} = \{\sum_i f_i \mid f_0 = 0\}$ of R. If S is an I_S-graded m.c.s of R, then, in view of Lemma 2.3, the induced subgraph of $\Gamma_S(R)$ on A is a totally disconnected graph.

(4) If S is an I_S-graded m.c.s of R and $I_S = \{0\}$, then $S \subseteq R_0$ and R is reduced. So, two elements $\sum_i f_i$ and $\sum_i g_i$ are adjacent if and only if f_0 is adjacent to g_0 and for each $i \neq 0$, $f_i = -g_i$.

(5) $\Gamma_{S_0}(R_0)$ is an induced subgraph of $\Gamma_S(R)$.

Proposition 2.9. (Compare [9, Lemma 2.3 and Proposition 2.13]). *Let S be a saturated multiplicatively closed subset of R. Then the following statements hold.*

(i) *For each $x \in I_S$ and $s \in S$, we have $x \pm s \in S$.*

(ii) *For each element $x \in I_S$, $deg_{\Gamma_S(R)}(x) = |S|$.*

(iii) *The induced subgraph of $\Gamma_S(R)$ on $R \setminus S$ is an $|A|$-partite graph, where A is the same as Remark 2.2.*

(iv) *If $2 \notin S$, then for each $f \in R$, $deg_{\Gamma_S(R)}(f) \geq |S|$.*

(v) *If S is an I_S-graded m.c.s of R and $f = \sum_i f_i \in R$, then $deg_{\Gamma_S(R)}(f) = |S_0| \prod_i |I_S \cap R_i|$.*

Proof. (i) Let $x \in I_S$ and $s \in S$. If $x \pm s \notin S$, then Remark 2.2 insures that $x \pm s \in P$ for some prime ideal P of R. Since $x \in I_S$, we should have $s \in P$ which is a contradiction. Therefore $x \pm s \in S$ as desired.

(ii) If $x \in I_S$ and $y \in S$, then by (i) $x + y \in S$. If $x \in I_S$ and $y \notin S$, then Remark 2.2 insures that $y \in P$ for some prime ideal P of R. Since $x \in P$, we have that $x + y \in P$ which implies that $x + y \notin S$. Hence each element x of I_S is just adjacent to each element of S. So $deg_{\Gamma_S(R)}(x) = |S|$.

(iii) Since $R \backslash S = \bigcup_{j \in A} P_j$, for two elements $x, y \in P_j$ for some $j \in A$, we have $x + y \in P_j$. This means that $x + y \notin S$. Therefore the induced subgraph of $\Gamma_S(R)$ on $R \setminus S$ is an $|A|$-partite graph with parts $P_1, P_2 \setminus P_1, P_3 \setminus (P_1 \cup P_2), \ldots, P_{|A|} \setminus \bigcup_{i=1}^{|A|-1} P_i$.

(iv) It is easily seen that $x \sim s - x$ for all $s \in S$. This immediately implies the result.

(v) Lemma 2.3 and part (i) imply that $f \sim \sum_i g_i$ if and only if $g_0 = s - f_0$ for some $s \in S_0$ and $g_i = x_i - f_i$ for some $x_i \in R_i \cap I_S$. This completes the proof. □

Proposition 2.10. *Let S be a saturated multiplicatively closed subset of R and a and b are two distinct non-zero elements in I_S. Then $\Gamma_S(R)$ is not planar.*

Proof. In view of Proposition 2.9(i), we have that the induced subgraph of $\Gamma_S(R)$ on the vertices $0, a, b, s, s + a, s + b$ form a $K_{3,3}$, for each $s \in S$. Hence, Kouratowski's Theorem insures that $\Gamma_S(R)$ can't be planar. □

3. Connectedness and dominating sets

In this section, we are going to study connectedness and some of the properties of $\Gamma_S(R)$. Also we study some of its relationships with $\Gamma_{S_0}(R_0)$ in special circumstances. We begin with the following theorem which is one of the main results of this paper and presents a *sufficient condition* for the disconnectedness of

$\Gamma_S(R)$. Note that since $J(R[x]) = \mathrm{Nil}(R[x])$, in view of Remark 2.6, the following result is a generalization of [3, Theorem 2.3] and [7, Proposition 4.6].

Theorem 3.1. (See [3, Theorem 2.3] and [7, Proposition 4.6]). *Let S be an I_S-graded m.c.s of R. If $(R \setminus I_S) \cap R_n \neq \emptyset$ for some non-zero integer n, then $\Gamma_S(R)$ is disconnected.*

Proof. By our assumption, we may choose $r \in (R \setminus I_S) \cap R_n$ for some non-zero integer n. We show that there is not any path between zero and r in $\Gamma_S(R)$. To this end, suppose to the contrary that $0 \sim \sum_i f_{i,1} \sim \cdots \sim \sum_i f_{i,m} \sim r$ is a path from zero to r in $\Gamma_S(R)$ such that $f_{i,j} \in R_i$ for each $1 \leq j \leq m$. Since $\sum_i f_{i,1}$ is adjacent to zero, $f_{i,1}$ is in I_S for all $i \neq 0$. Also, since $\sum_i f_{i,1} \sim \sum_i f_{i,2}$, $f_{i,1} + f_{i,2}$ is in I_S for all $i \neq 0$. These insure that $f_{i,2}$ is also in I_S. By a similar argument, we have that $f_{i,j}$ belongs to I_S for all $i \neq 0$ and $1 \leq j \leq m$. Now, since $\sum_i f_{i,m} \sim r$, we have $r + f_{n,m} \in I_S$, which implies that $r \in I_S$. This contradicts with our assumption. □

The following two examples illustrate that not only is the given condition in Theorem 3.10 not a *necessary condition*, but it is also an irredundant condition.

Example 3.2. Consider the trivial grading on $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ and the saturated multiplicatively closed subset $S = \{(1,1),(1,3),(3,1),(3,3)\}$ of R. It is clear that $\Gamma_S(R)$ is the union of two disjoint $K_{4,4}$ with parts

$$V_1 = \{(0,0),(0,2),(2,0),(2,2)\}, \quad V_2 = \{(1,1),(1,3),(3,1),(3,3)\},$$

and

$$V_1' = \{(1,0),(1,2),(3,0),(3,2)\}, \quad V_2' = \{(0,1),(0,3),(2,1),(2,3)\}.$$

Therefore $\Gamma_S(R)$ is disconnected, while $(R \setminus I_S) \cap R_n = \emptyset$ for all non-zero integers n.

Example 3.3. Consider $R = D[x]$ with the standard grading, where D is an integral domain, and $S = R \setminus (x)$. Then $I_S = (x)$ and $(R \setminus I_S) \cap R_n = \emptyset$ for all $n \neq 0$. Let $\sum_i f_i$ and $\sum_i g_i$ be two distinct elements of R. If $f_0, g_0 \neq 0$, we have that $\sum_i f_i \sim 0 \sim \sum_i g_i$. In addition, if $f_0, g_0 = 0$, then $\sum_i f_i \sim 1 \sim \sum_i g_i$. Moreover, if $f_0 = 0$ and $g_0 \neq 0$, then $\sum_i f_i \sim \sum_i g_i$. So, $\Gamma_S(R)$ is connected.

In the sequel of this section, we are going to provide some necessary or sufficient conditions for the connectedness of $\Gamma_S(R)$.

Theorem 3.4. *Let S be a multiplicatively closed subset of R such that $S = -S$ and $\mathcal{A} = \{\sum_i f_i | f_0 = 0\}$. If every element of R_0 is a finite sum of elements of S_0, then*

(i) *$\mathcal{A}$ intersects every connected component of $\Gamma_S(R)$;*

(ii) *$|\mathcal{A}|$ is an upper bound for the number of connected components of $\Gamma_S(R)$;*

(iii) *if $|\mathcal{A}| = 1$, then $\Gamma_S(R)$ is connected;*

(iv) *if $R_i \subseteq I_S$ for all $i \neq 0$, then $d(\sum_i f_i, 0) \leq s_0(R_0)$ in $\Gamma_S(R)$ for all $\sum_i f_i \in R$, where*

$$s_0(R_0) = \begin{cases} \min\{k \mid \textit{every element of } R_0 \textit{ is sum of } k \\ \ \textit{elements of } S_0\}; & \text{if min exists} \\ \omega; & \text{otherwise} \end{cases}$$

and so $\Gamma_S(R)$ is connected.

Proof. (i) Consider an arbitrary element $\sum_i f_i$ of R. Since every element of R_0 is a finite sum of elements of S_0, $f_0 = s_1 + \cdots + s_k$ for some $s_1, \ldots, s_k \in S_0$. It can be easily seen that

$$\begin{aligned} \sum_i f_i &\sim -(s_2 + \cdots + s_k) + \sum_{i\neq 0}(-f_i) \\ &\sim (s_3 + \cdots + s_k) + \sum_{i\neq 0} f_i \\ &\sim \cdots \\ &\sim (-1)^{k-1} s_k + \sum_{i\neq 0}(-1)^{k-1} f_i \\ &\sim \sum_{i\neq 0}(-1)^k f_i \end{aligned}$$

is a path in $\Gamma_S(R)$ which connects $\sum_i f_i$ to an element in $\mathcal{A}$.

(ii) and (iii) follow from (i).

(iv) Let $\sum_i f_i \in R$ and $s_0(R_0) = k$. Then $f_0 = s_1 + \cdots + s_k$ for some $s_1, \ldots, s_k \in S_0$. For every $1 \leq i \leq k$, set $b_i := (-1)^i \sum_{j=1}^{k-i} s_j$ (note that $b_k = 0$). Since $R_i \subseteq I_S$ for all $i \neq 0$, $f_i \in I_S$ for all $i \neq 0$. Hence by Lemma 2.3, there exists the path $\sum_i f_i \sim b_1 \sim \cdots \sim b_k$ from $\sum_i f_i$ to zero in $\Gamma_S(R)$. This completes the proof of (iv). □

Corollary 3.5. (Compare [3, Theorem 2.6], [9, Theorem 1.7] and [13, Proposition 3.2]). *Let S be a multiplicatively closed subset of R such that $S = -S$. Then $\Gamma_S(R)$ is connected if and only if every element of R is a finite sum of elements of S.*

Proof. Note that the trivial grading on R is equivalent to the fact that $|\{\sum_i f_i \mid f_0 = 0\}| = 1$. So Theorem 3.4(iii) proves the *if part*. Conversely, if $\Gamma_S(R)$ is connected, since every element in R is connected to zero, the result immediately follows. □

Corollary 3.6. *Assume that S is an I_S-graded m.c.s of R such that $S = -S$ and $\mathcal{A} = \{\sum_i f_i \mid f_0 = 0\}$. Let B be a dominating set for $\Gamma_S(R)$ such that for each element $\sum_i f_i$ in B, f_0 is a finite sum of elements of S_0. Then $\mathcal{A}$ intersects every connected component of $\Gamma_S(R)$.*

Proof. Since B is a dominating set for $\Gamma_S(R)$, for arbitrary element f_0 of $R_0 \setminus B$, there is a $\sum\limits_i g_i$ in B such that $f_0 + \sum\limits_i g_i \in S$. Thus $f_0 + g_0 \in S_0$ by Lemma 2.3. Therefore, there is $s_0 \in S_0$ such that $f_0 + g_0 = s_0$. Since $\sum\limits_i g_i \in B$, there exist $s_1, \dots, s_k \in S_0$ such that $g_0 = s_1 + \cdots + s_k$. Thus $f_0 = s_0 - (s_1 + \cdots + s_k)$. Therefore, every element of R_0 is finite sum of elements of S_0. Now, by Theorem 3.4(i), $\mathcal{A}$ intersects every connected component of $\Gamma_S(R)$. □

Proposition 3.7. (i) *Let S be a saturated multiplicatively closed subset of R such that $R_i \subseteq I_S$ for all $i \neq 0$. Then S_0 is a dominating set for $\Gamma_{S_0}(R_0)$ if and only if S is a dominating set for $\Gamma_S(R)$.*

(ii) *If R is a graded local ring and $S = U(R)$, then S_0 is a dominating set for $\Gamma_{S_0}(R_0)$ if and only if S is a dominating set for $\Gamma_S(R)$.*

Proof. (i) Let S_0 be a dominating set for $\Gamma_{S_0}(R_0)$ and $\sum_i f_i \in R \setminus S$. In light of Lemma 2.3, $f_0 \in R_0 \setminus S_0$. Hence there is an element $g_0 \in S_0$ such that $f_0 + g_0 \in S_0$. Using Lemma 2.3 again implies that $\sum_i f_i + g_0 \in S$. This shows that S is a dominating set for $\Gamma_S(R)$. Conversely, assume that S is a dominating set for $\Gamma_S(R)$ and $f_0 \in R_0 \setminus S_0$. Then there is an element $\sum_i g_i \in S$ such that $f_0 + \sum_i g_i \in S$. Now, Lemma 2.3 insures that $f_0 + g_0 \in S_0$, which completes the proof.

(ii) follows from (i) in conjunction with Remarks 2.6 and 2.8(2). □

Lemma 3.8. *Let S be a multiplicatively closed subset of R and $\mathcal{A} = \{\sum_i f_i \mid f_0 = 0\}$. If S_0 is a dominating set for $\Gamma_{S_0}(R_0)$, then $\mathcal{A}$ intersects every connected component of $\Gamma_S(R)$.*

Proof. Let $\sum_i f_i$ be an arbitrary element of R. If $f_0 \in S_0$, then $\sum_i f_i \sim \sum_{i\neq 0}(-f_i)$. Otherwise, since S_0 is a dominating set for $\Gamma_{S_0}(R_0)$, there exists an $s_0 \in S_0$ such that $f_0 + s_0 \in S_0$. So $\sum_i f_i \sim s_0 + \sum_{i\neq 0}(-f_i) \sim \sum_{i\neq 0} f_i$. This shows that $\mathcal{A}$ intersects every connected component of $\Gamma_S(R)$. □

The following theorem shows that for a special saturated multiplicatively closed subset of R, the converse of Theorem 3.4 also holds.

Theorem 3.9. *Let S be an I_S-graded m.c.s of R such that $S = -S$ and $\mathcal{A} = \{\sum_i f_i \mid f_0 = 0\}$. If $\mathcal{A}$ intersects every connected component of $\Gamma_S(R)$, then every element of R_0 is finite sum of elements of S_0.*

Proof. Let $\mathcal{A}$ intersect every connected component of $\Gamma_S(R)$. Then for an arbitrary element f_0 of R_0, there is an element $\sum\limits_i g_i$ in $\mathcal{A}$ such that there exists a path

$$f_0 \sim \sum_i f_{i,1} \sim \sum_i f_{i,2} \sim \cdots \sim \sum_i f_{i,k} \sim \sum_i g_i$$

from f_0 to $\sum\limits_i g_i$ in $\Gamma_S(R)$. Hence the elements

$$f_0 + \sum_i f_{i,1}, \sum_i f_{i,1} + \sum_i f_{i,2}, \ldots, \sum_i f_{i,k-1} + \sum_i f_{i,k}, \sum_i f_{i,k} + \sum_i g_i$$

are in S. Thus $f_0+f_{0,1}, f_{0,1}+f_{0,2}, \ldots, f_{0,k-1}+f_{0,k}, f_{0,k}+g_0$ are in S_0 by Lemma 2.3. Since $g_0 = 0$, we have $f_0 = (f_0 + f_{0,1}) - (f_{0,1} + f_{0,2}) + \cdots + (-1)^{k-1}(f_{0,k-1} + f_{0,k}) + (-1)^k(f_{0,k}+g_0)$. Hence f_0 is a finite sum of elements of S_0 as required. □

Corollary 3.10. *Let S be an I_S-graded m.c.s of R such that $S = -S$. If S_0 is a dominating set for $\Gamma_{S_0}(R_0)$, then every element of R_0 is a finite sum of elements of S_0.*

Proof. The result follows from Lemma 3.8 and Theorem 3.9. □

Corollary 3.11. (Compare [3, Corollary 2.7]). *Let S be a multiplicatively closed subset of R such that $S = -S$ and S is a dominating set of $\Gamma_S(R)$. Then $\Gamma_S(R)$ is connected.*

Proof. Since S is a dominating set for $\Gamma_S(R)$, by considering the trivial grading on R, Corollary 3.10 implies that every element of R is finite sum of elements of S. Hence Corollary 3.5 completes the proof. □

The next result provides conditions under which $\Gamma_{S_0}(R_0)$ inherits connectedness from $\Gamma_S(R)$.

Theorem 3.12. *Let S be an I_S-graded m.c.s of R such that $S = -S$. If $\Gamma_S(R)$ is connected, then every element of R_0 is finite sum of elements of S_0 and thus $\Gamma_{S_0}(R_0)$ is connected.*

Proof. Let $\Gamma_S(R)$ be a connected graph. For $f_0 \in R_0$, there exists a path

$$f_0 \sim \sum_i f_{i,1} \sim \sum_i f_{i,2} \sim \cdots \sim \sum_i f_{i,k} \sim 0$$

from f_0 to zero in $\Gamma_S(R)$. Hence the elements

$$f_0 + \sum_i f_{i,1}, \sum_i f_{i,1} + \sum_i f_{i,2}, \ldots, \sum_i f_{i,k-1} + \sum_i f_{i,k}, \sum_i f_{i,k}$$

are in S. Thus $f_0 + f_{0,1}, f_{0,1} + f_{0,2}, \ldots, f_{0,k-1} + f_{0,k}, f_{0,k}$ are in S_0. On the other hand, we have $f_0 = (f_0 + f_{0,1}) - (f_{0,1} + f_{0,2}) + \cdots + (-1)^{k-1}(f_{0,k-1} + f_{0,k}) + (-1)^k f_{0,k}$, which implies that f_0 is finite sum of elements of S_0. The last assertion follows from Corollary 3.5. □

Since studying $\Gamma_{S_0}(R_0)$ is usually simpler than $\Gamma_S(R)$, Theorem 3.12 provides a suitable criterion about the disconnectedness of $\Gamma_S(R)$, where R is a graded ring. The following example illustrates this fact.

Example 3.13. Consider $R = (\mathbb{Z}_4 \times \mathbb{Z}_4)[x]$ and the I_S-graded m.c.s

$$S = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_0 \in \{(1,1),(1,3),(3,1),(3,3)\},$$
$$a_i \in \{(0,0),(0,2),(2,0),(2,2)\}, \forall \ i = 1, \ldots, n\}$$

of R. Note that $S_0 = \{(1,1),(1,3),(3,1),(3,3)\}$. Therefore $\Gamma_{S_0}(R_0)$ is disconnected, by Example 3.2. Now, Theorem 3.12 yields that $\Gamma_S(R)$ is also disconnected.

Recall that a graph is said to be *Eulerian* if it has a closed trail containing all edges. It is well-known that a graph is Eulerian if and only if it is connected and the valency of its vertices is even (see [17, 1.2.10]). We end this section with the following result concerning $\Gamma_S(R)$, when it is Eulerian.

Corollary 3.14. *Let S be an I_S-graded m.c.s of R such that $S = -S$. If $\Gamma_S(R)$ is Eulerian, then*

(i) *every element of R_0 is a finite sum of elements of S_0,*
(ii) *$|S|$ is an even number,*
(iii) *$x + x \notin S$ for each $x \in R$.*

Proof. (i) Since $\Gamma_S(R)$ is Eulerian, it is connected. Now the result follows from Theorem 3.12.

(ii) immediately follows from Proposition 2.9(ii) and the fact that in an Eulerian graph the valency of all vertices is even.

(iii) Assume to the contrary that $x+x \in S$ for some $x \in R$. Then $\deg_{\Gamma_S(R)}(x) = |S| - 1$ by [9, Lemma 1.3(b)]. Now in view of (ii), $\deg_{\Gamma_S(R)}(x)$ is odd which is a contradiction. □

4. Diameter and girth

Suppose that G is a graph with vertex set V. Recall that the *diameter* of G, denoted by $\mathrm{diam}(G)$, is defined as follows:

$$\mathrm{diam}(G) := \sup\{d(a,b) | a, b \in V\}.$$

We set $\mathrm{diam}(G) = \infty$ if G is disconnected. Also recall that the *girth* of G, denoted by $\mathrm{girth}(G)$, is the length of a shortest cycle in G if G has a cycle; otherwise, $\mathrm{girth}(G) = \infty$.

By a similar argument to that used in [3] we have the following result.

Theorem 4.1. (Compare [3, Corollary 3.4]). *For a graded ring R and I_S-graded m.c.s S of R, we have the inequalities*

$$\mathrm{diam}(\Gamma_{S_0}(R_0)) \leq \mathrm{diam}(\Gamma_S(R)),$$

and

$$\mathrm{girth}(\Gamma_S(R)) \leq \mathrm{girth}(\Gamma_{S_0}(R_0)).$$

Lemma 4.2. (Compare [3, Lemma 3.3]) *Let S be a saturated multiplicatively closed subset of R such that $R_i \subseteq I_S$ for all $i \neq 0$ and $\sum_i f_i$, $\sum_i g_i$ be two distinct vertices of $\Gamma_S(R)$.*

(i) *If $f_0 = g_0$, then*

$$d_{\Gamma_S(R)}\left(\sum_i f_i, \sum_i g_i\right) = \begin{cases} 1 & 2f_0 \in S \\ 2 & 2f_0 \notin S\,, \end{cases}$$

(ii) *If $f_0 \neq g_0$, then*

$$d_{\Gamma_S(R)}\left(\sum_i f_i, \sum_i g_i\right) = d_{\Gamma_{S_0}(R_0)}(f_0, g_0)\,.$$

Proof. (i) If $2f_0 \in S$, then Lemma 2.3 implies that $\sum_i f_i \sim \sum_i g_i$. Otherwise, by Lemma 2.3, we have the path $\sum_i f_i \sim 1 - f_0 \sim \sum_i g_i$ in $\Gamma_S(R)$. Note that since $2f_0 \notin S$, we have $1 - f_0 \neq \sum_i f_i$ and $1 - f_0 \neq \sum_i g_i$.

(ii) Similar to the proof of [3, Lemma 3.3(ii)]. □

The following theorem is a natural generalization of [3, Theorem 3.5] and [9, Theorem 2.23]. Although some parts of its proof are similar to [3, Theorem 3.5], we give a complete proof for it here.

Theorem 4.3. (Compare [3, Theorem 3.5] and [9, Theorem 2.23]). *Let S be a saturated multiplicatively closed subset of R such that $R_i \subseteq I_S$ for all $i \neq 0$.*

(i) *If $S_0 = R_0 \setminus \{0\}$, then* $\mathrm{diam}(\Gamma_S(R)) \leq 2$.

(ii) *If $\Gamma_{S_0}(R_0)$ is complete, then* $\mathrm{diam}(\Gamma_S(R)) \leq 2$,

(iii) *If $\Gamma_{S_0}(R_0)$ is not complete, then* $\mathrm{diam}(\Gamma_{S_0}(R_0)) = \mathrm{diam}(\Gamma_S(R))$,

(iv) *If R_0 is a finite ring, then* $\mathrm{diam}(\Gamma_S(R)) \in \{1, 2, 3, \infty\}$.

Proof. (i) Let $\sum_i f_i$ and $\sum_i g_i$ be two distinct vertices of $\Gamma_S(R)$. If f_0 and g_0 are non-zero, then in view of Lemma 2.3, $\sum_i f_i \sim 0 \sim \sum_i g_i$ is a path from $\sum_i f_i$ to $\sum_i g_i$ in $\Gamma_S(R)$. In addition, if $f_0 = 0$ or $g_0 = 0$, but not both, then $\sum_i f_i$ is adjacent to $\sum_i g_i$. Moreover, if $f_0 = g_0 = 0$, then Lemma 2.3 implies that $\sum_i f_i \sim 1 \sim \sum_i g_i$ is a path from $\sum_i f_i$ to $\sum_i g_i$. Therefore, $\mathrm{diam}(\Gamma_S(R)) \leq 2$.

(ii) Let $\sum_i f_i$ and $\sum_i g_i$ be two distinct vertices of $\Gamma_S(R)$. Since $\Gamma_{S_0}(R_0)$ is complete, if $f_0 \neq g_0$, then $f_0 + g_0 \in S_0$. By Lemma 2.3, we have $\sum_i f_i + \sum_i g_i \in S$. If $f_0 = g_0$, then since $\Gamma_{S_0}(R_0)$ has no isolated vertex, there exists a vertex $h_0 \in R_0$ adjacent to f_0. Hence $\sum_i f_i \sim h_0 \sim \sum_i g_i$ by Lemma 2.3, and so $\mathrm{diam}(\Gamma_S(R)) \leq 2$.

(iii) Assume that for two natural numbers m and n, $\mathrm{diam}(\Gamma_{S_0}(R_0)) = n$ and

$$\mathrm{diam}(\Gamma_S(R)) = d_{\Gamma_S(R)}\left(\sum_i f_i, \sum_i g_i\right) = m.$$

(Note that, in view of Lemma 4.2, if $\mathrm{diam}(\Gamma_{S_0}(R_0))$ is finite, then $\mathrm{diam}(\Gamma_S(R))$ is also finite.) Since $\Gamma_{S_0}(R_0)$ is not complete, $n \geq 2$. Hence Theorem 4.1 yields that $m \geq 2$. Now, if $f_0 = g_0$, then by Lemma 4.2 (i), $d_{\Gamma_S(R)}(\sum_i f_i, \sum_i g_i) = 2$, and so the result follows from Theorem 4.1 in this case. Also, if $f_0 \neq g_0$, then by Lemma 4.2 (ii), $d_{\Gamma_{S_0}(R_0)}(f_0, g_0) = m$. Therefore, we should have $n \geq m$. Now, Theorem 4.1 completes the proof.

(iv) In view of Remarks 2.8(1) and [9, Theorem 2.23], we have

$$\mathrm{diam}(\Gamma_{S_0}(R_0)) \in \{1, 2, 3, \infty\}.$$

Now, (ii) and (iii) insure that $\mathrm{diam}(\Gamma_S(R)) \in \{1, 2, 3, \infty\}$ as desired. □

So, we are also able to improve Corollary 3.6 of [3] as follows.

Corollary 4.4. (Compare [3, Corollary 3.6]). *Let S be a saturated multiplicatively closed subset of R, $S_0 = R_0 \setminus \{0\}$, $\mathrm{char}(R_0) \neq 2$ and $R_i \subseteq I_S$ for all $i \neq 0$. Then*

$$\mathrm{diam}(\Gamma_{S_0}(R_0)) = \mathrm{diam}(\Gamma_S(R)) = 2.$$

Proof. The result follows from Proposition 2.1 of [9], and parts (i) and (iii) of Theorem 4.3. □

Theorem 4.3 provides an indirect method to calculate the diameter of $\Gamma_S(R)$. The following examples illustrate this fact.

Example 4.5. In Example 3.3, we saw that $\operatorname{diam}(\Gamma_S(R)) = 2$. Now, without calculating the diameter of $\Gamma_S(R)$, we show this. It is easily seen that $S_0 = D\backslash\{0\}$ and $R_0 = D$. Hence, by Theorem 4.3(i), we have $\operatorname{diam}(\Gamma_S(R)) \le 2$. Now since $\Gamma_S(R)$ is not a complete graph (e.g., $x \nsim x^2$), we have $\operatorname{diam}(\Gamma_S(R)) = 2$.

Example 4.6. In Example 3.13, we saw that $\Gamma_{S_0}(R_0)$ is not complete. Therefore, by Theorem 4.3(iii), $\operatorname{diam}(\Gamma_{S_0}(R_0)) = \operatorname{diam}(\Gamma_S(R))$. Since $\Gamma_{S_0}(R_0)$ is the union of two disjoint $K_{4,4}$s, $\operatorname{diam}(\Gamma_{S_0}(R_0)) = \infty$. Hence we also have $\operatorname{diam}(\Gamma_S(R)) = \infty$.

Theorem 4.7. *Assume that S is an I_S-graded m.c.s of R.*

(i) *If* $\operatorname{girth}(\Gamma_S(R)) \ge 4$ *and* $I_S = \{0\}$, *then* $\operatorname{girth}(\Gamma_S(R)) = \operatorname{girth}(\Gamma_{S_0}(R_0))$.

(ii) *If* $\operatorname{girth}(\Gamma_S(R)) = \operatorname{girth}(\Gamma_{S_0}(R_0)) \ge 4$, *then* $I_S = \{0\}$ *or* $2 \notin S$.

(iii) *(See* [9, Lemma 2.14].*) If* $I_S \neq 0$, *then* $\operatorname{girth}(\Gamma_S(R)) \le 4$.

(iv) *If* R_0 *is finite and* $I_S = \{0\}$, *then* $\operatorname{girth}(\Gamma_S(R)) \in \{3, 4, 6, \infty\}$.

Proof. (i) In view of Theorem 4.1, it is enough to show that $\operatorname{girth}(\Gamma_S(R)) \ge \operatorname{girth}(\Gamma_{S_0}(R_0))$. So, we may assume that $\operatorname{girth}(\Gamma_S(R)) = m$ for some integer $m \ge 4$. Let

$$\sum_i f_{i,1} \sim \sum_i f_{i,2} \sim \cdots \sim \sum_i f_{i,m-1} \sim \sum_i f_{i,m} \sim \sum_i f_{i,1}$$

be a cycle in $\Gamma_S(R)$. In the light of Lemma 2.3, we have

$$f_{0,1} \sim f_{0,2} \sim \cdots \sim f_{0,m-1} \sim f_{0,m} \sim f_{0,1}$$

in $\Gamma_{S_0}(R_0)$. This induces a cycle in $\Gamma_{S_0}(R_0)$ with length smaller than m, which implies that $\operatorname{girth}(\Gamma_{S_0}(R_0)) \le m$ unless $f_{0,j} = f_{0,j+1}$ for all $1 \le j \le m-2$. On the other hand, if $f_{0,j} = f_{0,j+1}$ for all $1 \le j \le m-2$, then in view of Remarks 2.8(4), we have $\sum_i f_{i,1} = \sum_i f_{i,3}$, which is a contradiction. So, $\operatorname{girth}(\Gamma_{S_0}(R_0)) \le m$ as required.

(ii) Suppose to the contrary that $I_S \neq \{0\}$ and $2 \in S$. Then for each $s \in S$ and $0 \neq r \in I_S$, $0 \sim s \sim r + s \sim 0$ is a cycle in $\Gamma_S(R)$, by Proposition 2.9(i). Hence $\operatorname{girth}(\Gamma_S(R)) = 3$, which is a contradiction.

(iii) Choose $s \in S$ and $0 \neq r \in I_S$. It is easily seen that $0 \sim s \sim r \sim s-r \sim 0$ is a cycle in $\Gamma_S(R)$ by Proposition 2.9(i).

(iv) follows from Theorem 2.15 in [9] and (i). □

Corollary 4.8. (Compare [3, Proposition 3.7(iii)]). *Let R be a positively graded ring or for each element $\sum_i f_i$ of R, $f_0 \notin P$ for all $P \in \mathrm{Nil}(R)$. Assume that $S = U(R)$ and $J(R) = 0$. Then*

(i) $\mathrm{girth}(\Gamma_S(R)) = \mathrm{girth}(\Gamma_{S_0}(R_0))$.

(ii) *If R_0 is finite, then* $\mathrm{girth}(\Gamma_S(R)) \in \{3, 4, 6, \infty\}$.

Proof. (i) By Remark 2.6 and Theorem 4.7(i), we only need to consider the case $\mathrm{girth}(\Gamma_S(R)) = 3$. Also, by Theorem 4.1, we have $\mathrm{girth}(\Gamma_S(R)) \leq \mathrm{girth}(\Gamma_{S_0}(R_0))$. So, let

$$\sum_i f_{i,1} \sim \sum_i f_{i,2} \sim \sum_i f_{i,3} \sim \sum_i f_{i,1}$$

be a triangle in $\Gamma_S(R)$. If $f_{0,1}, f_{0,2}$ and $f_{0,3}$ are distinct, then

$$f_{0,1} \sim f_{0,2} \sim f_{0,3} \sim f_{0,1}$$

is a triangle in $\Gamma_{S_0}(R_0)$ by Lemma 2.3 and Remark 2.6. Therefore, without loss of generality, we can assume that $f_{0,1} = f_{0,2}$ and so $2f_{0,1} \in S_0$. In view of Remarks 2.8(1), $2 \in S_0$ which implies that 2 is invertible. Since $I_S = \{0\}$, by Remarks 2.8(4), we have $f_{i,1} + f_{i,2} = 0, f_{i,2} + f_{i,3} = 0$ and $f_{i,3} + f_{i,1} = 0$ for all $i \neq 0$. Now, we have $2f_{i,1} = (f_{i,1} + f_{i,2} + f_{i,3}) + (f_{i,1} + f_{i,2} + f_{i,3}) = (f_{i,1} + f_{i,2}) + (f_{i,3} + f_{i,1}) + (f_{i,2} + f_{i,3}) = 0$. Since 2 is invertible, $f_{i,1} = 0$ for all $i \neq 0$ which implies that $f_{i,2} = 0$ for all $i \neq 0$. Hence $\sum_i f_{i,1} = \sum_i f_{i,2}$ which is a contradiction. Therefore $f_{0,1} \neq f_{0,2}$. Since $f_{0,1}, f_{0,2}$ were arbitrary, $f_{0,1} \sim f_{0,2} \sim f_{0,3} \sim f_{0,1}$ is a triangle in $\Gamma_{S_0}(R_0)$. Hence $\mathrm{girth}(\Gamma_{S_0}(R_0)) = 3$ as desired.

(ii) immediately follows from [9, Theorem 2.15] and (i). □

Corollary 4.9. *Let S be an I_S-graded m.c.s of R and* $\mathrm{girth}(\Gamma_{S_0}(R_0)) = 4$. *Then* $\mathrm{girth}(\Gamma_S(R)) = 4$ *if and only if* $I_S = \{0\}$ *or* $2 \notin S$.

Proof. In the light of Theorem 4.7(ii), we only need to prove the *if part.* By Theorem 4.1, we have $\mathrm{girth}(\Gamma_S(R)) \leq \mathrm{girth}(\Gamma_{S_0}(R_0)) = 4$; so $\mathrm{girth}(\Gamma_S(R))$ can be 3 or 4. Let $\mathrm{girth}(\Gamma_S(R)) = 3$ and seek a contradiction. Assume that $\sum_i f_{i,1} \sim \sum_i f_{i,2} \sim \sum_i f_{i,3} \sim \sum_i f_{i,1}$ is a triangle in $\Gamma_S(R)$. In view of Lemma 2.3, $f_{0,1} \sim f_{0,2} \sim f_{0,3} \sim f_{0,1}$ is a triangle in $\Gamma_{S_0}(R_0)$, which is a contradiction, unless $f_{0,i} =$

$f_{0,j}$ for some $1 \leq i, j \leq 3$ with $i \neq j$. We may assume that $f_{0,1} = f_{0,2}$, which insures that $2f_{0,1} \in S_0$. Therefore $2 \in S$. So, by our assumption we have $I_S = \{0\}$. Now Remarks 2.8(4) shows that $\sum_i f_{i,1} = \sum_i f_{i,3}$, which is a contradiction. Therefore $\mathrm{girth}(\Gamma_S(R)) = 4$ as desired. □

Remark 4.10. Note that by slight modifications similar to those we have used in this note, one can gain the graded version of some of the other results in [3], such as Theorems 4.4, 5.4 and 5.6.

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