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On weakly symmetric Riemannian manifolds

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1. Introduction

The notion of weakly symmetric Riemannian manifold has recently been introduced and investigated by T. Q. Binh and L. Tamássy [1], [4]. This is a non-flat Riemannian manifold whose curvature tensor R_{hijk} satisfies the condition

$$\nabla_r R_{hijk} = A_r R_{hijk} + B_h R_{rijk} + C_i R_{hrjk} + D_j R_{hirk} + E_k R_{hijr}, \tag{1.1}$$

where A, B, C, D, E are 1-forms which are not zero simultaneously and ∇ denotes covariant differentiation with respect to the Riemannian metric.

In the case $B = C = D = E = \frac{1}{2}A$, a weakly symmetric manifold is just a pseudo-symmetric manifold as introduced and investigated by M. C. Chaki [2], [3].

We mention still one case, namely the case $B = C = D = E \neq \frac{1}{2}A$, in which instead of (1.1), we have the condition

$$\nabla_r R_{hijk} = F_r R_{hijk} + D_h R_{rijk} + D_i R_{hrjk} + D_j R_{hirk} + D_k R_{hijr}. \tag{1.2}$$

Now, we recall the definition of a B-space, given by P. Venzi [5]. Let $\mathcal{L}(\Theta)$ be a vector space formed by all vectors Θ satisfying

$$\Theta_\ell R_{hijk} + \Theta_j R_{hik\ell} + \Theta_k R_{hi\ell j} = 0. \tag{1.3}$$

A Riemannian space is said to be a B-space if $\dim \mathcal{L}(\Theta) \geq 1$.

In §2 of the present paper, we prove that if a weakly symmetric Riemannian space is not a pseudo-symmetric manifold (in the sense of Chaki), then is a B-space. In §§3,4 and 5 we determine the necessary and sufficient conditions for a B-space to be weakly symmetric. Doing this we shall show that the condition (1.1) always reduces to (1.2).

2. Weakly symmetric Riemannian space as a B-space

Symmetrizing (1.1) with respect to h and i, we get

$$(B_h - C_h)R_{rijk} + (B_i - C_i)R_{rhjk} = 0. \tag{2.1}$$

This relation implies $B_h = C_h$. In fact, let us suppose $B_1 \neq C_1$. Then (2.1) with $h = i = 1$ gives $2(B_1 - C_1)R_{r1jk} = 0$, and therefore $R_{r1jk} = 0$ for all r, j, k. Putting now $h = 1$ in (2.1), we have $(B_1 - C_1)R_{rijk} = 0$, whence $R_{rijk} = 0$ for all r, i, j, k. But this contradicts our assumption that the manifold is non-flat. Thus $B_1 = C_1$. Repeating the procedure for each $h = 2, \dots, n$ we get $B_h = C_h$. In a similar manner, symmetrizing (1.1) with respect to j and k we get $D_j = E_j$. Thus, the condition (1.1) reduces to

$$\nabla_r R_{hijk} = A_r R_{hijk} + B_h R_{rijk} + B_i R_{hrjk} + D_j R_{hirk} + D_k R_{hijr}. \tag{2.2}$$

Applying the second Bianchi identity to (2.2), we get

$$(A_r - 2B_r)R_{hijk} + (A_h - 2B_h)R_{irjk} + (A_i - 2B_i)R_{rhjk} = 0, \tag{2.3}$$

and

$$(A_r - 2D_r)R_{hijk} + (A_j - 2D_j)R_{hikr} + (A_k - 2D_k)R_{hirj} = 0, \tag{2.4}$$

from which we find

$$(B_r - D_r)R_{hijk} + (B_h - D_h)R_{irjk} + (B_i - D_i)R_{rhjk} = 0. \tag{2.5}$$

We see that if $A_i \neq 2B_i$ or $A_i \neq 2D_i$, then the conditions (2.3) and (2.4) are of the form (1.3). Thus, we have proved

Theorem 1. *If a weakly symmetric Riemannian manifold is not pseudo-symmetric (in the sense of Chaki), then it is a B-space.*

In he sequel, we try to find the conditions for a B-space to be weakly symmetric. First, we note that ([5], Theorem 1) *for each B-space,* $\dim \mathcal{L}(\Theta) \leq 2$. Thus, for further investigation, we have to consider two cases: $\dim \mathcal{L}(\Theta) = 1$ and $\dim \mathcal{L}(\Theta) = 2$.

3. The case of $\dim \mathcal{L}(\Theta) = 1$

In view of (2.5), $B_i - D_i \in \mathcal{L}(\Theta)$ from which, taking into account the assumption $\dim \mathcal{L}(\Theta) = 1$, we find

$$B_i = \beta \Theta_i + D_i. \tag{3.1}$$

On the other hand, for each B-space, there exists a symmetric tensor T_{ij} such that ([5], Theorem 2)

$$R_{hijk} = T_{hk}\Theta_i\Theta_j + T_{ij}\Theta_h\Theta_k - T_{hj}\Theta_i\Theta_k - T_{ik}\Theta_h\Theta_j, \tag{3.2}$$

where Θ is the basis vector of the space $\mathcal{L}(\Theta)$.

Thus, if this B-space is simultaneously weakly symmetric, then we have

$$\begin{aligned}
&\Theta_i\Theta_j\nabla_r T_{hk} + \Theta_h\Theta_k\nabla_r T_{ij} - \Theta_i\Theta_k\nabla_r T_{hj} - \Theta_h\Theta_j\nabla_r T_{ik}\\
&\quad + \Theta_h(T_{ij}\nabla_r\Theta_k - T_{ik}\nabla_r\Theta_j) + \Theta_i(T_{hk}\nabla_r\Theta_j - T_{hj}\nabla_r\Theta_k)\\
&\quad + \Theta_j(T_{hk}\nabla_r\Theta_i - T_{ik}\nabla_r\Theta_h) + \Theta_k(T_{ij}\nabla_r\Theta_h - T_{hj}\nabla_r\Theta_i) =\\
&= A_r R_{hijk} + D_h R_{rijk} + D_i R_{hrjk} + D_j R_{hirk} + D_k R_{hijr}\\
&\quad + \beta(\Theta_h R_{rijk} + \Theta_i R_{hrjk}).
\end{aligned} \tag{3.3}$$

Now, let v^i be a vector field such that $\Theta_a v^a = 1$ and let us put

$$T_{hk}v^h = u_k, \quad T_{hk}v^h v^k = u_k u^k = \psi.$$

Then, transvecting (3.3) with $v^h v^k$ and using (3.2), we get

$$\begin{aligned}
\nabla_r T_{ij} =& s_r T_{ij} + t_r\Theta_i\Theta_j + \Theta_i H_{rj} + \Theta_j H_{ri} + u_j\nabla_r\Theta_i + u_i\nabla_r\Theta_j\\
&+ D_i(\psi\Theta_r\Theta_j + T_{rj} - u_j\Theta_r) + D_j(\psi\Theta_i\Theta_r + T_{ri} - u_i\Theta_r),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
s_r &= -2(\nabla_r\Theta_a)\Theta^a + A_r + (\beta + 2D_a v^a)\Theta_r,\\
t_r &= -(\nabla_r T_{ab})v^a v^b + \psi A_r + \beta\psi\Theta_r + 2D_a v^a u_r,\\
H_{rj} &= (\nabla_r T_{aj})v^a - \psi\nabla_r\Theta_j + (\nabla_r\Theta_a)v^a u_j - A_r u_j\\
&\quad - (\beta + D_a v^a)u_j\Theta_r - D_a v^a T_{rj} - D_j u_r.
\end{aligned}$$

On the other hand, differentiating (1.3) and using (2.2) and (1.3), we obtain

$$\begin{aligned}
(\nabla_r\Theta_\ell - D_\ell\Theta_r)R_{hijk} + (\nabla_r\Theta_j - D_j\Theta_r)R_{hik\ell}\\
+(\nabla_r\Theta_k - D_k\Theta_r)R_{hi\ell j} = 0,
\end{aligned} \tag{3.5}$$

from which, contracting with $v^h v^k$, using (3.2) and putting

$$p_r = (\nabla_r\Theta_a)\Theta^a - D_a v^a\Theta_r,$$

we get

$$
(3.6)\quad
\begin{aligned}
&T_{ij}\nabla_r\Theta_\ell - T_{i\ell}\nabla_r\Theta_j = (u_j\Theta_i + u_i\Theta_j)\nabla_r\Theta_\ell \\
&- (u_\ell\Theta_i + u_i\Theta_\ell)\nabla_r\Theta_j + \big[(T_{ij} + \psi\Theta_i\Theta_j \\
&- u_j\Theta_i - u_i\Theta_j)D_\ell - (T_{i\ell} + \psi\Theta_i\Theta_\ell - u_\ell\Theta_i \\
&- u_i\Theta_\ell)D_j\big]\Theta_r - \psi\big[(\nabla_r\Theta_\ell)\Theta_j - (\nabla_r\Theta_j)\Theta_\ell\big]\Theta_i \\
&- p_r\big[T_{i\ell}\Theta_j - T_{ij}\Theta_\ell + (u_j\Theta_\ell + u_\ell\Theta_j)\Theta_i\big].
\end{aligned}
$$

Thus, if the B-space considered is weakly symmetric, then T_{ij} and Θ_i satisfy the conditions (3.4) and (3.6).

Conversely, let us consider the Riemannian space whose curvature tensor can be expressed in the form (3.2) (it is easy to see that such a space is a B-space). Further, let us suppose that T_{ij} and Θ_i satisfy (3.4) and (3.5) where s_i, t_i, u_i, p_i and D_i are some vector fields while H_{ri} is some tensor field. Then we find that

$$\nabla_r R_{hijk} = (s_r + 2p_r)R_{hijk} + D_h R_{rijk} + D_i R_{hrjk} + D_j R_{hirk} + D_k R_{hijk}.$$

Thus, we can state

Theorem 2. *In a B-space there exists a symmetric tensor field T_{ij} such that the curvature tensor has the form* (3.2), *where Θ_i is the vector of the basis of $\mathcal{L}(\Theta)$. In order that such a space with $\dim\mathcal{L}(\Theta) = 1$ be weakly symmetric, it is necessary and sufficient that* (3.4) *and* (3.6) *hold. This weak symmetry is of the form* (1.2).

4. The case when $\dim\mathcal{L}(\Theta) = 1$ and the basis for $\mathcal{L}(\Theta)$ is not a null vector field

If a Riemannian manifold is a B-space, then ([5], Theorem 3)

$$\Theta_a\Theta^a R_{hijk} = \Theta_i\Theta_j R_{hk} + \Theta_h\Theta_k R_{ij} - \Theta_h\Theta_j R_{ik} - \Theta_i\Theta_k R_{hj},$$

where R_{ij} is the Ricci tensor. If $\dim\mathcal{L}(\Theta) = 1$ and the basis for $\mathcal{L}(\Theta)$ is not a null vector, then we can suppose $\Theta_a\Theta^a = \varepsilon$, $\varepsilon = 1$ or -1, i.e. without loss of generality, the preceding relation can be written in the form

$$(4.1)\qquad R_{hijk} = \varepsilon(\Theta_i\Theta_j R_{hk} + \Theta_h\Theta_k R_{ij} - \Theta_h\Theta_j R_{ik} - \Theta_i\Theta_k R_{hj}),$$

where now Θ is a unit basis vector for $\mathcal{L}(\Theta)$.

Transvecting (2.2) with g^{hk}, we have

$$(4.2)\qquad \nabla_r R_{ij} = A_r R_{ij} + B_i R_{rj} + D_j R_{ir} + B_a R^a{}_{jir} + D_a R^a{}_{ijr}.$$

But, transvecting (2.5) with g^{hk} we obtain

$$B_a R^a{}_{jir} = (B_r - D_r)R_{ij} - (B_i - D_i)R_{rj} + D_a R^a{}_{jir}.$$

Substituting this into (4.2), we get

$$\nabla_r R_{ij} = (A_r + B_r - D_r)R_{ij} + D_i R_{rj} + D_j R_{ir} + D_a(R^a{}_{jir} + R^a{}_{ijr}). \tag{4.3}$$

In view of (2.4) and because of the assumption $\dim \mathcal{L}(\Theta) = 1$, we have $A_i - 2D_i \in \mathcal{L}(\Theta)$. Thus, besides (3.1) we have $A_i = \alpha\Theta_i + 2D_i$. Using this, (3.1) and (4.1), we can rewrite (4.3) into the form

$$\begin{aligned} \nabla_r R_{ij} &= (\gamma\Theta_r + 2D_r)R_{ij} + (D_i - \varepsilon D_a\Theta^a\Theta_i)R_{jr} \\ &\quad + (D_j - \varepsilon D_a\Theta^a\Theta_j)R_{ir} + 2\varepsilon D_a R^a{}_r\Theta_i\Theta_j \\ &\quad - \varepsilon(\Theta_j\Theta_a R^a{}_i + \Theta_i\Theta_a R^a{}_j)\Theta_r, \end{aligned} \tag{4.4}$$

where γ is a scalar function.

We recall that in this section the vector field Θ satisfies $\Theta_a\Theta^a = \varepsilon$ and so we have $(\nabla_r\Theta_a)\Theta^a = 0$. Also, transvecting (1.3) with g^{hk}, we find

$$\Theta_a R^a{}_{i\ell j} = \Theta_j R_{i\ell} - \Theta_\ell R_{ij},$$

from which

$$\Theta_a R^a{}_j = \frac{1}{2}R\Theta_j \quad \text{and} \quad \Theta^a\Theta^b R_{iabj} = \varepsilon R_{ij} - \frac{1}{2}R\Theta_i\Theta_j, \tag{4.5}$$

where R is the scalar curvature of the manifold. Now, transvecting (3.5) with $\Theta^h\Theta^k$ we get

$$\begin{aligned} & R_{ij}\nabla_r\Theta_\ell - R_{i\ell}\nabla_r\Theta_j = \\ &= [R_{ij}(D_\ell - \varepsilon D_a\Theta^a\Theta_\ell) - R_{i\ell}(D_j - \varepsilon D_a\Theta^a\Theta_j)]\Theta_r \\ &+ \frac{1}{2}\varepsilon R[(\nabla_r\Theta_\ell - D_\ell\Theta_r)\Theta_j - (\nabla_r\Theta_j - D_j\Theta_r)\Theta_\ell]\Theta_i, \end{aligned} \tag{4.6}$$

while (4.4) can be rewritten as follows:

$$\begin{aligned} \nabla_r R_{ij} =& (\gamma\Theta_r + 2D_r)R_{ij} + (D_i - \varepsilon D_a\Theta^a\Theta_i)R_{rj} \\ &+ (D_j - \varepsilon D_a\Theta^a\Theta_j)R_{ir} + \varepsilon(2D_a R^a{}_r - R\Theta_r)\Theta_i\Theta_j. \end{aligned} \tag{4.7}$$

Conversely, substituting (4.6) and (4.7) into

$$\begin{aligned} \nabla_r R_{hijk} =& \varepsilon[\Theta_i\Theta_j\nabla_r R_{hk} + \Theta_h\Theta_k\nabla_r R_{ij} - \Theta_h\Theta_j\nabla_r R_{ik} - \Theta_i\Theta_k\nabla_r R_{hj} \\ &+ \Theta_h(R_{ij}\nabla_r\Theta_k - R_{ik}\nabla_r\Theta_j) + \Theta_i(R_{hk}\nabla_r\Theta_j - R_{hj}\nabla_r\Theta_k) \\ &+ \Theta_j(R_{kh}\nabla_r\Theta_i - R_{ki}\nabla_r\Theta_h) + \Theta_k(R_{ji}\nabla_r\Theta_h - R_{jh}\nabla_r\Theta_i)] \end{aligned}$$

and using (4.1), we get

$$\nabla_r R_{hijk} = [(\gamma - 2\varepsilon D_a \Theta^a)\Theta_r + 2D_r]R_{hijk} + D_h R_{rijk} + D_i R_{hrjk} + D_j R_{hirk} + D_k R_{hijr}.$$

Thus we obatain

Theorem 3. *Let us consider a B-space such that* $\dim \mathcal{L}(\Theta) = 1$ *and the basis for* $\mathcal{L}(\Theta)$ *is a unit vector field. In order that such a space be weakly symmetric, it is necessary and sufficient that the Ricci tensor and the basis vector* Θ *satisfy* (4.6) *and* (4.7). *This weak symmetry is of the form* (1.2).

5. The case of $\dim \mathcal{L}(\Theta) = 2$

P. Venzi proved in [5] that a Riemannian manifold is a B-space characterized by $\dim \mathcal{L}(\Theta) = 2$ if and only if there exists a coordinate system such that

$$R_{hijk} = \phi(\Theta_h \tilde{\Theta}_i - \tilde{\Theta}_h \Theta_i)(\Theta_j \tilde{\Theta}_k - \tilde{\Theta}_j \Theta_k), \tag{5.1}$$

where ϕ is a scalar function while the basis vectors Θ and $\tilde{\Theta}$ satisfy

$$\begin{aligned} \nabla_r \tilde{\Theta}_h &= a_r \tilde{\Theta}_h + b_r \Theta_h + c_h \tilde{\Theta}_r + d_h \Theta_r, \\ \nabla_r \Theta_h &= e_r \tilde{\Theta}_h + f_r \Theta_h + g_h \tilde{\Theta}_r + r_h \Theta_r. \end{aligned} \tag{5.2}$$

Thus, for a B-space which is weakly symmetric, we have

$$\begin{aligned} &\left[\frac{\nabla_r \phi}{\phi} + 2(a_r + f_r) - A_r\right](\Theta_h \tilde{\Theta}_i - \tilde{\Theta}_h \Theta_i)(\Theta_j \tilde{\Theta}_k - \tilde{\Theta}_j \Theta_k) \\ &+ \big[c_i \Theta_h \tilde{\Theta}_r + d_i \Theta_h \Theta_r + g_h \tilde{\Theta}_i \tilde{\Theta}_r + r_h \tilde{\Theta}_i \Theta_r \\ &- c_h \Theta_i \tilde{\Theta}_r - d_h \Theta_i \Theta_r - g_i \tilde{\Theta}_h \tilde{\Theta}_r - r_i \tilde{\Theta}_h \Theta_r \\ &- B_h(\Theta_r \tilde{\Theta}_i - \tilde{\Theta}_r \Theta_i) - B_i(\Theta_h \tilde{\Theta}_r - \tilde{\Theta}_h \Theta_r)\big](\Theta_j \tilde{\Theta}_k - \tilde{\Theta}_j \Theta_k) \\ &+ [c_k \Theta_j \tilde{\Theta}_r + d_k \Theta_j \Theta_r + g_j \tilde{\Theta}_k \tilde{\Theta}_r + r_j \tilde{\Theta}_k \Theta_r \\ &- c_j \Theta_k \tilde{\Theta}_k - d_j \Theta_k \Theta_r - g_k \tilde{\Theta}_j \tilde{\Theta}_r - r_k \tilde{\Theta}_j \Theta_r \\ &- D_j(\Theta_r \tilde{\Theta}_k - \tilde{\Theta}_r \Theta_k) - D_k(\Theta_j \tilde{\Theta}_r - \tilde{\Theta}_j \Theta_k)](\Theta_h \tilde{\Theta}_i - \tilde{\Theta}_h \Theta_i) = 0 \end{aligned} \tag{5.3}$$

Let v and $\tilde{v}$ be vector fields such that

$$\begin{aligned} \Theta_a v^a = 1, &\quad \tilde{\Theta}_a v^a = 0, \\ \Theta_a \tilde{v}^a = 0, &\quad \tilde{\Theta}_a \tilde{v}^a = 1. \end{aligned}$$

(If the basis vectors Θ and $\tilde{\Theta}$ are orthogonal and not null vectors, we can choose them to be unit vectors and then we can put $v^a = \Theta^a$, $\tilde{v}^a = \tilde{\Theta}^a$.)

Transvecting (5.3) with $v^j\tilde{v}^k\tilde{v}^r v^i$ and $v^j\tilde{v}^k v^r\tilde{v}^i$ and subtracting, we find

$$r_h = \varrho\Theta_h + \tilde{\varrho}\tilde{\Theta}_h + c_h,$$

while transvecting with $v^j\tilde{v}^k\tilde{v}^r\tilde{v}^i$ and $v^j\tilde{v}^k v^r v^i$ we get respectively

$$g_h = \gamma\Theta_h + \tilde{\gamma}\tilde{\Theta}_h, \quad d_h = \delta\Theta_h + \tilde{\delta}\tilde{\Theta}_h,$$

where ϱ, $\tilde{\varrho}$, γ, $\tilde{\gamma}$, δ, $\tilde{\delta}$ are scalar functions. Thus, the conditions (5.2) reduce to

$$\begin{aligned} \nabla_r\tilde{\Theta}_h &= \bar{a}_r\tilde{\Theta}_h + \bar{b}_r\Theta_h + c_h\tilde{\Theta}_r \\ \nabla_r\Theta_h &= \bar{e}_r\tilde{\Theta}_h + \bar{f}_r\Theta_h + c_h\Theta_r. \end{aligned} \tag{5.4}$$

Conversely, let us consider a B-space statisfying $\dim\mathcal{L}(\Theta) = 2$ and (5.4). Then it is easy to see that $\nabla_r R_{hijk}$ has the form (1.2). Thus we can state

Theorem 4. *Let us consider a B-space characterized by* $\dim\mathcal{L}(\Theta) = 2$ *Then there exists a coordinate system such that* (5.1) *holds. In order that this B-space be weakly symmetric, it is necessary and sufficient that the conditions* (5.4) *are satisfied, too. The weak symmetry is of the form* (1.2).

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