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On finite groups with metacyclic Sylow p-subgroups

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Abstract. Let p be a prime and let g be a finite group with metacyclic Sylow p-subgroups. In this paper, we give some structural properties of G when $G/\mathrm{O}_{p'p}(G)$ is of odd order. As a consequence, we show that g is p-supersoluble if its order is coprime to $p+1$ and has very restricted structure in the case when $p \neq 2$ and its order is coprime to $p-1$.

1. Introduction

All groups considered in this paper are finite. In the sequel, p will denote a prime number; p' will be the set of all primes different from p.

The results of the present paper are further contributions to the project which studies global properties of the groups that are determined by the structure of their Sylow p-subgroups. One of the most classical results in this context asserts that a group G is p-nilpotent, that is, G possesses a normal p-complement, if p is the smallest prime dividing the order of G and the Sylow p-subgroups of G are cyclic. This is a consequence of Burnside's well-known p-nilpotency criterion and can be found, e. g., in [H, IV, 2.8]. In fact the proof works whenever $(|G|, p-1) = 1$. The alternating group of degree 5 is a non-2-nilpotent group with abelian metacyclic Sylow 2-subgroups. Hence the hypothesis on the Sylow subgroups in the above result is essential. However, if $(|G|, p^2-1) = 1$ and the Sylow p-subgroups of G are metacyclic, then G is also p-nilpotent (see [H, IV, 5.10]).

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The case $p = 2$ was analysed in [CG]. It is proved there that a group G with a metacyclic Sylow 2-subgroup S is soluble provided that S has a cyclic normal subgroup T such that S/T is cyclic of order at least 4. Futhermore, results on groups whose Sylow 2-subgroups have cyclic commutator groups where obtained by Chabot in [C].

The results of this paper spring from these sources and deal with groups G with metacyclic Sylow p-subgroups. We will show that some structural propeties can be completely determined by some arithmetical ones.

As usual $\mathrm{O}_p(G)$ (resp. $\mathrm{O}_{p'}(G)$) is the largest normal p-subgroup (resp. p'-subgroup) of a group G ; $\mathrm{O}_{p'p}(G)$ is the largest p-nilpotent normal subgroup of G and $\mathrm{O}_{p'p}(G)/\mathrm{O}_{p'}(G) = \mathrm{O}_p(G/\mathrm{O}_{p'}(G))$; it is known that $\mathrm{O}_{p'p}(G)$ is the intersection of the centralisers of the chief factors of G whose orders are divisible by p (see [DH, A, 13.8]).

Our main theorem gives a precise structure of a group with metacyclic Sylow p-subgroups provided that $G/\mathrm{O}_{p'p}(G)$ is of odd order.

Theorem A. *Let G be a group with metacyclic Sylow p-subgroups. Suppose that $G/\mathrm{O}_{p'p}(G)$ is of odd order, and let $G^* = G/\mathrm{O}_{p'}(G)$. Then $G^* = P \rtimes (H_1 \times H_2)$, where P is a Sylow p-subgroup of G^*, H_1 is an abelian group of exponent dividing $p-1$, H_2 is a cyclic group with exponent dividing $p+1$. Moreover, $H_1 \times H_2$ can be generated by two elements.*

The following two corollaries are direct consequences of Theorem A.

Corollary 1. *Let G be a group such that $G/\mathrm{O}_{p'p}(G)$ is of odd order. Assume that a Sylow p-subgroup P of G is metacyclic. Then G' is p-nilpotent.*

Corollary 2. *Let G be a group of odd order. If every Sylow subgroup of G is metacyclic, then G' is nilpotent. In particular, the Fitting length of G is at most 2.*

Let G be a group with metacyclic Sylow p-subgroups. In the following, we apply Theorem A to obtain some results on the structure of G when $|G|$ is coprime with $p+1$ or $p-1$. Recall that a group G is said to be *p-supersoluble* if every chief factor of G is either a cyclic group of order p or a p'-group.

Theorem B. *Let G be a group with metacyclic Sylow p-subgroups. If $|G/\mathrm{O}_{p'p}(G)|$ and $p+1$ are coprime, then G is p-supersoluble.*

The converse of Theorem B is not true: the symmetric group G of degree four has cyclic Sylow 3-subgroups, G is 3-supersoluble and $(|G/\mathrm{O}_{p'p}(G)|, 3+1) = 2$.

Suppose that G is a p-supersoluble group. Then $G/C_G(A/B)$ is a cyclic group of exponent dividing $p-1$, for each p-chief factor A/B of G. Hence $G/\operatorname{O}_{p'p}(G)$ is an abelian group of exponent dividing $p-1$. Since $(p-1, p+1) = 1$ or 2, it follows that $(\exp(G/\operatorname{O}_{p'p}(G)), p+1) = 1$ or 2.

Combining the above fact with Theorem B, we have:

Corollary 3. *Suppose G is a group of odd order with metacyclic Sylow p-subgroups. Then G is p-supersoluble if and only if $|G/\operatorname{O}_{p'p}(G)|$ and $p+1$ are coprime.*

The following result, due to Berkovich [B], was recently extended by Asaad and Monakhov [AM, 1.1].

Corollary 4. *Let the group $G = AB$ be the product of the subgroups A and B. If G is of odd order and the Sylow p-subgroups of A and B are cyclic, then G is p-supersoluble.*

Next, we consider what happens if G is a group with metacyclic Sylow p-subgroups such that $(|G|, p-1) = 1$. In this case, G is not necessarily p-supersoluble as the following example shows:

Example. Let p, q be two primes such that $2 < q$ divides $p+1$. Let C be a cyclic group of order q and let V be an irreducible and faithful C-module over the finite field of p-elements. Then V is an elementary abelian group of order p^2. Let $G = V \rtimes C$ be the corresponding semidirect product. Then G is a non p-supersoluble group with a metacyclic Sylow p-subgroup such that $(|G|, p-1) = 1$.

As a consequence of Theorem A, we have

Theorem C. *Let p be an odd prime and let G be a group with metacyclic Sylow p-subgroups. Set $G^* = G/\operatorname{O}_{p'}(G)$. Then $(|G/\operatorname{O}_{p'p}(G)|, p-1) = 1$ if and only if G^* satisfies the following properties:*

(1) *A Sylow p-subgroup of G^* is normal in G^*.*

(2) *The Hall p'-subgroups of G^* are cyclic groups of odd order dividing $p+1$.*

Remark. The alternating group of degree 5 is an example of how the case $p = 2$ can differ from the case when p is an odd prime.

2. Proof of the main results

We mention that if some group G has metacyclic Sylow p-subgroups, then the same is true for every homomorphic image of G. In particular, the hypotheses of our main results are inherited by factor groups. This fact will be used in the sequel without any further reference.

Before we can begin with the proofs of our main theorems, we need two preliminary results. The first of these is a fairly technical lemma about the general linear group $\mathrm{GL}(2,p)$.

Lemma 2.1. *Let M be a subgroup of $\mathrm{GL}(2,p)$ of odd order. Then there exist subgroups A and B of $\mathrm{GL}(2,p)$ satisfying:*

(1) *A is in the center of $\mathrm{GL}(2,p)$ and B is isomorphic to a subgroup of $\mathrm{PSL}(2,p)$.*

(2) *$\pi(M)=\pi(A)\cup\pi(B)$.*

(3) *M is a subgroup of AB.*

Proof. Write $Z=Z(\mathrm{GL}(2,p))$ and $S=\mathrm{SL}(2,p)$. Then M is a subgroup of $\mathrm{O}^2(Z)S=\mathrm{O}^2(Z)\times S$. Let A and B be the images of M of the projections of M into $\mathrm{O}^2(Z)$ and S respectively. Then M is a subgroup of AB, $\pi(M)=\pi(A)\cup\pi(B)$ and $B\cap(Z\cap S)=1$. Hence $B\cong[B(Z\cap S)]/(Z\cap S)$, which is isomorphic to a subgroup of $\mathrm{PSL}(2,p)$. □

Lemma 2.2. *Let p be an odd prime. Suppose that G is a p-soluble group with metacyclic Sylow p-subgroups. Then the p-length of G is at most one.*

Proof. Let P be a Sylow p-subgroup of G. Then, by [S, 2.3.4], P is a modular group, that is, every two subgroups of P are permutable in P. It then follows from [SW, Corollary 4.2] that the p-length of G is at most one. □

Proof of Theorem A. We first show that the p-length of G is at most one. Clearly G is p-soluble. If $p\neq 2$, then the p-length of G is at most one by Lemma 2.2. If $p=2$, then the hypothesis that $G/\mathrm{O}_{p'p}(G)$ is of odd order yields that the Sylow p-subgroups of G are contained in $\mathrm{O}_{p'p}(G)$. In this case we also have the p-length of G is at most one.

Clearly $\mathrm{O}_{p'}(G^*)=1$. Since the p-length of G is at most one, the Sylow p-subgroup P of G^* is normal in G^*. By [H, VI, 6.5], $C_{G^*}(P)\leq P$. Let H be a Hall p'-subgroup of G^*. Then $C_H(P)=H\cap C_{G^*}(P)=1$. By [G, 5, Theorem 1.4], we know that $C_H(P/\Phi(P))=C_H(P)=1$.

Assume that P is cyclic. Then $H=H/C_H(P/\Phi(P))$ is a group of automorphisms of a cyclic group of prime order. In this case H is a cyclic group of order dividing $p-1$ and thus the theorem holds.

Now assume that P is not cyclic, then $P/\Phi(P)$ is an elementary abelian group of order p^2, since P is metacyclic. In this case $H=H/C_H(P/\Phi(P))$ is a subgroup of $\mathrm{GL}(2,p)$. Since $H\cong G/\mathrm{O}_{p'p}(G)$, H is of odd order. By Lemma 2.1,

$H \leq AB$, where A and B are subgroups of $\mathrm{GL}(2,p)$ satisfying the properties (1)–(3) of Lemma 2.1. By (1) and (2) of Lemma 2.1, B is a subgroup of $\mathrm{PSL}(2,p)$ and B is a p'-group of odd order. By [H, II, 8.27], B must be a cyclic group with exponent dividing $p-1$ or $p+1$. Since $A \leq Z(\mathrm{GL}(2,p))$, $H \leq AB$ is an abelian group with exponent dividing p^2-1. Since $(p+1,p-1)=2$ and $2 \notin \pi(H)$, $H = H_1 \times H_2$, where the exponent of H_1 divides $p-1$ and the exponent of H_2 divides $p+1$. Since $(|H_2|,|A|)=1$, $H_2 \leq B$ is a cyclic group. Moreover, since AB is an abelian group of rank at most two, it follows that H can be generated by two elements. □

Proof of Corollary 4. G is soluble by the Feit–Thompson odd order theorem. Applying [AFG, 1.3.3], there exist Sylow p-subgroups G_p, A_p and B_p of G, A and B respectively, such that $G_p = A_pB_p$. According to [H, III, 11.5], G_p is metacyclic. Since the class of all p-supersoluble groups is a saturated formation ([H, VI, §9]), it suffices to prove that every primitive epimorphic image of G is p-supersoluble ([DH, A, 15.9]). Note that the hypotheses of the corollary hold in every factor group of G. Therefore, arguing by induction on the order of G, we may assume that G is primitive. According to [DH, A, 15.2], G has a unique minimal normal subgroup N such that $C_G(N) = N = \mathrm{O}_{p'p}(G)$ which is an elementary abelian p-group. By Theorem A, $G_p = N$ is the Sylow p-subgroup of G. Assume that $A_p = 1$. Then N is cyclic of order p and G is p-supersoluble. Hence we may assume that $A_p \neq 1$, $B_p \neq 1$ are of order p and $|N| = p^2$. Assume there exists a prime q dividing $|G/N|$ and $p+1$. Without loss of generality we may assume that q divides $|A|$. Let $a \in A$ be an element of order q and $Z = \langle a\rangle$. Then, N as a $Z = \langle a\rangle$-module, is factorised as $N = A_p \times L$ with $|A_p| = |L| = p$. This means that $Z/C_Z(N) = Z$ is abelian of exponent dividing $p-1$, which contradicts the fact $(p-1,p+1)=2$. Consequently, $(|G/N|,p+1)=1$ and G is p-supersoluble by Corollary 3. □

Proof of Theorem B. Assume that $|G/\mathrm{O}_{p'p}(G)|,p+1)=1$. If $p=2$, then G is p-nilpotent by [H, IV, 5.10]. So we may assume that $p \neq 2$. It is clear that $G/\mathrm{O}_{p'p}(G)$ is of odd order, since $|G/\mathrm{O}_{p'p}(G)|$ and $p+1$ are coprime.

Let $G^* = G/\mathrm{O}_{p'}(G)$. By Theorem A, $G^* = P \rtimes (H_1 \times H_2)$, where P is a Sylow p-subgroup of G^*, H_1 is an abelian group of exponent dividing $p-1$, H_2 is a cyclic group with exponent dividing $p+1$. Since $|G/\mathrm{O}_{p'p}(G)| = |H_1 \times H_2|$ and $p+1$ are coprime, we have $H_2 = 1$ and thus $G^* = P \rtimes H_1$.

Let H/K be any chief factor of G^* whose order is divisible by p. Since G^* is soluble, H/K is a p-group. Then $\mathrm{O}_p(G^*/C_{G^*}(H/K)) = 1$. Since P is a normal Sylow p-subgroup of G^*, $P \leq C_{G^*}(H/K)$. It follows that $G^*/C_{G^*}(H/K)$ is isomophic to a subgroup of H_1 and so it is an abelian group of exponent dividing $p-1$. By [DH, B, 9.8], $|H/K| = p$. Therefore $G^* = G/\operatorname{O}_{p'}(G)$ is p-supersoluble, and so is G. □

Proof of Theorem C. Suppose $(|G^*|, p-1) = 1$. Since p is an odd prime, $G/\operatorname{O}_{p'p}(G)$ is of odd order. Now properties (1) and (2) follow directly from Theorem A.

Conversely, suppose G satisfies properties (1) and (2). Let H be a Hall p'-subgroup of G^*. Since H is a cyclic group of odd order of exponent dividing $p+1$ and $(p-1, p+1) = 2$, we have $(|G/\operatorname{O}_{p'p}(G)|, p-1) = (|H|, p-1) = 1$. □

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