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Trans-Sasakian manifolds homothetic to Sasakian manifolds

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Abstract. In this paper, we obtain necessary and sufficient conditions for a 3-dimensional compact and connected trans-Sasakian manifold of type (α, β) to be homothetic to a Sasakian manifold. We also show that if a compact trans-Sasakian manifold admits an isometric immersion in the Euclidean space R^4 with Reeb vector field being transformation of unit normal vector field under the complex structure of R^4, then it is homothetic to a Sasakian manifold. We also introduce the axiom of flat torus for a 3-dimensional trans-Sasakian manifold and show that a 3-dimensional connected trans-Sasakian manifold with Ricci curvature in the direction of Reeb vector field a nonzero constant, satisfying axiom of flat torus is homothetic to a Sasakian manifold.

1. Introduction

Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold (cf. [1]). Then the product $\overline{M} = M \times R$ has natural almost complex structure J with the product metric G being almost Hermitian metric. The geometry of the almost Hermitian manifold $(\overline{M}, J, G)$ dictates the geometry of the almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ and gives different structures on M, a Sasakian structure, a quasi-Sasakian structure, a Kenmotsu structure and others (cf. [1], [2], [12]). It is known that there are sixteen different types of structures on the almost Hermitian manifold $(\overline{M}, J, G)$ (cf. [10]), using the structure in the class $\mathcal{W}_4$ on $(\overline{M}, J, G)$ a structure $(\varphi, \xi, \eta, g, \alpha, \beta)$ on M called a trans-Sasakian structure

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is introduced (cf. [16]) which generalizes Sasakian structure and Kenmotsu structure on an almost contact metric manifold (cf. [2], [14]), where α, β are smooth functions defined on M. Since the introduction of trans-Sasakian manifolds, very important contributions of Blair and Oubiña [2] and Marrero [14] have appeared studying the geometry of trans-Sasakian manifolds. In general, a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakain manifold of type (α, β). The trans-Sasakian manifolds of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are called the cosymplectic, α-Sasakian and β-Kenmotsu manifolds, respectively. Some authors have studied α-Sasakian and β-Kenmotsu manifolds with α, β as constants, however in this paper we consider α-Sasakian and β-Kenmotsu manifolds with both α, β as functions. Marrero [14] has shown that a trans-Sasakian manifold of dimension ≥ 5 is either a cosymplectic manifold, an α-Sasakian manifold or a β-Kenmotsu.

Since then there have been an emphasis on studying the geometry of 3-dimensional trans-Sasakian manifolds, putting some restrictions on the smooth functions α, β appearing in the definition of trans-Sasakian manifolds or the Reeb vector field ξ. There are several examples of trans-Sasakian manifolds constructed mostly on 3-dimensional non-compact simply connected Riemannian manifolds (cf. [2], [15]). Recall that a trans-Sasakian manifold of type (α, β) is said to be proper if neither of the functions α or β is zero. As Marrero [14] has classified trans-Sasakian manifolds in dimension ≥ 5 and has shown that there are no proper trans-Sasakian manifolds in these dimensions, one naturally raises the question: 'under what conditions a 3-dimensional trans-Sasakian manifold is not proper?'.

This question was taken up in [9], and in this paper we continue answering this question by obtaining two different necessary and sufficient conditions for a trans-Sasakian manifold to be homothetic to a Sasakian manifold.

It is well known that a Killing vector field is a Jacobi-type vector field and the converse is not true (see [7] for a definition of Jacobi-type vector fields) and that the Reeb vector field on a Sasakian manifold being Killing is a Jacobi-type vector field. We use this fact to show that the Reeb vector field of a 3-dimensional compact and connected trans-Sasakian manifold with the Ricci curvature $\mathrm{Ric}(\xi, \xi)$ a positive constant, is a Jacobi-type vector field if and only if the trans-Sasakian manifold is homothetic to a Sasakian manifold (see Theorem 3.1).

We also show that the Reeb vector field ξ of a 3-dimensional compact and connected trans-Sasakian manifold with Ricci curvature $\mathrm{Ric}(\xi, \xi)$ a constant, is a conformal vector field if and only if the trans-Sasakian manifold is homothetic to a Sasakian manifold (see Theorem 3.2). It is known that a compact 3-dimensional smooth manifold can be immersed in the Euclidean space R^4 (cf. [5]); we use this result that a 3-dimensional compact trans-Sasakian manifold can be immersed in the Euclidean space R^4 and under the condition that this immersion is an isometric immersion with Reeb vector field ξ is related to the unit normal vector field N of the immersion by $\xi = -JN$, to show that the trans-Sasakian manifold is homothetic to a Sasakian manifold (In fact, isometric to $S^3(c)$ see Theorem 4.1).

Finally, we introduce the axiom of flat torus for a 3-dimensional compact trans-Sasakian manifold analogous to such axioms in [3], [4] and [17] and show that a trans-Sasakian manifold with nonzero constant $\mathrm{Ric}(\xi,\xi)$, satisfying this axiom, is homothetic to a Sasakian manifold (see Theorem 5.1).

2. Preliminaries

Let (M,φ,ξ,η,g) be a $(2n+1)$-dimensional contact metric manifold, with φ is a $(1,1)$-tensor field, ξ is a unit vector field and η is smooth 1-form dual to ξ with respect to the Riemannian metric g such that

$$\varphi^2 = -I + \eta\otimes\xi, \varphi(\xi) = 0, \eta\circ\varphi = 0, g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y) \quad (2.1)$$

$X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of smooth vector fields on M (cf. [1]). If there are smooth functions α, β on an almost contact metric manifold (M,φ,ξ,η,g) satisfying

$$(\nabla\varphi)(X,Y) = \alpha\left(g(X,Y)\xi - \eta(Y)X\right) + \beta\left(g(\varphi X,Y)\xi - \eta(Y)\varphi X\right), \quad (2.2)$$

then it is called a trans-Sasakian manifold. ($(\nabla\varphi)(X,Y) = \nabla_X\varphi Y - \varphi(\nabla_X Y)$, $X, Y \in \mathfrak{X}(M)$ and ∇ is the Levi Civita connection with respect to the metric g, cf. [2], [7], [10].) We shall denote this trans-Sasakian manifold by $(M,\varphi,\xi,\eta,g,\alpha,\beta)$ and call it a trans-Sasakian manifold of type (α,β). From equations (2.1) and (2.2) it follows that

$$\nabla_X\xi = -\alpha\varphi(X) + \beta(X - \eta(X)\xi), \quad X \in \mathfrak{X}(M). \quad (2.3)$$

It is clear that a trans-Sasakian manifold of type $(1,0)$ is a Sasakian manifold (cf. [1]) and a trans-Sasakian manifold of type $(0,1)$ is Kenmotsu manifold (cf. [10]). A trans-Sasakian manifold of type $(0,0)$ is called a cosymplectic manifold (cf. [9]).

Let Ric be the Ricci tensor of a Riemannian manifold (M,g). Then the Ricci operator Q is a symmetric tensor field of type $(1,1)$ defined by $\mathrm{Ric}(X,Y) = g(QX,Y)$, $X, Y \in \mathfrak{X}(M)$. We state following results, which we need in the sequel.

Lemma 2.1. [9] *Let* $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ *be a 3-dimensional trans-Sasakian manifold. Then* $\xi(\alpha) + 2\alpha\beta = 0$.

Lemma 2.2. [9] *Let* $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ *be a 3-dimensional trans-Sasakian manifold. Then its Ricci operator satisfies*

$$Q(\xi) = \varphi(\nabla\alpha) - \nabla\beta + 2(\alpha^2 - \beta^2)\xi - g(\nabla\beta, \xi)\xi,$$

where $\nabla\alpha$, $\nabla\beta$ *are the gradients of the smooth functions* α, β.

Theorem 2.1. [15] *Let* (M, g) *be a Riemannian manifold. If* M *admits a Killing vector field* ξ *of constant length satisfying*

$$k^2 (\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi) = g(Y, \xi)X - g(X, Y)\xi; \quad X, Y \in \mathfrak{X}(M)$$

for a nonzero constant k, *then* M *is homothetic to a Sasakian manifold.*

Recall that a smooth vector field u on a Riemannian manifold (M, g) is said to be a Jacobi-type vector field if it satisfies (cf. [7], [8])

$$\nabla_X \nabla_X u - \nabla_{\nabla_X X} u + R(u, X)X = 0, \quad X \in \mathfrak{X}(M), \tag{2.4}$$

where R is the curvature tensor field. It is clear that each Killing vector field is a Jacobi-type vector field, however a Jacobi-type vector field need not be a Killing vector field. For example, the position vector field on the Euclidean space R^n is a Jacobi-type vector field which is not a Killing vector field.

3. The Reeb vector field ξ as Jacobi-type vector field

Recall that the Reeb vector field ξ on a $(2n+1)$-dimensional Sasakian manifold is a Killing vector field and therefore it is a Jacobi-type vector field. Note that an α-Sasakian manifold with constant α satisfies the hypothesis of Theorem 2.1 and is therefore homothetic to a Sasakian manifold. However, owing to the importance of Theorem 2.1, we shall refer to it for proving that a trans-Sasakian manifold is homothetic to a Sasakian manifold instead of the above observation about an α-Sasakian manifold with constant α.

Theorem 3.1. *Let* $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ *be a 3-dimensional compact and connected trans-Sasakian manifold. Suppose that the Ricci curvature* $\operatorname{Ric}(\xi, \xi)$ *of* (M, g) *is a nonzero constant. Then* M *is homothetic to a Sasakian manifold if and only if the vector field* ξ *is a Jacobi-type vector field.*

Proof. Suppose ξ is a Jacobi-type vector field. Then (2.4) gives

$$\nabla_X\nabla_X\xi - \nabla_{\nabla_X X}\xi + R(\xi, X)X = 0, \quad X \in \mathfrak{X}(M).$$

Using (2.3) in the equation above, after an easy calculation we obtain

$$\begin{aligned} &- X(\alpha)\varphi X + X(\beta)X + 2\alpha\beta\eta(X)\varphi X + (\alpha^2 - \beta^2)\eta(X)X \\ &- \left\{X(\beta) + (\alpha^2 + \beta^2)g(X, X) + 2\beta^2\eta(X)^2\right\}\xi \\ &+ R(\xi, X)X \ = \ 0. \end{aligned}$$

Taking trace, we find

$$Q(\xi) = \varphi\,(\nabla\alpha) - \nabla\beta + \left\{2(\alpha^2 + \beta^2) + \xi(\beta)\right\}\xi.$$

Now, combining this equation with Lemma 2.2, we obtain

$$\xi(\beta) = -2\beta^2. \tag{3.1}$$

Using (2.3), it follows that

$$\operatorname{div}\xi = 2\beta. \tag{3.2}$$

Equations (3.1) and (3.2) give

$$\operatorname{div}\left(\beta^3\xi\right) = 3\beta^2\xi(\beta) + \beta^3 \operatorname{div}\xi = -4\beta^4,$$

so by Stokes' theorem $\beta = 0$. Hence, by Lemma 2.2, $\operatorname{Ric}(\xi, \xi) = 2\alpha^2$ is nonzero constant, therefore α is a nonzero constant and thus equations (2.2) and (2.3) give

$$\alpha^{-2}\left(\nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi\right) = g(Y, \xi)X - g(X, Y)\xi,$$

thus proving that M is homothetic to a Sasakian manifold (cf. Theorem 2.1). The converse is obvious.

Recall that a smooth vector field ξ on a Riemannian manifold (M, g) is said to be a conformal vector field if

$$\left(\pounds_\xi g\right)(X, Y) = 2\rho g(X, Y), \quad X, Y \in \mathfrak{X}(M), \tag{3.3}$$

where $\pounds_\xi$ is the Lie derivative with respect to ξ and ρ is a smooth function on M. Now, we prove the following:

Theorem 3.2. *Let $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ be a 3-dimensional compact and connected trans-Sasakian manifold whose Ricci curvature $\mathrm{Ric}(\xi, \xi)$ is nonzero constant. Then M is homothetic to a Sasakian manifold if and only if the vector field ξ is a conformal vector field.*

Proof. Suppose ξ is a conformal vector field. Using equations (2.3) and (3.3), we get

$$\beta g(X, Y) - \beta\eta(X)\eta(Y) = \rho g(X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{3.4}$$

Taking $X = Y = \xi$ in the equation above, we obtain $\rho = 0$. Hence ξ is a Killing vector field and, consequently, is a Jacobi-type vector field. Thus we get the result by Theorem 3.1. Conversely, if $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ is homothetic to a Sasakian manifold, the vector field ξ is Killing and therefore a conformal vector field.

4. Trans-Sasakian manifolds isometrically immersed in R^4

It is well known that if $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ is a 3-dimensional compact trans-Sasakian manifold, then there exists a smooth immersion $\Psi : M \to R^4$ (cf. [3]). This immersion need not be an isometric immersion in to the Euclidean space $(R^4, \langle , \rangle)$. It is known that this Euclidean space has a complex structure J such that $(R^4, J, \langle , \rangle)$ is a Kaehler manifold. In this section, we show that if the immersion $\Psi : M \to R^4$ is an isometric immersion with unit normal N with $\xi = -JN$, then M is homothetic to a Sasakian manifold. The main result of this section is the following:

Theorem 4.1. *Let $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ be a 3-dimensional compact and connected trans-Sasakian manifold. Then there exists an isometric immersion of M in the Euclidean space R^4 with unit normal N satisfying $\xi = -JN$ if, and only if, M is isometric to the Sasakian manifold $S^3(\alpha^2)$.*

Proof. Let $\Psi : M \to R^4$ be the isometric immersion. The Euclidean space $(R^4, J, \langle , \rangle)$ is a Kaehler manifold with complex structure J and the Euclidean metric $\langle , \rangle$. We denote by A the shape operator of the hypersurface M. Define an operator $\psi : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $JX = \psi(X) + \eta(X)N$, where $\psi(X)$ is tangential component of JX to M. Then using the properties of complex structure J and Gauss–Wiengarten formulae for the hypersurface we immediately get the following:

$$\psi^2(X) = -X + \eta(X)\xi, \psi(\xi) = 0, \eta(\psi(X)) = 0, \quad X \in \mathfrak{X}(M), \tag{4.1}$$

$$g(\psi X,\psi Y)=g(X,Y)-\eta(X)\eta(Y),\quad X,Y\in\mathfrak{X}(M), \tag{4.2}$$

and

$$\nabla_X\xi=\psi AX,\quad (\nabla_X\psi)(Y)=\eta(Y)AX-g(AX,Y)\xi,\quad X,Y\in\mathfrak{X}(M). \tag{4.3}$$

Using (2.3) in the first equation of (4.3), we get

$$-\alpha\varphi X-\beta\varphi^2X=\psi AX. \tag{4.4}$$

Since A is symmetric and ψ is skew-symmetric, we have $Tr(\psi A)=0$. Taking trace in (4.4), we get $\beta=0$. Then equations (2.3) and (4.3) give $\psi AX=-\alpha\varphi X$, that is $g(\psi AX,X)=0$. Polarizing the equation $g(\psi X,AX)=0$, we get $\psi AX=A\psi X$, $X\in\mathfrak{X}(M)$, which leads to $\psi A\xi=0$. Hence, $A\xi=\lambda\xi$ for a smooth function λ. Since $\beta=0$, equation (2.3) assures that ξ is a Killing vector field and the one-parameter group $\{f_t\}$ of ξ consists of isometries which satisfy $df_t\circ A=A\circ df_t$. Hence

$$[\xi,AX]=A[\xi,X],\quad X\in\mathfrak{X}(M).$$

Using equation (2.3) in the above equation, we get

$$(\nabla A)(\xi,X)=\alpha A\varphi X-\alpha\varphi AX,\quad X\in\mathfrak{X}(M),$$

which, together with the Codazzi equation for hypersurfaces, equation (2.3) and $A\xi=\lambda\xi$, gives

$$X(\lambda)\xi-\lambda\alpha\varphi X=-\alpha\varphi AX.$$

Taking inner product with ξ in the above equation we get $X(\lambda)=0$. Thus λ is a constant and

$$\alpha\varphi(AX-\lambda X)=0,\quad X\in\mathfrak{X}(M). \tag{4.5}$$

Note that the Ricci curvature of the hypersurface M, by Lemma 2.2 is given by

$$\mathrm{Ric}(\xi,\xi)=2\alpha^2, \tag{4.6}$$

and on a compact hypersurface of the Euclidean space, there exists a point where the Ricci curvature is strictly positive. Hence $\alpha\neq0$, thus equation (4.5) on connected M gives $\varphi AX=\lambda\varphi X$. Operating φ on the last equation and using $A\xi=\lambda\xi$, we get $AX=\lambda X$, $X\in\mathfrak{X}(M)$. Thus $A=\lambda I$, and, consequently M is isometric to $S^3(\lambda^2)$. However, equation (4.6) gives $\alpha=\lambda$, therefore M is isometric to $S^3(\alpha^2)$.

The converse is trivial as $S^3(\alpha^2)$ has a Sasakian structure.

5. Axiom of flat torus for trans-Sasakian manifolds

A Riemannian manifold (M, g) satisfies the axiom of planes if there exists a 2-dimensional totally geodesic submanifold tangent to any 2-dimensional section of the tangent bundle TM at every point of the manifold (cf. [4]). Also, a Riemannian manifold (M, g) satisfies the axiom of 2-spheres, if for each $p \in M$ and each 2-dimensional subspace $\pi \subset T_pM$ of the tangent space T_pM, there exists a 2-dimensional umbilical submanifold N with parallel mean curvature vector field such that $p \in N$ and $\pi = T_pN$ (cf. [17]). Similarly, axioms of holomorphic and antiholomorphic planes are defined for Kaehler manifolds (cf. [3], [17]). These axioms are used to characterize the real and complex space forms. In this section we introduce the axiom of flat torus for a 3-dimensional trans-Sasakian manifold and show that a connected 3-dimensional trans-Sasakian manifold whose Ricci curvature in the direction of the Reeb vector field ξ a nonzero constant and which satisfies the axiom of flat torus is homothetic to Sasakian manifold.

Let $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ be a 3-dimensional trans-Sasakian manifold and $T^2 = S^1 \times S^1$ be the 2-dimensional flat torus with product metric of constant curvature 0. We say that the trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ satisfies the axiom of flat torus if for each $p \in M$, there exists an isometric immersion $f : T^2 \to M$ tangential to ξ and $p \in f(T^2)$.

Theorem 5.1. *Let $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ be a 3-dimensional compact and simply connected trans-Sasakian manifold with nonzero constant* $\mathrm{Ric}(\xi, \xi)$. *If M satisfies the axiom of flat torus, then M is homothetic to a Sasakian manifold.*

Proof. We denote by the same letter g the metric of constant curvature 0 on the flat torus T^2 and by $\widetilde{\nabla}$ the Riemannian connection on the Riemannian manifold (T^2, g). For an isometric immersion of T^2 into the trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$, we denote by N and A the local unit normal vector field and the shape operator, respectively. Then we have the following Gauss and Wiengarten formulae

$$\nabla_X Y = \widetilde{\nabla}_X Y + g(AX, Y)N, \quad \nabla_X N = -AX, \quad X, Y \in \mathfrak{X}(T^2). \tag{5.1}$$

Since φ is skew symmetric, we have $\varphi N \in \mathfrak{X}(T^2)$, and thus we get a vector field $u \in \mathfrak{X}(T^2)$ defined by $u = -\varphi N$. Since the vector field ξ is tangential to T^2 and $\varphi\xi = 0$, we get $\eta(u) = 0$, and, consequently, the vector field u is a unit vector field and hence $\{u, \xi\}$ is a local orthonormal frame on T^2. Let ω be the smooth 1-form dual to the unit vector field u. We set

$$\varphi X = \psi X + \omega(X)N, \quad X \in \mathfrak{X}(T^2), \tag{5.2}$$

where ψX is the tangential component of φX to T^2. As $\omega(\xi) = 0$, equation (5.2) gives $\psi(\xi) = 0$. Also, using $\varphi u = N$ in equation (5.2), we get $\psi u = 0$. Thus the orthonormal frame $\{u, \xi\}$ annihilates ψ, consequently the equation (5.2) reduces to

$$\varphi X = \omega(X)N, \quad X \in \mathfrak{X}(T^2). \tag{5.3}$$

Now, using equations (2.2), (5.1) and (5.3), we get

$$\begin{aligned}\nabla_X u &= -\left(\nabla\varphi\right)(X, N) + \varphi AX \\ &= -\beta\omega(X)\xi + \omega(AX)N, \quad X \in \mathfrak{X}(T^2),\end{aligned}$$

which on equating tangential and normal components gives

$$\widetilde{\nabla}_X u = -\beta\omega(X)\xi, \quad X \in \mathfrak{X}(T^2), \tag{5.4}$$

where, by abuse of notation, β means the restriction of the given β to T^2. Also, equations (2.3), (5.1) and (5.3) give

$$\widetilde{\nabla}_X\xi + g(AX, \xi)N = -\alpha\omega(X)N + \beta(X - \eta(X)\xi),$$

that is,

$$\widetilde{\nabla}_X\xi = \beta(X - \eta(X)\xi), \quad X \in \mathfrak{X}(RP^2) \text{and} A\xi = -\alpha u. \tag{5.5}$$

Equations (5.4) and (5.5) give, in particular,

$$\widetilde{\nabla}_\xi u = 0, \widetilde{\nabla}_u\xi = \beta u, \widetilde{\nabla}_\xi\xi = 0, \widetilde{\nabla}_u u = -\beta\xi,$$

consequently the curvature tensor field $\widetilde{R}$ of the Riemannian manifold (T^2, g) satisfies

$$\widetilde{R}(u, \xi)\xi = 0 - \xi(\beta)u - \beta^2 u = -(\xi(\beta) + \beta^2)u.$$

Taking inner product with u in the above equation and using the fact that (T^2, g) is of constant curvature 0, we get

$$\xi(\beta) = -\beta^2 \tag{5.6}$$

on T^2. Since, through each point of M, there passes T^2, the above equation is valid on the whole M. Using the equation (5.6) in Lemma 2.2, we get

$$\mathrm{Ric}(\xi, \xi) = 2\alpha^2,$$

which proves that α is a nonzero constant. Then, on a connected M, Lemma 2.1, gives $\beta = 0$. Finally, equations (2.2) and (2.3) together with Theorem 2.1, prove that M is homothetic to Sasakain manifold.

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