

Rings with unipotent units

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Abstract. We systematically study rings whose units are all unipotent. The first main result is that a ring R has this property if and only if R has a 2-power characteristic and the unit group of R is a (possibly infinite) 2-group. The second main result is that R is an exchange ring with all units unipotent if and only if its Jacobson radical $\text{rad}(R)$ is nil and $R/\text{rad}(R)$ is a Boolean ring. The rings in the second main result are precisely Diesl's strongly nil-clean rings, for which several new properties are obtained.

1. Introduction

Throughout this paper, $U(R)$ denotes the group of units of a unital ring R , and $\text{nil}(R)$ denotes the set of nilpotent elements in R . It is well known that $1 + \text{nil}(R) \subseteq U(R)$. The elements in $1 + \text{nil}(R)$ are called the *unipotents*, or the *unipotent units* in R . Unipotent subgroups of $U(R)$ are known to be very important in the study of algebraic groups when R is a matrix ring $M_n(K)$ over a field K , in which case $U(R)$ is just the general linear group $GL_n(K)$. For instance, every subgroup $H \subseteq GL_n(K)$ has a unique maximal normal unipotent subgroup, called the *unipotent radical* of H ; see, for instance, HUMPHREYS' book [Hu], or the exposition in [La, Theorem 9.22].

For an arbitrary ring R , much less is known, and so far no general theory of unipotent groups of units has emerged. Nevertheless, one can study the set of unipotent units of R , and try to see to what extent this set would impact on the structure of the ring R . The work of DIESL [Di] on nil-clean rings (rings in which every element is the sum of an idempotent and a nilpotent element) has

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led CĂLUGĂREANU [Ca] to define the following interesting notion: a ring R is called a **UU ring** if all units of R are unipotent. For instance, *reduced UU rings are precisely the rings with trivial unit groups*. Obvious examples of these include for instance all Boolean rings, and all commutative and noncommutative free algebras over \mathbb{F}_2 (the field of two elements). Non-reduced examples, on the other hand, include $\mathbb{Z}/2^t\mathbb{Z}$ and $\mathbb{F}_2[x]/(x^t)$ (where $t \geq 2$). Also, CĂLUGĂREANU has shown in [Ca, 2.2] that, if a commutative ring R is UU, then so is any polynomial ring over R (in any number of commuting variables). Finally, DIESL has proved in [Ca, 3.11] that R is a *strongly nil-clean ring* (in the sense that every element of R is the sum of an idempotent and a nilpotent element which commute) if and only if R is a UU ring that is strongly π -regular (defined by the property that the chain $aR \supseteq a^2R \supseteq \dots$ stabilizes for every $a \in R$).¹ Various other examples of UU rings will be given later in Section 2.

In this paper, we prove two principal results about UU rings. The first result below shows that the UU condition on a ring R amounts to a characteristic condition on R along with a group-theoretic condition on $U(R)$, as follows.

Theorem A. *A ring R is a UU ring if and only if $\text{char}(R) = 2^t$ for some integer t and $U(R)$ is a (possibly infinite) 2-group. Such a ring R has a finite nilpotence index if and only if its unit group $U(R)$ has a finite exponent.*

In general, of course, we do not expect that the UU assumption on a ring R would lead to a structural determination of R itself. (The case of a trivial unit group is a case in point.) However, if we turn our attention to the class of *exchange rings* in the sense of WARFIELD [Wa] (and NICHOLSON [Ni₂]), the situation becomes much more amenable. In fact, the following second main result in this paper “identifies” the exchange UU rings with DIESL’s strongly nil-clean rings in [Di], and recovers for these rings a simple structural characterization proved first by HIRANO–TOMINAGA–YAQUB in [HTY], and later by KOŞAN–WANG–ZHOU in [KWZ].

Theorem B. *A ring R is an exchange UU ring if and only if R is a strongly nil-clean ring, if and only if $\text{rad}(R)$ (the Jacobson radical of R) is a nil ideal and $R/\text{rad}(R)$ is a Boolean ring.*

In particular, R is a semiprimitive exchange UU ring (or a semiprimitive strongly nil-clean ring) if and only if R is a Boolean ring. Indeed, our proof of Theorem B rests squarely on proving it first in this special semiprimitive case.

¹This result of Diesl will be re-proved independently in Section 4; see Theorem 4.3.

The many interesting connections between Theorem B and various other results in the literature will be explained in the proof of Theorem 4.3.

Before proving the two theorems above, we devote Section 2 to developing a number of basic facts on UU rings, some of which are improved forms of the results of CĂLUGĂREANU in [Ca]. For the convenience of the reader, however, all results in Section 2 are stated and proved in an entirely self-contained manner. In the two remaining sections, Theorem A and Theorem B will be proved (respectively, in (3.4) and (4.3)) along with an assortment of other results on the notions of unipotent units, UU rings, and (strongly) nil-clean rings.

The notations and terminology introduced above will be used consistently in the rest of this work. At a few places, to make connections to other papers in the literature, we will also invoke the basic notions of clean rings, strongly clean rings, and uniquely clean rings. These classes of rings were defined in [Ni₂], [Ni₃], and [NZ] respectively. Other standard notations and terminology in ring theory can be found in [La]. Henceforth, we will use the widely accepted shorthand “*iff*” for “*if and only if*” in the text.

2. Basic properties of UU rings

In this section, we prove a number of useful properties of UU rings. To begin with, we have the three facts below whose routine proofs will be left as exercises.

- (2.1) *A subdirect product of finitely many UU rings is a UU ring.*
 (2.2) *A direct product $R_1 \times \cdots \times R_n$ is UU iff each R_i is UU.*
 (2.3) *If R is a UU ring and S is a factor ring of R such that units of S lift to units of R , then S is also a UU ring. [Thus, for instance, if R is a UU ring of stable range one in the sense of BASS [Ba] (see also [Va]), so is every factor ring of R .]*

We come now to several results which require some verifications. The first two parts of the following theorem were first proved by DIESEL (in [Di, (3.15)–(3.16)]) in the case where “UU” is replaced by “nil-clean”; here we have the complete analogues for the “UU” case.

- Theorem 2.4.** (1) *For any nil ideal $I \subseteq R$, R is UU iff R/I is UU.*
 (2) *A ring R is UU iff $J := \text{rad}(R)$ is nil and R/J is UU.*
 (3) *A commutative ring R is UU iff $J = \text{rad}(R)$ is nil and $U(R/J) = \{\bar{1}\}$. In this case, if $R \neq 0$, the units of R generate a local ring $J \cup (1 + J)$ in R , which contains all local subrings of R .*

PROOF. (1) The “only if” part follows from (2.3) since we have here $I \subseteq \text{rad}(R)$, which implies that $U(R) \rightarrow U(R/I)$ is surjective. For the “if” part, assume that R/I is UU. For any $u \in U(R)$, $(u - 1) + I \in R/I$ is nilpotent. Since I is nil, $1 - u \in R$ is also nilpotent. This shows that R is UU.

(2) The “if” part of (2) follows from the “if” part of (1). For the “only if” part of (2), assume R is UU. Since units of $\bar{R} := R/J$ lift to units of R , (2.3) shows again that \bar{R} is UU. Next, from $1 + J \subseteq U(R) = 1 + \text{nil}(R)$, we have $J \subseteq \text{nil}(R)$, so J is a nil ideal.

(3) The “if” part is clear from (2), since $U(R/J) = \{\bar{1}\}$ certainly implies that R/J is UU. Conversely, if R is commutative and UU, then $J = \text{rad}(R)$ being nil yields $J = \text{nil}(R)$. Since $U(R) = 1 + \text{nil}(R) = 1 + J$, we have clearly $U(R/J) = \{\bar{1}\}$. In particular, $\text{char}(R/J) = 2$, so $2 \in J$. This finally implies that $S := J \cup (1 + J)$ is a local ring (if $R \neq 0$). Recalling that any local ring is generated by its units,² it follows that S is the subring generated by $U(R)$, and that any local subring of R is contained in S . \square

Example 2.5. If R is a commutative UU ring, we have observed in (3) above that $\text{rad}(R) = \text{nil}(R)$. However, if R is a general UU ring, $\text{rad}(R) \subseteq \text{nil}(R)$ may be a strict inclusion. Such a UU ring R , constructed by G. Bergman, is presented here with his kind permission. Let R be the \mathbb{F}_2 -algebra generated by x, y with the single relation $x^2 = 0$. Using his result on coproducts from [Be, Corollary 2.16], BERGMAN showed that $U(R) = 1 + \mathbb{F}_2x + xRx$. Since $(\mathbb{F}_2x + xRx)^2 = 0$, we have $\mathbb{F}_2x + xRx \subseteq \text{nil}(R)$, so R is a UU ring. For any nonzero $r \in R$, it is easy to see that $1 + yry \notin 1 + \mathbb{F}_2x + xRx$. Therefore, $\text{rad}(R) = \{0\}$, which is properly contained in the set $\text{nil}(R)$.

In (2.6) and (2.7) below, we revisit a few key properties of UU rings obtained by CĂLUGĂREANU [Ca], with somewhat simplified proofs.

Theorem 2.6. *For any UU ring R , the following hold.*

- (1) $2 \in \text{rad}(R)$, and $\text{char}(R) = 2^t$ for some integer $t \geq 0$.³
- (2) For any idempotent $e \in R$, the corner ring eRe is a UU ring.
- (3) Any (unital) subring of R is a UU ring.

PROOF. (1) Since R is UU, $-1 = 1 + a$ for some $a \in \text{nil}(R)$. Thus, $2 \in \text{nil}(R)$, and so $1 + 2R \in 1 + \text{nil}(R) \subseteq U(R)$. This implies that $2 \in \text{rad}(R)$ by [La: (4.1)], and now Theorem 2.4(2) shows that $2^t = 0 \in R$ for some $t \geq 0$.

²Indeed, any element in a local ring is either a unit, or the sum of 1 and a unit.

³Here, we use the convention that the zero ring has characteristic 1.

(2) Let $u \in U(eRe)$ with inverse v . Then $u + (1 - e) \in U(R)$ with inverse $v + (1 - e)$, so $u + (1 - e) = 1 + a$ for some $a \in \text{nil}(R)$. Thus $a = u - e \in eRe \cap \text{nil}(R) \subseteq \text{nil}(eRe)$. We have therefore $u = e + a$, which is a unipotent unit in eRe .

(3) For any subring $S \subseteq R$ and any $u \in U(S)$, we have $u \in U(R)$, so $1 - u \in \text{nil}(R) \cap S \subseteq \text{nil}(S)$. Thus, $u \in 1 + \text{nil}(S)$, which checks that S is also a UU ring. \square

Theorem 2.7. *For any ring $S \neq 0$ and any integer $n \geq 2$, $\mathbb{M}_n(S)$ is not a UU ring.*

PROOF. Since $\mathbb{M}_2(S)$ is isomorphic to a corner ring of $\mathbb{M}_n(S)$ (for $n \geq 2$), it suffices to show that $\mathbb{M}_2(S)$ is not a UU ring by virtue of Theorem 2.6(2). Consider the matrix $U = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \text{GL}_2(S)$. Since $I - U = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \in \text{GL}_2(S)$ too, it cannot be nilpotent. This means that U is *not* unipotent, so $\mathbb{M}_2(S)$ is indeed not a UU ring. \square

From Theorem 2.7 above, we see that if R is a ring having an idempotent e such that eRe and $(1-e)R(1-e)$ are both (commutative) UU, it *does not* follow that R itself is UU. This is in sharp contrast with the cases of exchange rings and clean rings where positive results were obtained, respectively, in [Ni₂: (2.6)] and [HN: p. 2590].

Our next result gives a complete description of all UU rings that are semisimple, semilocal, or local. (The local case was first handled by CĂLUGĂREANU in [Ca: (2.6)].)

Theorem 2.8. (1) *A semisimple ring R is UU iff $R \cong \mathbb{F}_2 \times \cdots \times \mathbb{F}_2$.*

(2) *A semilocal ring R is UU iff $\text{rad}(R)$ is a nil ideal and $R/\text{rad}(R) \cong \mathbb{F}_2 \times \cdots \times \mathbb{F}_2$. In this case, R is automatically a semiperfect ring.*

(3) *A local ring (R, \mathfrak{m}) is UU iff \mathfrak{m} is a nil ideal and $R/\mathfrak{m} \cong \mathbb{F}_2$.*

PROOF. (1) This follows from the Wedderburn–Artin theorem [La: (3.5)], upon applying (2.2), Theorem 2.7, and the easy fact that a division ring D is a UU ring iff $D \cong \mathbb{F}_2$.

(2) The “if” part is clear from Theorem 2.4(1). The “only if” part follows from Theorem 2.4(2) and part (1) above, since a semilocal ring R is defined (e.g. in [La: (20.1)]) by the condition that $R/\text{rad}(R)$ is semisimple. Finally, if R is semilocal and UU, then $\text{rad}(R)$ being a nil ideal implies that idempotents of

$R/\text{rad}(R)$ can be lifted to idempotents in R by [La: (21.28)], so R is a semiperfect ring.

(3) follows easily by specializing the first conclusion in (2) to the case of local rings. \square

We note in passing that Theorem B stated in the Introduction is essentially the extension of part (2) above from the case of semilocal UU rings to the case of exchange UU rings.⁴ As for part (3), we see that $R_1 = \mathbb{Z}/2^t\mathbb{Z}$ is a commutative local UU ring with $\text{char}(R_1) = 2^t$. However, the infinite direct product $R_2 = \prod_{t=1}^{\infty} \mathbb{Z}/2^t\mathbb{Z}$ is *not* UU, since the unit $u = (1, -1, -1, \dots) \in R$ is not in $1 + \text{nil}(R)$.

Turning our attention now to more general rings, the following result gives some quick noncommutative examples of UU rings (starting from any UU ring).

Theorem 2.9. *Let T_n be the ring of $n \times n$ upper triangular matrices over a ring R , where $n \geq 1$ is a fixed integer. Then R is a UU ring iff T_n is a UU ring.*

PROOF. Let $I = \{(a_{ij}) \in T_n : \text{all } a_{ii} = 0\}$. This is a nil ideal in T_n , with $T_n/I \cong R^n$. Therefore, the desired result follows from (2.2) and Theorem 2.4(1). \square

Again, the theorem above is a complete UU ring analogue of DIESL's corresponding result [Di: (4.1)] for nil-clean and strongly nil-clean rings.

We close this section with another quick observation on the behavior of unipotent units under the formation of Jacobson pairs. Recall that two elements $1 - ab$ and $1 - ba$ in a ring R (with $a, b \in R$) are said to be a *Jacobson pair* in R .

Proposition 2.10. *If $(u, v) = (1 - ab, 1 - ba)$ form a Jacobson pair in R , then u is a unipotent unit of R iff so is v .*

PROOF. It suffices to prove the "only if" part, so assume that $u = 1 - ab$ is a unipotent unit. This implies that $(ab)^n = 0$ for some integer n . But then $(ba)^{n+1} = b(ab)^na = 0$, so $v = 1 - ba$ is also a unipotent unit. \square

⁴In fact, semiperfect rings are exactly the *orthogonally finite* exchange rings, according to a well-known theorem of CAMILLO and YU in [CY].

3. Some characterizations of UU rings

In classical ring theory, there is a “circle operation” on any ring $(R, +, \times)$ defined by $a \circ b := a + b - ab$ which makes R into a monoid with identity element 0. Moreover, the “Jacobson map” $\varepsilon : (R, \circ) \rightarrow (R, \times)$ defined by $\varepsilon(a) = 1 - a$ is a monoid isomorphism, sending 0 to 1, and mapping the set $\text{nil}(R)$ bijectively onto the set of unipotent units of R . (See, for instance, JACOBSON’s book [Ja: p. 8].) In the case where R is a commutative ring, it is easy to check that $(\text{nil}(R), \circ)$ is a group (with identity 0), and that ε maps this group isomorphically onto the group of unipotent units of R . However, if R is an arbitrary ring, then in most cases the set $\text{nil}(R)$ is no longer closed under the circle operation, and the unipotent units may not form a subgroup of $U(R)$. For instance, for the matrix units $E_{12}, E_{21} \in \text{nil}(\mathbb{M}_2(\mathbb{Z}))$, we have $E_{12} \circ E_{21} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ which is not nilpotent, and $(I_2 - E_{12})(I_2 - E_{21}) = I_2 - E_{12} \circ E_{21} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ which is not unipotent.

The following proposition provides the first conceptual characterization of a UU ring in terms of the circle operation.

Proposition 3.1. *A ring R is a UU ring iff $\text{nil}(R)$ is a subgroup of the monoid (R, \circ) (with a common identity element 0) and the map $\varepsilon : (\text{nil}(R), \circ) \rightarrow (U(R), \times)$ is a group isomorphism.*

PROOF. To begin with, as long as the map $\varepsilon : \text{nil}(R) \rightarrow U(R)$ is surjective, R is (by definition) a UU ring. Conversely, let R be a UU ring, and let $a, a' \in \text{nil}(R)$. Then $(1 - a)(1 - a') \in U(R)$. Since R is UU, we have $1 - (1 - a)(1 - a') \in \text{nil}(R)$, which amounts to $a \circ a' \in \text{nil}(R)$, with $\varepsilon(a \circ a') = \varepsilon(a)\varepsilon(a')$. Thus, the bijective map $\varepsilon : (\text{nil}(R), \circ) \rightarrow (U(R), \times)$ is a monoid isomorphism, and hence a group isomorphism. \square

Next, we come to Theorem 3.2 below, the proof of which consists of an exploitation of the interplay between the three binary operations “+”, “ \times ”, and “ \circ ” on the ring R . We thank D. Khurana for pointing out that, without the quantitative information, the special case of part (B) of this result for prime-power characteristic was contained in a “Test-Exercise” T1.3 in the “Algebra” notes of D. P. PATIL [Pa].

Theorem 3.2. *Let R be a ring with $\text{char}(R) = m < \infty$.*

(A) *If $a \in R$ is such that $a^{s+1} = 0$ for some integer $s \geq 0$, then $(1 - a)^{m^s} = 1$.*

(B) *If a unit $u \in R$ is unipotent, then $u^{m^s} = 1$ for some integer s . The converse is true if m is a power of a prime, but is not true in general.*

PROOF. (A) Let G be the additive subgroup of R generated by $\{a, a^2, \dots, a^s\}$. Then $(G, +)$ is a factor group of $(\mathbb{Z}_m)^s$, so $|G|$ divides m^s . Next, observe that $G \subseteq \text{nil}(R)$, and that (G, \circ) is a submonoid of (R, \circ) with identity 0. Since the map ε embeds (G, \circ) into the group $(U(R), \times)$, (G, \circ) is a cancellation monoid. As $|G| < \infty$, (G, \circ) is a group. In particular, $a \circ \dots \circ a = 0$ if the number of “factors” is $|G|$, or any multiple thereof. Applying the Jacobson map ε , we conclude that $(1 - a)^{m^s} = 1$.

(B) The first conclusion in (B) follows by applying (A) to $a := 1 - u$. Conversely, assume that $m = p^t$ for some prime p , and consider any $u \in R$ with $u^{m^s} = 1$ for some integer s . By Frobenius’ Law (applied to R/pR),

$$(u - 1)^{p^{ts}} \equiv u^{p^{ts}} - 1 \equiv u^{m^s} - 1 \equiv 0 \pmod{pR}.$$

Since pR is a nil ideal, we have $u - 1 \in \text{nil}(R)$, so u is unipotent. To see that this converse part may fail if m is not a prime power, let $R = \mathbb{M}_2(\mathbb{Z}_6)$, with $\text{char}(R) = 6$. We check easily that the unit $u = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ satisfies $u^6 = 1$ (over any ground ring). However, $u - 1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \in U(R)$ is *not* nilpotent, so the unit u is not unipotent. □

To make the quantitative information in the above theorem more explicit and more precise, we state the following.

Corollary 3.3. *Let R be a ring with $\text{char}(R) = m < \infty$.*

- (1) *If R has nilpotence index $s+1 < \infty$ (that is, $a^{s+1} = 0$ for every $a \in \text{nil}(R)$), then any unipotent unit of R has multiplicative order dividing m^s .*
- (2) *If the multiplicative orders of all unipotent units in R divide a fixed integer n , then R has nilpotence index $\leq m^n$.*

PROOF. (1) is clear from Theorem 3.2(A) (by fixing s and varying $a \in \text{nil}(R)$). To prove (2), assume that the integer n exists. For any $a \in \text{nil}(R)$, we have then $(1 - a)^n = 1$. Since $\text{char}(R) = m < \infty$, $S := \mathbb{Z} \cdot 1 \subseteq R$ is a subring of m elements. Therefore, the fact that $a^n \in S + Sa + \dots + Sa^{n-1}$ implies that the unital ring $T := S[a]$ has at most m^n elements. As a is nilpotent in T , it follows easily (by considering the descending chain $T \supseteq aT \supseteq a^2T \supseteq \dots$) that $a^{m^n} = 0$, so R has nilpotence index $\leq m^n$. □

In the rest of this paper, the term “2-group” means a possibly infinite 2-group. Using this notion, we can now prove our second characterization theorem for UU rings in part (1) below, with a supplement in part (2). The proof is short since most of the needed work was already done in the verification of Theorem 3.2 and Corollary 3.3.

Theorem 3.4. (1) *A ring R is a UU ring iff $\text{char}(R) = 2^t$ (for some integer $t \geq 0$) and $U(R)$ is a 2-group.*
 (2) *Let R be a UU ring, with $\text{char}(R) = 2^t$. If R has nilpotence index $s+1 < \infty$, then $U(R)$ has exponent dividing 2^{ts} . Conversely, if $U(R)$ has finite exponent n , then R has nilpotence index $\leq 2^{tn}$.*

PROOF. To prove (1), first assume that $\text{char}(R) = 2^t$ and $U(R)$ is a 2-group. For any $u \in U(R)$, we have $u^{2^{ts}} = 1$ for some s . The “converse” part of (B) in Theorem 3.2 shows that u is unipotent, so R is a UU ring. Next, assume that R is a UU ring. Then $\text{char}(R) = 2^t$ (for some t) by Theorem 2.6(1). For any $u \in U(R)$, Theorem 3.2(B) shows that $u^{2^{ts}} = 1$ for some integer s . Thus, $U(R)$ is a 2-group. Finally, (2) follows from Corollary 3.3 (applied to the case of 2-power characteristic). \square

Remark 3.5. (A) For part (2) above, better bounds can be gotten by *ad hoc* methods in some special cases. For instance, if $U(R)$ has exponent ≤ 2 , then R has nilpotence index ≤ 3 (even without assuming R to be UU). Indeed, for any $a \in \text{nil}(R)$, we have by assumption $(1-a)^2 = 1$; hence $a^2 = 2a$. Replacing a by $-a$ gives $a^2 = -2a$, so $4a = 0$. It follows that $a^3 = 2a^2 = 4a = 0$. On the other hand, if a UU ring R has nilpotence index ≤ 3 , then $8 = 0 \in R$, so for any $a \in \text{nil}(R)$, we have $0 = (2+a)^3 = 4a + 6a^2$. This implies that $(1+a)^4 = 1 + 4a + 6a^2 = 1$, so $U(R)$ has exponent dividing 4.

(B) In general, a UU ring may not have a finite nilpotence index. For instance, the commutative local ring generated over \mathbb{F}_2 by x_2, x_3, \dots , with the relations $x_i^2 = 0$ for all $i \geq 2$ is such an example.

(C) In Theorem 3.4(1), the condition that $U(R)$ is a 2-group alone need not imply that R is UU. The ring \mathbb{Z} and the fields \mathbb{F}_5 and \mathbb{F}_9 are obvious examples. We note also that, for the “only if” part of Theorem 3.4(1) to hold, it is essential that the ring R itself is assumed to be UU. In general, if $H \subseteq U(R)$ is just a multiplicative group of unipotent units, it need not follow that H is a 2-group. For instance, if $R = \mathbb{M}_2(\mathbb{Z})$ and H is the unipotent group $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subseteq \text{GL}_2(\mathbb{Z})$, then every non-identity element of H has infinite order.

The following easy consequence of Theorem 3.4(1) will prove to be useful later toward the end of Section 4. (See the proof of Corollary 4.8.)

Corollary 3.6. *A nil-clean ring R is a UU ring iff $U(R)$ is a 2-group.*

PROOF. By [Di: (3.14)], $\text{char}(R) = 2^t$ for some t , so Theorem 3.4(1) applies. \square

The last two results in this section are motivated by the fact that, although nil-clean rings are clean by [Di: (1.4)], a nil-clean element in a ring need not be clean according to [AC]. In [Ca: (2.4)], however, CĂLUGĂREANU pointed out that a UU ring is clean iff it is nil-clean. Theorem 3.7 below, due to A. Diesl, extends Călugăreanu's ring-theoretic statement to two *element-wise* statements. We thank A. Diesl and P. P. Nielsen for showing us the proof of the following result (as well as its consequence (3.11)).

Theorem 3.7. *For any UU ring R , the following hold.*

- (1) *An element $a \in R$ is clean iff it is nil-clean.*
- (2) *An element $a \in R$ is strongly clean iff it is strongly nil-clean. In this case, a is, in fact, uniquely strongly clean in the sense of CHEN, WANG and ZHOU [CWZ].*

PROOF. (1) First assume $a \in R$ is clean, say $a = e + u$ where $e = e^2$ and $u \in U(R)$. Writing $u = 1 + b$ where $b \in \text{nil}(R)$, we have

$$a = e + 1 + b = (1 - e) + (2e + b). \quad (3.8)$$

In $\bar{R} := R/2R$, $\overline{2e + b} = \bar{b}$ is a nilpotent. Since $2 \in \text{nil}(R)$ too (by Theorem 2.6(1)), this shows that $2e + b \in \text{nil}(R)$, and so (3.8) above implies that a is nil-clean. Conversely, if $a \in R$ is nil-clean, say $a = f + b$ where $f = f^2$ and $b \in \text{nil}(R)$, then

$$a = (1 - f) + (b - 1 + 2f). \quad (3.9)$$

In $\bar{R} = R/2R$ again, $\overline{b - 1 + 2f} = \bar{b} - \bar{1}$ is a unit. As $2R \subseteq \text{rad}(R)$ (by Theorem 2.6(1)), we have in fact $b - 1 + 2f \in U(R)$, so (3.9) implies that a is clean.

(2) The analogue of (1) in the “strong” case can be drawn from the same arguments as above, upon adding a commuting condition $eu = ue$, or $fb = bf$. Finally, if $a = e + u = f + v$ are two strongly clean decompositions for an element $a \in R$, the fact that R is UU implies that

$$a - 1 = e + (u - 1) = f + (v - 1) \quad (3.10)$$

are two strongly nil-clean decompositions for $a - 1$. By the uniqueness of a strongly nil-clean decomposition for any ring element (see [HTY: Theorem 3], or [Di: Corollary 3.8]), we have $e = f$, and hence $u = v$. This shows that $a \in R$ is (automatically) uniquely strongly clean. \square

The theorem above leads to a third characterization of UU rings (part (1) below) that is less direct but easier to prove than Theorem 3.4(1).

Corollary 3.11. (1) *A ring R is a UU ring iff strongly clean elements in R are strongly nil-clean.*

(2) *Any strongly nil-clean ring is UU and uniquely strongly clean in the sense of [CWZ].*

PROOF. (1) The “only if” part follows from Theorem 3.7(2). For the “if” part, assume that strongly clean elements in R are strongly nil-clean. In particular, all units are strongly nil-clean. According to [Di: (3.10)], this is tantamount to saying that all units are unipotent. Thus, R is a UU ring.

(2) Let R be any strongly nil-clean ring. By part (1) above, R is a UU ring. Thus, the last statement in Theorem 3.7(2) shows that R is uniquely strongly clean. \square

4. Exchange UU rings and strongly nil-clean rings

In this section, we come to our second main result, which proves the equivalence of exchange UU rings and Diesl’s strongly nil-clean rings, and recovers for them a sharp structural characterization obtained in [HTY] and [KWZ]. We will do this first in the *semiprimitive* case, for which the following result shows that “exchange UU” boils down to “Boolean”, and therefore also to “uniquely clean” as per the earlier results of NICHOLSON and ZHOU in [NZ]. This result will then be used later (in Theorem 4.3) to derive the equivalence of exchange UU rings and strongly nil-clean rings in general.

Theorem 4.1. *For any ring R , the following statements are equivalent:*

- (1) *R is uniquely clean and semiprimitive.*
- (2) *R is clean, $\text{char}(R) = 2$, and $U(R) = \{1\}$.*
- (3) *R is a Boolean ring.*
- (4) *R is regular and uniquely clean.*
- (5) *R is a regular UU ring.*
- (6) *R is a semiprimitive exchange UU ring.*

PROOF. The equivalence of the first four statements is due to NICHOLSON and ZHOU (see [NZ: Theorem 19]), and (3) \Rightarrow (5) is clear. Next, (5) \Rightarrow (6) follows from the standard fact that a regular ring is a semiprimitive exchange ring. We can now complete the proof with the following nontrivial implication.

(6) \Rightarrow (3). Assuming (6), we first check that $\text{nil}(R) = \{0\}$. If not, then there would exist a nonzero $a \in R$ with $a^2 = 0$. Since R is an I_0 -ring⁵ (in the sense of NICHOLSON [Ni₁]) and $\text{rad}(R) = 0$, a classical result LEVITZKI [Le: Theorem 2.1] implies that R has a nonzero corner ring $eRe \cong \mathbb{M}_2(T)$ for some (nonzero) ring T . (For an alternative treatment of Levitzki's result needed here, see JACOBSON's book [Ja: (X.11.1)].) By Theorem 2.6(2), $eRe \cong \mathbb{M}_2(T)$ is a UU ring, which contradicts Theorem 2.7. Since $\text{nil}(R) = \{0\}$, R is now a *reduced* exchange ring, with $U(R) = 1 + \text{nil}(R) = \{1\}$. In particular, R is an abelian ring, and therefore a clean ring by [Ni₂]. For any $r \in R$, $r + 1$ being clean means that $r + 1 = e + u$ where $e = e^2$ and $u \in U(R)$. Therefore, $r + 1 = e + 1$, and so $r = e$, proving that R is a Boolean ring. \square

Corollary 4.2. *An exchange ring R has a trivial unit group iff it is Boolean.*

PROOF. It suffices to prove the “only if” part. For this, assume that $U(R) = \{1\}$. Clearly, R is a UU ring, and $1 + \text{rad}(R) \subseteq U(R) = \{1\}$ implies that $\text{rad}(R) = \{0\}$. Applying (6) \Rightarrow (3) in Theorem 4.1, we see that R is a Boolean ring. \square

Since the class of exchange rings is quite large, Corollary 4.2 applies to many rings, including for instance all semiperfect rings (such as left or right artinian rings), and all π -regular rings (such as algebraic algebras over fields). The special case of Corollary 4.2 for *finite rings* was recently noted by R. COLEMAN in [Co]. Corollary 4.2 also implies that, if all (von Neumann) regular elements of an exchange ring R are idempotents, then R is Boolean. This fact has been pointed out before by an anonymous referee of a paper of CHEN and LI; see the proof of [CL: Theorem 8].

By extending Theorem 4.1 to the non-semiprimitive case, we can now prove the following expanded version of the “Theorem B” in the Introduction. This result identifies exchange UU rings precisely as the strongly nil-clean rings of A. Diesl, and retrieves for the latter rings the very simple structural characterization (5) below that was proved first in [HTY], and later in [KZW]. Note that the interesting equivalence (4) \Leftrightarrow (5) was also proved in the commutative case in [Di: (3.20)].

⁵An I_0 -ring is sometimes also referred to as a “*semipotent ring*”.

Theorem 4.3. *For any ring R , the following statements are equivalent:*

- (1) R is an exchange UU ring.
- (2) R is a clean UU ring.
- (3) R is a strongly π -regular UU ring.
- (4) R is a strongly nil-clean ring.
- (5) $\text{rad}(R)$ is a nil ideal, and $R/\text{rad}(R)$ is a Boolean ring.

PROOF. For proper credits, (3) \Leftrightarrow (4) is due to DIESL [Di: (3.11)], and (4) \Leftrightarrow (5) was recently proved by KOŞAN, WANG and ZHOU [KWZ: (2.6)] (assuming an earlier result of CHEN, WANG and ZHOU [CWZ: Corollary 18]), although the same equivalence was known earlier to HIRANO, TOMINAGA and YAQUB in [HTY: Theorem 3].⁶ Also, (1) \Leftrightarrow (5) is related to a result of LEE and ZHOU in [LZ: Theorem 13] (which characterized a somewhat broader class of rings). To make our proof self-contained, we shall not assume *any* of the aforementioned results. Instead, we give below a completely independent proof for Theorem 4.3 in the quickest possible way, using a single 5-cycle of implications (with even one trivial step).

(2) \Rightarrow (1). This is clear, as clean rings are exchange rings by [Ni₂: (1.8)].

(1) \Rightarrow (5). Under (1), $\bar{R} := R/\text{rad}(R)$ remains an exchange ring by [Ni₂: (1.4)]. Also, $\text{rad}(R)$ is nil and \bar{R} is UU by Theorem 2.4. Knowing now that \bar{R} is a *semiprimitive* exchange UU ring, Theorem 4.1 implies that \bar{R} is a Boolean ring.

(5) \Rightarrow (3). Assume (5). For any $r \in R$, we have $r - r^2 \in \text{rad}(R)$, so $(r - r^2)^n = 0$ for some $n \geq 1$. By expansion, we have $r^n \in r^{n+1}R \cap Rr^{n+1}$, so R is strongly π -regular. Now consider any $u \in U(R)$. Then $\bar{u} = \bar{1} \in R/\text{rad}(R)$ since $R/\text{rad}(R)$ is Boolean. Thus, $1 - u \in \text{rad}(R) \subseteq \text{nil}(R)$, and so $u \in 1 + \text{nil}(R)$, showing that R is UU.

(3) \Rightarrow (4). Under (3), R is strongly clean by [BM: Proposition 2.6] or [Ni₃: Theorem 1]. To show that R is strongly nil-clean, consider any $a \in R$. We can write $1 + a = e + u$ where $e = e^2$, $u \in U(R)$, and $eu = ue$. Thus, $a = e + (u - 1)$, with $u - 1 \in \text{nil}(R)$ (since R is UU). As e commutes with $u - 1$, this checks that R is strongly nil-clean.

(4) \Rightarrow (2). This implication is already well covered by Corollary 3.11(2), but the latter depended on [Di: (3.10)]. A self-contained proof of (4) \Rightarrow (2) is as follows. Assuming (4), any $u \in U(R)$ can be written in the form $e + b$ where $e = e^2$, $b \in \text{nil}(R)$, and $eb = be$. Then $e = u - b = u(1 - u^{-1}b) \in U(R)$ (since

⁶In [HTY], it was also proved that (5) is equivalent to “ $a - a^2 \in \text{nil}(R)$ for every $a \in R$ ”.

$u^{-1}b \in \text{nil}(R)$). Thus, $e = 1$, so $u = 1 + b \in 1 + \text{nil}(R)$, showing that R is UU. Finally, for any $a \in R$, we can write $1 + a = e + b$ as above. Then $a = e - (1 - b)$ shows that a is clean. \square

Remark 4.4. Somewhat surprisingly, in the general (possibly non-semiprimitive) case, the class of exchange UU rings in Theorem 4.3 is *independent* of the class of the uniquely clean rings of NICHOLSON and ZHOU in [NZ]. First, a local ring (R, \mathfrak{m}) with $R/\mathfrak{m} \cong \mathbb{F}_2$ is uniquely clean by [NZ: Theorem 15], but $\mathfrak{m} = \text{rad}(R)$ need not be a nil ideal, so R may fail to be a UU ring. Second, the ring T_n of $n \times n$ upper triangular matrices over \mathbb{F}_2 is a UU ring as in Theorem 2.9, but if $n \geq 2$, T_n is not an abelian ring so it cannot be uniquely clean according to NICHOLSON and ZHOU [NZ: Lemma 4]. In summary, an exchange UU ring is uniquely clean iff it is abelian, and a uniquely clean ring R is an exchange UU ring iff $\text{rad}(R)$ is nil. These claims can be easily checked by using the characterizations of uniquely clean rings in [NZ: Theorem 20].

We record now some consequences of Theorem 4.3 which give further useful information on the structure of an exchange UU ring (or a strongly nil-clean ring).

Corollary 4.5. *For any exchange UU ring R , the following hold.*

- (A) $\text{rad}(R) = \text{nil}(R)$.
- (B) *The center Z of R is a uniquely clean UU ring, with $\text{rad}(Z) = Z \cap \text{rad}(R)$.*
- (C) *Any factor ring of R is an exchange UU ring.*
- (D) *Any nonzero idempotent $e \in R$ is not a sum of two units in R ; that is, e is not “2-good” in the terminology of VÁMOS [Va].*
- (E) *If $R \neq 0$, $U(R)$ generates a local ring $\text{rad}(R) \cup (1 + \text{rad}(R))$ in R , which contains all local subrings of R .*

PROOF. (A) Any $a \in \text{nil}(R)$ maps to zero in $R/\text{rad}(R)$ (since $R/\text{rad}(R)$ is a reduced ring by (5) of Theorem 4.3). Thus, we must have $a \in \text{rad}(R)$, so $\text{nil}(R) = \text{rad}(R)$. (We note in passing that an *ad hoc* proof for (A) when R is a strongly nil-clean ring has appeared earlier in NIELSEN’s review [Ni] of DIESL’s paper [Di].)

(B) Since R is a (strongly) π -regular (by Theorem 4.3), a classical result of MCCOY [MC: Theorem 1] implies that the center Z of R is π -regular, and hence an exchange ring. Also, Z is a UU ring by Theorem 2.6(3), and it is of course abelian. Thus, it follows from Remark 4.4 that Z is uniquely clean.⁷

⁷In fact, Z is also uniquely nil-clean, by an application of Theorem 5.9(5) of [Di].

Finally, in view of Theorem 4.3, we have

$$\text{rad}(Z) = \text{nil}(Z) = Z \cap \text{nil}(R) = Z \cap \text{rad}(R).$$

(C) This follows from (1) \Leftrightarrow (4) in Theorem 4.3, since any factor ring of a strongly nil-clean ring is obviously also strongly nil-clean.

(D) The property (D) was first brought to light by LEE and ZHOU in [LZ: Theorem 13]. Here, we observe, more generally, that *the property (D) holds for any ring R for which $R/\text{rad}(R)$ is Boolean*. Indeed, if $e \in U(R) + U(R)$, then e maps to $\bar{1} + \bar{1} = \bar{0}$, so $e \in \text{rad}(R)$. As is well known, this is possible only when $e = 0$. (We leave it as an easy exercise to show that the property (D) also holds in any abelian UU ring.)

(E) Since $\text{rad}(R) = \text{nil}(R)$ and $U(R/\text{rad}(R)) = \{\bar{1}\}$, the proof given earlier for part (3) of Theorem 2.4 carries over verbatim to prove (E). \square

Remark 4.6. The reason the above corollary is of interest is that it holds for *exchange UU rings*, while without the UU assumption, the conclusions of the Corollary largely do not hold. For instance, if R is an exchange ring (or even a clean ring), its center may not be an exchange (or equivalently, clean) ring; see [HKL] and [BR]. Also, if R is just a UU ring but not assumed to be an exchange ring, then a factor ring of R may fail to be a UU ring. For instance, any free \mathbb{F}_2 -algebra F is UU, and a suitable choice of F will map onto $M_2(\mathbb{F}_2)$, which is not UU by Theorem 2.7. Finally, we note that part (B) and part (C) of Corollary 4.5 are formal analogues of results of NICHOLSON and ZHOU on uniquely clean rings [NZ], which showed that the uniquely clean property is preserved by going down to the center and by passing to factor rings.

While clean rings are not nil-clean in general, we can draw the following conclusion from Theorem 4.3 on (abelian) clean rings.

Corollary 4.7. (i) *Let R be an abelian clean ring. Then R is nil-clean iff all units of R are nil-clean.*

(ii) *Let R be a clean ring. Then R is strongly nil-clean iff all units of R are strongly nil-clean.*

PROOF. (i) We need only prove the “if” part, so assume that all units of R are nil-clean. For any $u \in U(R)$, write $u = e + r$ where $e^2 = e$ and $r \in \text{nil}(R)$. Since e is central, we have $ur = ru$, so $e = u - r \in U(R)$, and hence $e = 1$. This shows that R is a UU ring. By (2) \Rightarrow (4) in Theorem 4.3, R is (strongly) nil-clean. (We note incidentally that this conclusion can also be proved by a direct

argument as that in Theorem 3.7 (1), not using (4.3); in fact, each element being clean is the sum of a unipotent and an idempotent and hence of a nilpotent and an idempotent, as required.)

(ii) In the notations above, since $er = re$, we have that $ur = ru$ and thus the same trick works. \square

Remark 4.8. Although Corollary 4.7 can be established directly by using the natural element-wise manipulation of (strongly) clean and (strongly) nil-clean elements, the next question seems to be reasonably difficult in general, because even the sum of two nontrivial central idempotents need not be again an idempotent.

Problem 4.9. Does it follow that a clean ring is nil-clean iff every unit is nil-clean?

The last consequence of Theorem 4.3 is the following result relating nil-clean rings to strongly nil-clean rings in terms of the unit group $U(R)$.

Corollary 4.10. *A ring R is strongly nil-clean iff R is nil-clean and $U(R)$ is a 2-group.*

PROOF. If R is strongly nil-clean, of course it is nil-clean. And it is a UU ring by Theorem 4.3, so Theorem 3.4(1) implies that $U(R)$ is a 2-group. Conversely, if R is nil-clean and $U(R)$ is a 2-group, then R is a UU ring by Corollary 3.6. On the other hand, the nil-clean ring R is clean by [Di: Proposition 3.4]. Applying (2) \Rightarrow (4) in Theorem 4.3 shows that R is strongly nil-clean, as desired. \square

As a simple illustration for Corollary 4.10, consider a matrix ring $R = \mathbb{M}_n(S)$ over a ring S . If $S \cong \mathbb{F}_2$, R is known to be a nil-clean ring by [BCDM] or [KLZ]. However, as long as $S \neq 0$ and $n \geq 2$, R has a unit $\text{diag}(V, I_{n-2})$ of order 3 for $V = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. And indeed, we know (from either Theorem 2.7 or Theorem 3.4(1)) that R is not a UU ring, and in particular not a strongly nil-clean ring.

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