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On groups with small verbal conjugacy classes

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Abstract. Given a group G and a word w, we denote by G_w the set of all w-values in G and by $w(G)$ the corresponding verbal subgroup. The main result of the paper is the following theorem. Let n be a positive integer and let w be either the lower central word γ_n or the derived word δ_n. Let G be a group in which for any element $g \in G$ there exist finitely many Chernikov subgroups whose union contains g^{G_w}. Then $\langle g^{w(G)} \rangle$ is Chernikov for all $g \in G$.

1. Introduction

Let w be a word in n variables, and let G be a group. The verbal subgroup $w(G)$ of G determined by w is the subgroup generated by the set G_w consisting of all values $w(g_1, \dots, g_n)$, where $g_1, \dots, g_n$ are elements of G. A word w is said to be concise if whenever G_w is finite for a group G, it always follows that $w(G)$ is finite. P. Hall asked whether every word is concise, but it was later proved that this problem has a negative solution in its general form (see [5, p. 439]). On the other hand, many important words are known to be concise. For instance, Turner-Smith [9] showed that the lower central words γ_n and the derived words δ_n are concise; here the words γ_n and δ_n are defined by the formulae $\gamma_1 = \delta_0 = x$, $\gamma_n = [\gamma_{n-1}, \gamma_1]$ and $\delta_n = [\delta_{n-1}, \delta_{n-1}]$. The corresponding verbal subgroups for these words are the familiar nth term of the lower central series of G denoted by $\gamma_n(G)$ and the nth derived group of G denoted by $G^{(n)}$.

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There are several natural ways to look at Hall's question from a different angle. The circle of problems arising in this context can be characterized as follows.

Given a word w and a group G, assume that certain restrictions are imposed on the set G_w. How does this influence the properties of the verbal subgroup $w(G)$?

If X and Y are non-empty subsets of a group G, we will write X^Y to denote the set $\{y^{-1}xy \mid x \in X, y \in Y\}$. In [2] groups G with the property that x^{G_w} is finite for all $x \in G$ were called $FC(w)$-groups. Recall that FC-groups are precisely groups with finite conjugacy classes. The main result of [2] tells us that if w is a concise word, then a group G is an $FC(w)$-group if and only if $x^{w(G)}$ is finite for all $x \in G$. In particular, it follows that if w is a concise word and G is an $FC(w)$-group, then the verbal subgroup $w(G)$ is FC. Later it was shown in [1] that there exists a function $f = f(m, w)$ such that if, under the hypothesis of the above theorem, x^{G_w} has at most m elements for all $x \in G$, then $x^{w(G)}$ has at most f elements for all $x \in G$. In relation with the above results, the following question was considered in [4].

Given a concise word w and a group G, assume that for all $x \in G$ the subgroup $\langle x^{G_w} \rangle$ satisfies a certain finiteness condition. Is it true that a similar condition is also satisfied by $\langle x^{w(G)} \rangle$ for all $x \in G$?

Here and throughout the paper $\langle M \rangle$ denotes the subgroup generated by the set M. The following theorem is the main result of [4].

Theorem 1.1. *Let n be a positive integer and let w be either the word γ_n or the word δ_n. Suppose that G is a group in which $\langle g^{G_w} \rangle$ is Chernikov for all $g \in G$. Then $\langle g^{w(G)} \rangle$ is Chernikov for all $g \in G$ as well.*

Recall that a group G is Chernikov if it has a subgroup of finite index that is a direct product of finitely many groups of type C_{p^∞} for various primes p (quasicyclic p-groups, or Prüfer p-groups). By a deep result obtained independently by Shunkov [8], and Kegel and Wehrfritz [3] Chernikov groups are precisely the locally finite groups satisfying the minimal condition on subgroups, that is, any non-empty set of subgroups possesses a minimal subgroup.

The purpose of the present paper is to strengthen Theorem 1.1 in the following way.

Theorem 1.2. *Let n be a positive integer and let w be either the word γ_n or the word δ_n. Let G be a group in which for any element $g \in G$ there exist finitely many Chernikov subgroups whose union contains g^{G_w}. Then $\langle g^{w(G)} \rangle$ is Chernikov for all $g \in G$.*

A proof of Theorem 1.2 in the case where $w = \gamma_n$ can be obtained from the case $w = \delta_n$ by simply replacing everywhere in the proof the term "δ_n-commutators" by "γ_n-commutators". That is why we do not provide an explicit proof for the case $w = \gamma_k$ concentrating instead on proving Theorem 1.2 in the case $w = \delta_n$.

The hypothesis in Theorem 1.2 is reminiscent of the situation considered in [7] where it was proved that if the set of δ_n-commutators in a group G is contained in a union of finitely many Chernikov subgroups, then $G^{(n)}$ is Chernikov. As a by-product of the proof of Theorem 1.2 we obtain a considerably stronger result – Corollary 2.11 in the next section says that for any word w if the set of w-values in a group G is contained in a union of finitely many Chernikov subgroups, then $w(G)$ is Chernikov.

2. Preliminaries

Let G be a group acted on by a group A. As usual, $[G, A]$ denotes the subgroup generated by all elements of the form $x^{-1}x^a$, where $x \in G, a \in A$. It is well-known that $[G, A]$ is a normal subgroup of G. If B is a normal subset of A such that $A = \langle B \rangle$, then $[G, A] = \langle [G, b]; b \in B \rangle$. In particular, if A is cyclic, then $[G, A] = [G, a]$, where a is a generator of A.

The minimal subgroup of finite index of a Chernikov group T is called the radicable part of T. Throughout the article we denote this subgroup by T^0. In general a group T is called radicable if the equation $x^n = a$ has a solution in T for every positive integer n and every $a \in T$. It is well-known that a periodic abelian radicable group is a direct product of quasicyclic p-subgroups. Suppose the radicable part of a Chernikov group T has index i and is a direct product of precisely j groups of type C_{p^∞} (for various primes p). The ordered pair (j, i) is called the size of T. The set of all pairs (j, i) is endowed with the lexicographic order. It is easy to check that if H is a proper subgroup of T, the size of H is necessarily strictly less than that of T. Also, if N is an infinite normal subgroup of T, the size of T/N is necessarily strictly less than that of T.

The following lemma is well-known (see for example [6, Part 1, Lemma 3.13]).

Lemma 2.1. *Suppose that R is a radicable abelian normal subgroup of the group G and suppose that H is a subgroup of G such that $[R, \underbrace{H, \dots, H}_{r}] = 1$ for some natural number r. If H/H' is periodic, then $[R, H] = 1$.*

The next few lemmas can be easily deduced from the above. The interested reader can find their proofs for example in [4].

Lemma 2.2. *In a periodic nilpotent group G every radicable abelian subgroup Q is central.*

Lemma 2.3. *Let A be a periodic group acting on a periodic radicable abelian group G. Then $[G,A,A]=[G,A]$.*

Lemma 2.4. *Let A be a finite group acting on a periodic radicable abelian group G. Then $[G,A]$ is radicable.*

Lemma 2.5. *Let A be a radicable group acting on a Chernikov group B. Then $[B,A,A]=1$.*

Lemma 2.6. *Let G be a Chernikov group for which there exists a positive integer m such that G can be generated by elements of order dividing m. If $G^0 \leq Z(G)$, then G is finite.*

Proof. Essentially, this is Lemma 2.7 in [4]. □

Lemma 2.7. *Let G be a group, y an element of G, and x is a δ_n-commutator for some $n \geq 0$. Then $[y,x,x]$ is a δ_{n+1}-commutator.*

Proof. This follows from the fact that $[y,x,x]$ can be written as $[x^{-y},x]^x$. □

Lemma 2.8. *Let G be a group generated by an element g and an abelian radicable subgroup S. Suppose that G has finitely many Chernikov subgroups whose union contains g^S. Then the subgroup $\langle g^S\rangle$ is Chernikov.*

Proof. Suppose that the lemma is false and the subgroup $\langle g^S\rangle$ is not Chernikov. Let $C_1,\dots,C_k$ be finitely many Chernikov subgroups such that $g^S \subseteq \cup C_i$. Without loss of generality we assume that the subgroups $C_1,\dots,C_k$ are chosen in such a way that the sum of the sizes of $C_1,\dots,C_k$ is as small as possible. In that case, of course, each subgroup C_i is generated by $C_i\cap g^S$. Remark that $\langle g^G\rangle = \langle g^S\rangle$ and therefore the subgroup $\langle g^S\rangle$ is normal. If all subgroups $C_1,\dots,C_k$ are finite, then so is the set g^S. In that case the index $[S:C_S(g)]$ is finite. Being radicable, S does not have proper subgroups of finite index and so we deduce that $g^S=g$ and $\langle g^S\rangle=\langle g\rangle$. Since g is contained in a Chernikov subgroup, g must be of finite order and so $\langle g\rangle$ is finite. Therefore, at least one of the subgroups $C_1,\dots,C_k$ is infinite. Without loss of generality assume that C_1 is infinite. Among all infinite subgroups of C_1 that can be generated by elements of g^S we choose a minimal one, say K. Let $Y = K \cap g^S$ and so $K = \langle Y \rangle$. If x is an arbitrary element in S, the set Y^x has infinite intersection with at least one of the subgroups C_i. Suppose that $C_j \cap Y^x$ is infinite and set $L = \langle C_j \cap Y^x \rangle$. It is clear that $L^{x^{-1}}$ is an infinite subgroup of K generated by a subset of Y. Because of minimality of K we conclude that $L = K^x$. Thus, for any $x \in S$ there exists j such that $K^x \le C_j$. Choose $a \in K^0$. It follows that for any $x \in S$ there exists j such that $a^x \le {C_j}^0$. Since a radicable Chernikov group has only finitely many elements of any given order, we deduce that the class a^S is finite. Taking into account that S has no proper subgroups of finite index and that a was taken in K^0 arbitrarily we now deduce that $[K^0, S] = 1$. Since Y normalizes K^0 and since $G = \langle S, Y \rangle$, it follows that K^0 is normal in G. The size of the image of C_1 in G/K^0 is strictly less than that of C_1 and therefore, by induction, $\langle g^S \rangle / K^0$ is Chernikov. Since also K^0 is Chernikov, so is $\langle g^S \rangle$. The proof is complete. □

An idea from the proof of Lemma 2.8 can be used to significantly improve the result that if the set of δ_n-commutators in a group G is contained in a union of finitely many Chernikov subgroups, then $G^{(n)}$ is Chernikov [7]. We will now show that for any word w if the set of w-values in a group G is contained in a union of finitely many Chernikov subgroups, then $w(G)$ is Chernikov. In fact we have the following rather general proposition.

Proposition 2.9. *Let X be a normal subset of a group G and suppose that G has Chernikov subgroups $C_1, \ldots, C_k$ whose union contains X. Then $\langle X \rangle$ is Chernikov.*

Recall that a group having an ascending central series is called hypercentral. For the proof of Proposition 2.9 we will require the following well-known lemma whose proof can be easily deduced for example from [6, Part 2, Theorem 9.23 and Corollary 2, page 125].

Lemma 2.10. *Let G be a hypercentral group generated by its quasicyclic subgroups. Then G is abelian.*

Proof of Proposition 2.9. Without loss of generality we assume that all subgroups C_i are generated by elements of X. Let C be the normal closure of the subgroups $C_1^0, \ldots, C_k^0$. It is clear that C has no subgroups of finite index. If $C = 1$, then the set X is finite. Since the elements of X are contained in Chernikov subgroups, it follows that all elements of X have finite order. In that case $\langle X \rangle$ is finite by Dietzmann's Lemma on elements of finite order having finitely many conjugates (see [6, Part 1, p. 45]). So we assume that $C \ne 1$. In particular, we assume that $C_1^0 \ne 1$. Let K be a minimal infinite subgroup of C_1 generated by elements of X. Because of minimality, for every $x \in G$ there exists i such that $K^x \leq C_i$. Let a be an element of K^0. It follows that every conjugate a^x belong to C_i^0 for some i. Since each subgroup C_i^0 has only finitely many elements of any given order, we conclude that the conjugacy class a^G is finite. Since C has no subgroups of finite index, $a \in Z(C)$. Thus, we have shown that $K^0 \leq Z(C)$. Next, we can repeat the argument with G replaced by $G/Z(C)$ and conclude that if $C \neq Z(C)$, then $Z_2(C) \neq Z(C)$. Thus, we see that C is hypercentral. Since C is generated by quasicyclic subgroups, Lemma 2.10 tells us that C is abelian. Recall that every conjugate $(K^0)^x$ belongs to C_i^0 for some i. Hence, the normal closure $\langle (K^0)^G \rangle$ is Chernikov. Note that the sum of sizes of the images of $C_1, \dots, C_k$ in the quotient $G/\langle (K^0)^G \rangle$ is strictly smaller than that of $C_1, \dots, C_k$. Thus, by induction, the image of $\langle X \rangle$ in $G/\langle (K^0)^G \rangle$ is Chernikov. Therefore $\langle X \rangle$ is Chernikov, as desired. □

The following corollary is now straightforward.

Corollary 2.11. *Let w be a group-word and G a group in which the set of w-values is contained in a union of finitely many Chernikov subgroups. Then $w(G)$ is Chernikov.*

3. Proof of Theorem 1.2

We will now assume the hypothesis of Theorem 1.2 with $w = \delta_n$. Thus, n is a positive integer and G is a group in which for any element $g \in G$ there exist finitely many Chernikov subgroups whose union contains g^{G_w}. We denote by X the set of all δ_n-commutators in G and by H the nth derived group of G. In other words, $H = \langle X \rangle$. Our goal is to prove that $\langle g^H \rangle$ is Chernikov for all $g \in G$.

Let B be the subgroup of G generated by all subgroups of the form $[T, x]$, where T is an abelian radicable subgroup, $x \in X$ and x normalizes T.

Lemma 3.1. *The subgroup B is abelian.*

Proof. Let $S = [T, x]$, where T is an abelian radicable subgroup, $x \in X$ and x normalizes T. By Lemma 2.4 S is radicable. Lemma 2.3 shows that $S = [T, x, x]$. In view of Lemma 2.7 every element in $[T, x, x]$ is a δ_n-commutator. Thus, S is an abelian radicable subgroup contained in X. Choose an arbitrary element $g \in G$. By Lemma 2.8 $\langle g^S \rangle$ is Chernikov. It follows from Lemma 2.5 that $[\langle g^S \rangle, S, S] = 1$. In particular $[g, S, S] = 1$ and so S commutes with S^g. This happens for every $g \in G$ and therefore the normal subgroup $\langle S^G \rangle$ is abelian. Lemma 2.2 shows that in any group a product of normal abelian radicable periodic subgroups is abelian. Being a product of such subgroups, B is abelian. □

Lemma 3.2. *The quotient H/B is an FC-group.*

Proof. Since every element of H is a product of finitely many elements from X, it is sufficient to show that under the additional hypothesis that $B = 1$ the index $[H : C_H(x)]$ is finite for every $x \in X$. Thus, we assume that $B = 1$. Suppose that the lemma is false and choose $x \in X$ such that $[H : C_H(x)]$ is infinite. Set $Y = x^X$. Let $C_1, \dots, C_k$ be finitely many Chernikov subgroups such that $Y \subseteq \cup C_i$. Without loss of generality we assume that the subgroups $C_1, \dots, C_k$ are chosen in such a way that the sum of the sizes of $C_1, \dots, C_k$ is as small as possible. In that case, of course, each subgroup C_i is generated by $C_i \cap Y$. If the subgroups $C_1, \dots, C_k$ were all finite, then in view of the main result of [2] $[H : C_H(x)]$ would be finite. Thus, at least one of the subgroups $C_1, \dots, C_k$ is infinite. Assume that C_1 is infinite and let $Y_1 = Y \cap C_1$. For any $y \in Y_1$ we have $[C_1^0, y] \leq B$. Since $B = 1$ and $C_1 = \langle Y_1 \rangle$, it follows that $C_1^0 \leq Z(C_1)$ whence, by Lemma 2.6, C_1 is finite, a contradiction. □

Lemma 3.3. *For each $g \in G$ the image of $\langle g^H \rangle$ in G/B is Chernikov.*

Proof. It follows from Lemma 3.2 that G is locally finite. Let us assume that $B = 1$. Then H is an FC-group and, since radicable groups have no proper subgroups of finite index, all radicable subgroups of H are contained in the center. Choose $g \in G$ and let $C_1, \dots, C_k$ be finitely many Chernikov subgroups such that $g^X \subseteq \cup C_i$. The subgroup $J = \langle C_1^0, \dots, C_k^0 \rangle$ is Chernikov since it is generated by finitely many commuting Chernikov subgroups. Since g has finite order, it is clear that $J_1 = \prod_i J^{g^i}$ is Chernikov, too. Set $M = H\langle g \rangle$. We remark that J_1 is normal in M. The subgroups $C_1, \dots, C_k$ all have finite images in M/J_1 and therefore the image of the verbal conjugacy class g^X is finite. By [4, Lemma 2.9] the image of the conjugacy class g^H is finite as well. Since g is of finite order, by Dietzmann's lemma the image of $\langle g^H \rangle$ in M/J_1 is finite. Since J_1 is Chernikov, the result follows. □

Lemma 3.4. *The subgroup $[B, h]$ is Chernikov for every $h \in H$.*

Proof. Suppose first that $h \in X$. Then, as we have remarked earlier, $[B, h] \subseteq X$. Let $C_1, \dots, C_k$ be finitely many Chernikov subgroups such that $h^{[B,h]} \subseteq \cup C_i$. Then $[B, h] = [B, h, h] \subseteq \cup (C_i \cap [B, h])$. In view of Lemma 3.1, the subgroups $C_i \cap [B, h]$ commute. Thus, $[B, h]$ is contained in a union of commuting Chernikov subgroups and hence is Chernikov itself.

We now drop the assumption that $h \in X$. Since $h \in H$, we can write h as a product of several elements from X. Suppose that $h = x_1 \cdots x_s$, where $x_i \in X$. Then it is clear that $[B, h] \leq \prod_i [B, x_i]$. Since each $[B, x_i]$ is Chernikov and all $[B, x_i]$ commute, the result follows. □

Lemma 3.5. *Let A be a subgroup of H whose image in G/B is abelian and radicable. Then $[B, A] = 1$.*

Proof. Let $a \in A$. Then, since B is abelian, A/B naturally acts on $[B, a]$ and of course $[B, a, A/B] = [B, a, A]$. By Lemma 3.4 the subgroup $[B, a]$ is Chernikov. According to Lemma 2.5 $[B, a, A, A] = 1$. In particular $[B, a, a, a] = 1$ and so Lemma 2.3 shows that $[B, a] = 1$. This happens for every $a \in A$ and therefore $[B, A] = 1$. □

Lemma 3.6. *For every $g \in G$ the subgroup $[B, g]$ is Chernikov.*

Proof. It was mentioned in the proof of Lemma 3.1 that if T is an abelian radicable subgroup, $x \in X$ and x normalizes T, then $[T, x]$ is an abelian radicable subgroup contained in X. Therefore B is the product of its subgroups $S_1, S_2, \ldots$ each of which is contained in X. Given $g \in G$, let $C_1, \ldots, C_k$ be finitely many Chernikov subgroups such that $g^X \subseteq \cup C_i$ and $B_i = C_i \cap B$ for $i = 1, \ldots, k$. Denote by D the product of all subgroups of the form $(B_i)^{g^j}$ for $i \leq k$ and $j = 0, 1, \ldots$. Since g has finite order, D is a product of finitely many commuting Chernikov subgroups and so is Chernikov itself. It is clear that D is normal in $B\langle g \rangle$.

Since each S_l is contained in X, it follows that $g^{S_l} \subseteq \cup C_i$ for every $l = 1, 2, \ldots$. We look at the image of the class g^{S_l} in the quotient $B\langle g \rangle / D$ and conclude the image is finite since B has finite index in $B\langle g \rangle$. It follows that modulo D the element g centralizes a subgroup of finite index in S_l. Taking into account that S_l has no proper subgroups of finite index we conclude that $[S_l, g] \leq D$. This happens for every $l = 1, 2, \ldots$. Because $[B, g]$ is the product of subgroups of the form $[S_l, g]$, we have $[B, g] \leq D$. □

Lemma 3.7. *For every $g \in G$ the subgroup $[B, \langle g^H \rangle]$ is Chernikov.*

Proof. Choose $g \in G$ and set $K = \langle g^H \rangle$ and $C = C_K(B)$. Then K/C naturally acts on B and $[B, K] = [B, K/C]$. By Lemma 3.3 the image of K in G/B is Chernikov. Let A be the subgroup of $K \cap H$ whose image in G/B is the radicable part of the image of $K \cap H$. By Lemma 3.5 the subgroup A is contained in C. Obviously, $K \cap H$ has finite index in K and therefore the index of A in K is finite. Thus, K/C is finite and so $[B, K]$ is a product of finitely many subgroups of the form $[B, u]$ for suitable elements $u \in K$. By Lemma 3.6 each of the subgroups $[B, u]$ is Chernikov and the result follows. □

We are now ready to complete the proof of Theorem 1.2. Choose $g \in G$ and set $K = \langle g^H \rangle$. By Lemma 3.7 the subgroup $[B, K]$ is Chernikov. We remark that $[B, K]$ is normal in HK and pass to the quotient $\bar{V} = HK/[B, K]$. The image of a subgroup T of HK in $\bar{V}$ will be denoted by $\bar{T}$.

We have $[\bar{B}, \bar{K}] = 1$. It follows from Lemma 3.3 that $\bar{K}/Z(\bar{K})$ is Chernikov. A theorem of Polovickii [6, Part 1, p. 129] now tells us that $\bar{K}'$, the derived group of $\bar{K}$, is Chernikov.

Therefore K' is Chernikov as well. The subgroup $\langle g^X \rangle$ is generated by finitely many Chernikov subgroups and has Chernikov derived group $\langle g^X \rangle'$. We conclude that $\langle g^X \rangle$ is Chernikov for all $g \in G$. The main theorem of [4] now tells us that $\langle g^H \rangle$ is Chernikov for all $g \in G$. The proof is now complete.

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