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How many Ricci flat affine connections are there with arbitrary torsion?

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Abstract. In a previous paper, we solved the question how many real analytic connections with torsion exist locally in dimension n. In the present paper, we solve the questions how many of these connections are Ricci flat. This family of affine connections is described in terms of the number of arbitrary functions of n variables. Suprisingly, the condition "Ricci flat" is not too restrictive when the dimension n is approaching to infinity.

1. Introduction

When we consider an infinite family of well-determined geometric objects, it is natural to put the question about "how many" such objects there exist. In the real analytic case, the Cauchy–Kowalevski Theorem is the standard tool ([4], [8], [12]). Hence a natural way how to measure an infinite family of real analytic geometric objects is a finite family of arbitrary functions of k variables and (optionally) a family of arbitrary functions of $k-1$ variables, and, optionally, "modulo" another family of arbitrary functions of $k-1$ variables. The last (optional) family of functions corresponds to the family of automorphisms of any geometric object from the given family. A good example is the following question: How many real analytic Riemannian metrics are there in dimension 3? It is known (see [5], [9]) that every such metric can be put locally into a diagonal form and that all coordinate transformations preserving diagonal form of the given metric depend on 3 arbitrary functions of two variables. Hence all Riemannian metrics

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in dimension 3 can be locally described by 3 arbitrary functions of 3 variables modulo 3 arbitrary functions of 2 variables.

An immediate question arise if we can "calculate the number" of more basic geometric objects, namely the affine connections in an arbitrary dimension n. Here, we can put, in addition, some condition on the torsion and on the Ricci tensor. To the authors' knowledge, [1], [2], [3] may be the first contributions in this direction. We shall be occupied with real analytic affine connections given locally in arbitrary dimension n.

In the paper [2] the authors proved that the class of all real analytic affine connections with torsion can be described using $n(n^2-1)$ arbitrary functions of n variables modulo $2n$ arbitrary functions of $n-1$ variables. Further, it was proved that the class of all real analytic affine connections (with arbitrary torsion) with skew-symmetric Ricci form depends on $n(2n^2-n-3)/2$ arbitrary functions of n variables and $n(n+1)/2$ arbitrary functions of $n-1$ variables, modulo $2n$ arbitrary functions of $n-1$ variables. The class of real analytic connections (with arbitrary torsion) with symmetric Ricci form depends on $n(2n^2-n-1)/2$ arbitrary functions of n variables and $n(n-1)/2$ arbitrary functions of $n-1$ variables, modulo $2n$ arbitrary functions of $n-1$ variables. The analogous results for affine connections without torsion were obtained in [1].

In the paper [3], equiaffine connections with torsion were studied. An affine connection is *equiaffine* if it admits a parallel volume form. It is well known (see e.g. [11]) that a connection with zero torsion is equiaffine if and only if the Ricci tensor is symmetric. Hence, for the case of a connection with zero torsion, the results obtained in [1] can be applied. In the paper [3], the class of equiaffine connections in dimension n with arbitrary torsion, and its natural subclasses with symmetric, or skew-symmetric Ricci tensor, respectively, were characterized in terms of arbitrary functions of n variables and arbitrary functions of $n-1$ variables.

2. Preliminaries

For the aim of the next sections, and to remain self-contained, we shall formulate the important special case of order one of the Cauchy–Kowalevski Theorem.

Theorem 1. *Consider a system of partial differential equations for unknown functions $U^1(x^1,\dots,x^n),\dots,U^N(x^1,\dots,x^n)$ on an open domain in $\mathbb{R}^n$ and of the* *form*

$$\begin{aligned}
\frac{\partial U^1}{\partial x^1} &= H^1\left(x^1,\dots,x^n,U^1,\dots,U^N,\frac{\partial U^1}{\partial x^2},\dots,\frac{\partial U^1}{\partial x^n},\dots,\frac{\partial U^N}{\partial x^2},\dots,\frac{\partial U^N}{\partial x^n}\right),\\
\frac{\partial U^2}{\partial x^1} &= H^2\left(x^1,\dots,x^n,U^1,\dots,U^N,\frac{\partial U^1}{\partial x^2},\dots,\frac{\partial U^1}{\partial x^n},\dots,\frac{\partial U^N}{\partial x^2},\dots,\frac{\partial U^N}{\partial x^n}\right),\\
&\cdots\\
\frac{\partial U^N}{\partial x^1} &= H^N\left(x^1,\dots,x^n,U^1,\dots,U^N,\frac{\partial U^1}{\partial x^2},\dots,\frac{\partial U^1}{\partial x^n},\dots,\frac{\partial U^N}{\partial x^2},\dots,\frac{\partial U^N}{\partial x^n}\right),
\end{aligned}$$

where $H^i, i=1,\dots,N$, *are real analytic functions of all variables in a neighbourhood of* $(x_0^1,\dots,x_0^n,a^1,\dots,a^N,a_2^1,\dots,a_n^1,\dots,a_2^N,\dots,a_n^N)$, *where* x_0^j, a^i, a_j^i *are arbitrary constants.*

Further, let the functions $\varphi^1(x^2,\dots,x^n),\dots,\varphi^N(x^2,\dots,x^n)$ *be real analytic in a neighbourhood of* $(x_0^2,\dots,x_0^n)$ *and satisfy* $\varphi^i(x_0^2,\dots,x_0^n)=a^i$ *for* $i=1,\dots,N$ *and*

$$\Big(\frac{\partial\varphi^1}{\partial x^2},\dots,\frac{\partial\varphi^1}{\partial x^n},\dots,\frac{\partial\varphi^N}{\partial x^2},\dots,\frac{\partial\varphi^N}{\partial x^n}\Big)(x_0^2,\dots,x_0^n)=(a_2^1,\dots,a_n^1,\dots,a_2^N,\dots,a_n^N).$$

Then the system has a unique solution $(U^1(x^1,\dots,x^n),\dots,U^N(x^1,\dots,x^n))$ *which is real analytic around* $(x_0^1,\dots,x_0^n)$, *and satisfies*

$$U^i(x_0^1,x^2,\dots,x^n)=\varphi^i(x^2,\dots,x^n),\qquad i=1,\dots,N.$$

We now recall the results from the previous paper [1], which will be used in further sections. We work locally with the spaces $\mathbb{R}[u^1,\dots,u^n]$, or $\mathbb{R}[x^1,\dots,x^n]$, respectively and we use the notation $\mathbf{u}=(u^1,\dots,u^n)$ and $\mathbf{x}=(x^1,\dots,x^n)$.

Lemma 2. ([1]) *For any affine connection determined by* $\Gamma_{ij}^h(\mathbf{x})$, *there exists a local transformation of coordinates determined by* $\mathbf{x}=f(\mathbf{u})$ *such that the connection in new coordinates satisfies* $\bar\Gamma_{11}^h(\mathbf{u})=0$, *for* $h=1,\dots,n$. *All such transformations depend on* $2n$ *arbitrary functions of* $n-1$ *variables.*

The system of coordinates with the property from the above lemma is called *pre-semigeodesic* system of coordinates, see for example [10]. We finish this paragraph with the following existence therorem, which is a corollary of Lemma 2.

Theorem 3. ([2]) *All affine connections with torsion in dimension* n *depend locally on* $n(n^2-1)$ *arbitrary functions of* n *variables, modulo* $2n$ *arbitrary functions of* $(n-1)$ *variables.*

Proof. After the transformation into pre-semigeodesic coordinates, we obtain n Christoffel symbols equal to zero. We are left with $n^3 - n = n(n^2 - 1)$ functions. The transformations into pre-semigeodesic coordinates is uniquely determined up to the choice of $2n$ functions $\varphi_0^i(u^2, \dots, u^n), \varphi_1^i(u^2, \dots, u^n)$ of $n-1$ variables. □

We also recall some standard facts and formulas about the Ricci tensor. In the space $\mathbb{R}^n[u^i]$ with coordinate vector fields $E_i = \frac{\partial}{\partial u^i}$, we denote the derivatives with respect to u^i by the bottom index i. Using the definition

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1}$$

we calculate the curvature operators

$$R(E_i, E_j)E_k \quad = \quad (\Gamma^\alpha_{jk})_i E_\alpha - (\Gamma^\beta_{ik})_j E_\beta + \Gamma^\alpha_{jk}\Gamma^\gamma_{i\alpha} E_\gamma - \Gamma^\beta_{ik}\Gamma^\delta_{j\beta} E_\delta.$$

For the Ricci form

$$\operatorname{Ric}(X,Y) = \operatorname{trace}\big[W \mapsto R(W,X)Y\big] \tag{2}$$

we obtain

$$\operatorname{Ric}(E_i, E_j) \quad = \quad \sum_{k,l=1}^{n} \Big[(\Gamma^k_{ij})_k - (\Gamma^k_{kj})_i + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{kj}\Gamma^k_{il}\Big]. \tag{3}$$

3. Ricci flat affine connections

We investigate the conditions

$$\operatorname{Ric}(E_i, E_j) = 0, \qquad\qquad i,j = 1, \dots, n, \tag{4}$$

and we want to show that the Cauchy–Kowalevski Theorem is applicable to this system of equations. We rewrite it as

$$\sum_{k=1}^{n} \Big[(\Gamma^k_{ij})_k - (\Gamma^k_{kj})_i\Big] = \sum_{k,l=1}^{n} \Big[\Gamma^l_{kj}\Gamma^k_{il} - \Gamma^l_{ij}\Gamma^k_{kl}\Big], \quad i,j = 1, \dots, n. \tag{5}$$

We denote the sums of the terms on the right-hand sides by Λ_{ij} and rewrite the system into the more suitable form

$$\Big[(\Gamma^1_{ij})_1 + \dots + (\Gamma^n_{ij})_n\Big] - \Big[(\Gamma^1_{1j})_i + \dots + (\Gamma^n_{nj})_i\Big] = \Lambda_{ij}, \quad i,j = 1, \dots, n.$$

For $i = 1$ and $j = 1, \ldots, n$, we keep each derivative $(\Gamma^n_{nj})_1$ on the left-hand side of the corresponding equation. We denote the sum of all remaining terms on the left-hand side of the corresponding equation by Λ'_{1j} and move it to the right-hand side. For $i > 1$ and $j = 1, \ldots, n$, we keep each derivative $(\Gamma^1_{ij})_1$ on the left-hand side of the corresponding equation. We denote the sum of all remaining terms on the left-hand side of the corresponding equation by Λ'_{ij} and move it to the right-hand side. We obtain the new system

$$\begin{aligned} (\Gamma^n_{nj})_1 &= \Lambda_{1j} - \Lambda'_{1j}, \quad j = 1, \ldots, n, \\ (\Gamma^1_{ij})_1 &= \Lambda_{ij} - \Lambda'_{ij}, \quad i = 2, \ldots, n, \quad j = 1, \ldots, n. \end{aligned} \tag{6}$$

We note that the first derivatives which are on the left-hand sides of this system are not present in any terms Λ'_{ij} on the right-hand sides. Now, it is clear that we can choose all Christoffel symbols, except those whose derivatives appear on the left-hand sides of the system, as arbitrary functions and determine the other Christoffel symbols using the Cauchy–Kowalevski Theorem.

Theorem 4. *The family of real analytic Ricci flat affine connections with torsion in dimension n depends on $n(n^2 - n - 1)$ functions of n variables and n^2 functions of $n - 1$ variables modulo $2n$ functions of $n - 1$ variables.*

Proof. The family of all real analytic affine connections depends on $n(n^2 - 1)$ Christoffel symbols. (The n Christoffel symbols are zero in pre-semigeodesic coordinates.) The n^2 Christoffel symbols are determined from the system of equations (6). Hence, we can choose arbitrarily the $n(n^2-1)-n^2 = n(n^2-n-1)$ functions. The n^2 functions of $n-1$ variables appear by solving the system (6) by the Cauchy–Kowalevski Theorem and the $2n$ functions of $n - 1$ variables appear because we have used pre-semigeodesic coordinates. □

4. Conclusions

Convention. Let $f(n)$ and $h(n)$ be two sequences depending on natural numbers and let $\lim_{n\to\infty} \frac{f(n)}{h(n)} = 1$. Then we say that $f(n)$ and $h(n)$ are *asymptotically equal at infinity.*

Now, we can conclude with the following

Theorem 5. *The number of all real analytic Ricci flat affine connections with arbitrary torsion is asymptotically equal at infinity to the number of all real analytic affine connections with arbitrary torsion.*

Proof. The result follows from Theorems 3 and 4 because real analytic functions of $(n-1)$ variables form a set of measure zero among the real analytic functions of n variables (a result of Hilbert) and they need not be counted. □

Open problems. a) Find an analogue of Theorem 4 for the torsion-free case. b) Find an analogue of Theorem 4 for Ricci parallel affine connections.

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