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Continued fractional algebraic independence of sequences

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There are a lot of papers concerning algebraic independence ([3], [4]), the transcendence of continued fractions, ([2], [4], [5], [11], [12], [13], [15]) and the irrationality of infinite series ([6], [7], [8], [9], [10], [14], [16], [17], [18]), however there is no criterion describing the continued fractional algebraic independence of sequences. This paper deals with such a criterion.

Definition. Let $\{a_{in}\}_{n=1}^{\infty}$ $(i = 1, 2, \dots, k)$ be sequences of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers, the continued fractions $[a_{i1}c_1, a_{i2}c_2, \dots]$ (where $i = 1, 2, \dots, k$) are algebraically independent, then the sequences $\{a_{in}\}_{n=1}^{\infty}$ are continued fractional algebraically independent.

Theorem (Bundschuh [3]). *Let $\beta_1, \dots, \beta_k$ be given complex numbers, and let $g : \mathbb{N} \to \mathbb{R}_+$ satisfy $g(n) \to \infty$ as $n \to \infty$. Suppose that for each $\tau \in \{1. \dots, t\}$ there exists an infinite set $N_\tau \subset \mathbb{N}$ and τ sequences $\{\beta_{1n}\}_{n \in N_\tau}, \dots, \{\beta_{\tau n}\}_{n \in N_\tau}$ of algebraic numbers such that for each $n \in N_\tau$ the inequalities*

$$g(n) \sum_{\sigma=1}^{\tau-1} |\beta_\sigma - \beta_{\sigma n}| < |\beta_\tau - \beta_{\tau n}| \leq$$

$$\leq \exp(-g(n)\ [\mathbb{Q}(\beta_{1n}, \dots, \beta_{\tau n}) : \mathbb{Q}] \sum_{\sigma=1}^{\tau} \frac{s(\beta_{\sigma n})}{\partial(\beta_{\sigma n})}$$

hold, where $\partial(\beta)$ and $H(\beta)$ denote the degree and the height respectively of an algebraic number β and $s(\beta) = \partial(\beta) + \log H(\beta)$. Then $\beta_1, \dots, \beta_t$ are algebraically independent.

Theorem 1. *Let* $\{a_{in}\}_{n=1}^{\infty}$ $(i = 1, \dots, k)$ *be sequences of positive integers. If*

$$\limsup(\lg(\lg a_{1n}))/n = \infty \tag{1}$$

and

$$a_{i+1,n}\, 2^{2^n} > a_{in} > (1 + 1/n)\ (a_{i+1,n} + 1) \tag{2}$$

for $i = 1, \dots, k-1$ *hold, then* $\{a_{in}\}_{n=1}^{\infty}$ *are continued fractional algebraically independent sequences.*

Proof. It is sufficient to prove that the continued fractions $\alpha_i = [a_{i1}, a_{i2}, \dots]$ are algebraically independent. If $\{c_n\}_{n=1}^{\infty}$ denotes any sequence of positive integers, and $b_{ij} = c_j a_{ij}$, then the sequences $\{b_{in}\}_{n=1}^{\infty}$ $(i = 1, \dots, k)$ satisfy (1) and (2). (1) also implies that there is a monotonically increasing sequence $\{H_n\}_{n=1}^{\infty}$ of positive real numbers H_n, with $\lim_{n\to\infty} H_n = \infty$, such that $\limsup_{n\to\infty} a_{1n}^{1/H_n^n} = \infty$. Let us put $S_n = a_{1n}^{1/H_n^n}$. Thus

$$\limsup_{n\to\infty} S_n = \infty. \tag{3}$$

Then for infinitely many n we have

$$S_{n+1} > (1 + 1/n^2) \max_{1\le k\le n} S_k. \tag{4}$$

If not, then for a fixed N and for every positive integer $n > N$

$$S_{n+1} < (1 + 1/n^2) \dots (1 + 1/N^2) \max_{1\le k\le N} S_k.$$

This implies

$$S_{n+1} < K_1 \prod_{i=N}^{\infty} (1 + 1/n^2) = K_2,$$

a contradiction with (3). Thus (4) holds. Now for infinitely many n we have

$$
\begin{aligned}
(5)\qquad & a_{1,n+1}^{1/(H_n-1)} = S_{n+1}^{H_{n+1}^{n+1}/(H_n-1)} > \\
& > (1+1/n^2)^{H_n^{n+1}/(H_n-1)} \cdot \max_{1\le k\le n} S_k^{H_n^{n+1}/(H_n-1)} > \\
& > (1+1/n^2)^{H_n^{n+1}/(H_n-1)} \cdot \max_{1\le k\le n} S_k^{(H_n^{n+1}-1)/(H_n-1)} \ge \\
& \ge (1+1/n^2)^{H_n^{n+1}/(H_n-1)} \prod_{i=1}^{n} \max_{1\le k\le n} S_k^{H_n^i} \ge \\
& \ge (1+1/n^2)^{H_n^{n+1}/(H_n-1)} \prod_{i=1}^{n} a_{1i}.
\end{aligned}
$$

Using Bundschuh's Theorem it is enough to prove that for infinitely many n and for every $j = 1, \ldots, k$

$$
(6)\qquad g(n)\sum_{i=1}^{j-1} |\alpha_i - \alpha_{in}| < |\alpha_j - \alpha_{jn}| < H(\alpha_{1n})^{-g(n)}
$$

hold, where $\alpha_{in} = [a_{i1}, \ldots, a_{in}] = p_{in}/q_{in}$. It is well known that there is a constant $c = c(\alpha_1, \ldots, \alpha_k)$ such that

$$
(7)\qquad \frac{c}{a_{i,n+1}q_{in}^2} < |\alpha_i - \alpha_{in}| < \frac{1}{a_{i,n+1}q_{in}^2}
$$

(for the proof see e.g. [11] chapter 10) and

$$
(8)\qquad \prod_{i=1}^{n} a_{ji} < q_{jn} < \prod_{i=1}^{n} (a_{ji}+1), \quad (j = 1, \ldots, n)
$$

which can be proved by mathematical induction and using

$$
q_{j,n+1} = a_{n+1}q_{jn} + q_{j,n-1}.
$$

(6) and (7) imply that it is sufficient to prove that for infinitely many n

$$
(9)\qquad g(n)ja_{j,n+1}q_{jn}^2 < c\ a_{j-1,n+1}q_{j-1,n}^2
$$

and

$$
(10)\qquad q_{1n}^{g(n)} < a_{j,n+1}q_{jn}^2
$$

hold. (2) and (8) imply that

$$
\lim_{n\to\infty} ca_{j-1,n+1}q_{j-1,n}^2(a_{j,n+1}q_{jn}^2)^{-1} = \infty.
$$

Then we can put

$$g(n) = \min(H_n, ca_{j-1,n+1}q_{j-1,n}^2(a_{j,n+1}q_{jn}^2)^{-1}) \tag{11}$$

and this immediately implies (9). (5) implies

$$a_{1,n+1} > (1+1/n^2)^{H_n^{n+1}} \left(\prod_{i=1}^{n} a_{1i}\right)^{H_n - 1}.$$

Thus

$$\begin{aligned}\prod_{i=1}^{n+1} a_{1i} &> (1+1/n^2)^{H_n^{n+1}} \left(\prod_{i=1}^{n} a_{1i}\right)^{H_n} = \\ &= (1+1/n^2)^{H_n^{n+1}} 2^{-nH_n} \left(\prod_{i=1}^{n} 2a_{1i}\right)^{H_n} \geq \\ &\geq (1+1/n^2)^{H_n^{n+1}} 2^{-nH_n} \left(\prod_{i=1}^{n} (a_{1i}+1\right)^{H_n}.\end{aligned}$$

Using (8) we obtain

$$\prod_{i=1}^{n+1} a_{1i} \geq (1+1/n^2)^{H_n^{n+1}} 2^{-nH_n} \cdot q_{1n}^{H_n}.$$

This, (2) and (8) imply

$$\begin{aligned}(1+1/n^2)^{H_n^{n+1}} 2^{-nH_n} \cdot q_{1n}^{H_n} &\leq \prod_{i=1}^{n+1} a_{ji} 2^{(j-1)2^i} \leq \\ &\leq 2^{(j-1)2^{n+2}} \cdot \prod_{i=1}^{n+1} a_{ji} \leq 2^{(j-1)2^{n+2}} a_{j,n+1} q_{jn}^2\end{aligned} \tag{12}$$

Now we have

$$\lim_{n\to\infty} 2^{(j-1)2^{n+2}} \cdot 2^{nH_n} \cdot (1+1/n^2)^{-H_n^{n+1}} = 0.$$

This, (11) and (12) imply (10) and the proof is finished.

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