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Refinements of Hardy-type inequalities via superquadracity

By JAMES ADEDAYO OGUNTUASE (Abeokuta), LARS-ERIK PERSSON (Luleå)
and OLANREWAJU OLABIYI FABELURIN (Ile-Ife)

Abstract. Some new refinements of Hardy-type integral inequalities are derived, proved and discussed using the concept of superquadratic and subquadratic functions. The results obtained are generalizations and improvements of inequalities of this type in the literature.

1. Introduction

In a note published in 1920, Hardy [3] announced and proved in his famous 1925 note [4] (see also [6]) the following classical integral inequality

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx \le \Big(\frac{p}{p-1}\Big)^p \int_0^\infty f^p(x)dx, \tag{1.1}$$

where $p>1$ and f is a nonnegative p-integrable function on $(0,\infty)$.

In 1928, Hardy [5] (see also [6]) proved the first generalization of (1.1), namely

$$\int_0^\infty x^{-k}\Big(\int_0^x f(t)dt\Big)^p dx \le \Big(\frac{p}{k-1}\Big)^p \int_0^\infty x^{p-k} f^p(x)dx \tag{1.2}$$

for $p\ge 1, k>1$ and also the dual form of this inequality

$$\int_0^\infty x^{-k}\Big(\int_x^\infty f(t)dt\Big)^p dx \le \Big(\frac{p}{1-k}\Big)^p \int_0^\infty x^{p-k} f^p(x)dx \tag{1.3}$$

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for $p \geq 1, k < 1$.

Nowadays, there abounds in the literature a lot of information about Hardy's inequality comprising both its generalizations and applications in different ways (see e.g. [9], [10] and the references given therein). The first known refinements of inequalities (1.2)-(1.3) was obtained in 1971 by Shum [18] where the following results were stated and proved:

$$\int_0^b x^{-k}\left(\int_0^x f(t)dt\right)^p dx + \frac{p}{k-1}b^{1-k}\Big(\int_0^b f(t)dt\Big)^p \leq \Big(\frac{p}{k-1}\Big)^p \int_0^b x^{p-k} f^p(x)dx \tag{1.4}$$

for $p \geq 1, k > 1, 0 < b \leq \infty$, and

$$\int_b^\infty x^{-k}\left(\int_x^\infty f(t)dt\right)^p dx + \frac{p}{1-k}b^{1-k}\Big(\int_b^\infty f(t)dt\Big)^p \leq \Big(\frac{p}{1-k}\Big)^p \int_b^\infty x^{p-k} f^p(x)dx \tag{1.5}$$

for $p \geq 1, k < 1, 0 \leq b < \infty$.

Furthermore, Imoru [7] used mainly the convexity argument to give another proof of a generalized form of (1.4)–(1.5). Moreover, Persson and Oguntuase [16] presented another elementary proofs of inequalities (1.4)–(1.5) and also proved that some versions of (1.4)–(1.5) in fact holds also for $p < 0$. Abramovich *et al.* [1] introduced the concept of superquadratic and subquadratic functions and also proved the refined Jensen's inequality for "more convex" functions. For a detailed theory of convexity and its applications in the development of Hardy type inequalities (see e.g. the books [13], [14] and the references cited therein). In a more recent paper, Oguntuase and Persson [15] using mainly superquadracity argument obtained further refinements of (1.2)–(1.3). In particular, the following inequality is obtained:

$$\begin{aligned}&\int_0^b x^{-k}\left(\int_0^x f(t)dt\right)^p dx \\ &+ \frac{k-1}{p}\int_0^b\int_t^b \left|\frac{p}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} f(t) - \frac{1}{x}\int_0^x f(t)dt\right|^p x^{p-k-\frac{k-1}{p}}dx t^{\frac{k-1}{p}-1}dt \\ &\leq \left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)dx,\end{aligned} \tag{1.6}$$

where $p \geq 2$, $k > 1$, $0 < b \leq \infty$.

Furthermore, the dual of inequality (1.6) was similarly derived and proved.

For the general theory of Hardy type inequalities we refer to the books [6], [9] and [10] and the references given there. However, it is still a very active area. In particular, we refer to the following newer results and references in [2], [11], [12], [17], which cannot be found in these books.

In this paper we prove some new refined Hardy type inequalities which cannot be found in the standard literature on this subject e.e. (which can not be found in books and references mentioned above). Our main results are presented and proved in Section 3 but first we present some preliminaries in Section 2.

2. Preliminaries

In this Section, we present some basic definitions and results on superquadratic and subquadratic functions which are very useful to the proofs of our main results.

Definition 2.1. ([1, Definition 2.1]) A function $\Phi : [0, \infty) \to \Re$ is said to be superquadratic provided for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$\Phi(y) - \Phi(x) - C_x(y - x) - \Phi(|y - x|) \geq 0 \tag{2.1}$$

for all $y \geq 0$. Φ is subquadratic if $-\Phi$ is superquadratic.

The following lemma shows in particular that a nonnegative superquadratic function is necessary convex.

Lemma 2.2. *(*[1, Lemma 2.2]*) Let $\Phi(x)$ be a superquadratic function with C_x as in* (2.1)

(1) *Then $\Phi(0) \leq 0$.*

(2) *If $\Phi(0) = \Phi^{'}(0) = 0$, then $C_x = \Phi^{'}(x)$ whenever Φ is differentiable at $x > 0$.*

(3) *If $\Phi \geq 0$, then Φ is convex and $\Phi(0) = \Phi^{'}(0) = 0$.*

Lemma 2.3. *(*[1, Lemma 3.1]*) Suppose $\Phi : [0, \infty) \to \Re$ is continuously differentiable and that $\Phi(0) \leq 0$. If $\Phi^{'}$ is superadditive or $\frac{\Phi^{'}(x)}{x}$ is nondecreasing, then Φ is superquadratic.*

The next result gives a refined Jensen's inequality for superquadratic and subquadratic functions.

Theorem 2.4. ([1, Theorem 2.3]) *Let (Ω, Σ, μ) be a probability measure space. Then the inequality*

$$\Phi\left(\int_\Omega f(x)d\mu(x)\right) + \int_\Omega \Phi\left(\left|f(x) - \int_\Omega f(x)d\mu(x)\right|\right) d\mu(x) \leq \int_\Omega \Phi(f(x))d\mu(x) \tag{2.2}$$

holds for all probability measures μ and all non-negative μ-integrable functions f if and only if $\phi : [0, \infty) \to \Re$ is a superquadratic. Moreover, (2.2) *holds in the reversed direction if and only if ϕ is subquadratic.*

Proof. See [1] for details. □

Remark 2.5. By setting $\phi(u) = u^p, p \geq 2$ in Theorem 2.4 yields that

$$\left(\int_\Omega f(x)d\mu(x)\right)^p + \int_\Omega \left(\left|f(x) - \int_\Omega f(x)d\mu(x)\right|\right)^p d\mu(x) \leq \int_\Omega (f(x))^p d\mu(x), \tag{2.3}$$

while the sign of (2.3) is reversed if $1 < p \leq 2$.

3. Main results

By using Hölder's inequality we find that

$$\left(\int_0^l f(t)dt\right)^p \leq \frac{l^{(p-1)(1-a)}}{(1-a)^{p-1}} \int_0^l t^{a(p-1)} f^p(t)dt,$$

where $p > 1, 0 < a < 1, 0 < l < \infty$. First, we state the following complement to this result:

Lemma 3.1. *Let $p > 1, a < 1$, and $f \geq 0$ be a measurable function. Then, for $p \geq 2$ and each $l > 0$,*

$$\left(\int_0^l f(t)dt\right)^p + l^{a-1+p}(1-a)\int_0^l t^{-a}\left(\left|\left(\frac{1}{1-a}\right)\left(\frac{t}{l}\right)^a f(t) - \frac{1}{l}\int_0^l f(t)dt\right|\right)^p dt \leq \frac{l^{(p-1)(1-a)}}{(1-a)^{p-1}} \int_0^l t^{a(p-1)} f^p(t)dt. \tag{3.1}$$

If $1 < p \leq 2$, then the sign of (3.1) *is reversed.*

Proof. Let $p \geq 2$ and define the probability measure $d\mu$ on $(0,x)$ by $d\mu := (1-a)t^{-a}l^{a-1}dt$. Then by using Theorem 2.4 (c.f. Remark 2.5) we obtain that

$$\begin{aligned}
\left(\int_0^l f(t)dt\right)^p &= \left(\int_0^l \frac{1}{1-a}t^a l^{1-a} f(t)d\mu\right)^p = \frac{l^{p(1-a)}}{(1-a)^p}\left(\int_0^l t^a f(t)d\mu\right)^p \\
&\leq \frac{l^{p(1-a)}}{(1-a)^p}\left\{\int_0^l \left(t^a f(t)\right)^p d\mu - \int_0^x \left(\left|t^a f(t) - \int_0^l t^a f(t)d\mu\right|^p\right)d\mu\right\} \\
&= \frac{l^{(p-1)(1-a)}}{(1-a)^{p-1}}\left\{\int_0^l t^{a(p-1)} f^p(t)dt - \int_0^l t^{-a}\left(\left|t^a f(t) - (1-a)l^{a-1}\int_0^l f(t)dt\right|^p\right)dt\right\} \\
&= \frac{l^{(p-1)(1-a)}}{(1-a)^{p-1}}\int_0^l t^{a(p-1)} f^p(t)dt \\
&- \frac{l^{a-1+p}}{(1-a)^{-1}}\int_0^l t^{-a}\left(\left|\left(\frac{1}{1-a}\right)\left(\frac{t}{l}\right)^a f(t) - \frac{1}{l}\int_0^l f(t)dt\right|\right)^p dt.
\end{aligned}$$

For the case $1 < p \leq 2$, the proof is similar to the one given above except that the inequality sign is reversed. □

Theorem 3.2. (a) *Let $f \geq 0, g > 0, a < 1, p \geq 2$ and $q > p - a(p-1)$. If $\frac{x}{g(x)}$ is non-increasing and $F(x) = \int_0^x f(t)dt$, then*

$$\begin{aligned}
&\int_0^\infty \frac{F^p(x)}{g^q(x)}dx \\
&+ (1-a)\int_0^\infty \int_t^\infty \left(\left|\left(\frac{1}{1-a}\right)\left(\frac{t}{x}\right)^a f(t) - \frac{1}{x}\int_0^x f(t)dt\right|\right)^p \frac{x^{a-1+p}}{g^q(x)}dx t^{-a}dt \\
&\leq \frac{1}{[(a-1)(p-1)+q-1](1-a)^{p-1}}\int_0^\infty \frac{(xf(x))^p}{g^q(x)}dx. \qquad (3.2)
\end{aligned}$$

(b) *If instead $\frac{g(x)}{x}$ is non-decreasing and $1 < p \leq 2$, then (3.2) holds in the reversed direction.*

Proof. Let $p \geq 2$. By applying Lemma 3.1 and Fubini's theorem, we find that

$$\begin{aligned}
\int_0^\infty \frac{F^p(x)}{g^q(x)}dx &= \int_0^\infty g^{-q}(x)\left(\int_0^x f(t)dt\right)^p dx \\
&\leq \int_0^\infty g^{-q}(x)\frac{x^{(p-1)(1-a)}}{(1-a)^{p-1}}\int_0^x t^{a(p-1)} f^p(t)dtdx \\
&- \int_0^\infty g^{-q}(x)\frac{x^{a-1+p}}{(1-a)^{-1}}\int_0^x t^{-a}\left(\left|\left(\frac{1}{1-a}\right)\left(\frac{t}{x}\right)^a f(t) - \frac{1}{x}\int_0^x f(t)dt\right|\right)^p dtdx
\end{aligned}$$

$$= \frac{1}{(1-a)^{p-1}} \int_0^\infty t^{a(p-1)} f^p(t) \int_t^\infty x^{(1-a)(p-1)} g^{-q}(x) dx dt$$
$$- (1-a) \int_0^\infty \int_t^\infty \frac{x^{a-1+p}}{g^q(x)} \left(\left| \left(\frac{1}{1-a} \right) \left(\frac{t}{x} \right)^a f(t) - \frac{1}{x} \int_0^x f(t)dt \right| \right)^p t^{-a} dx dt$$
$$\leq \frac{1}{(1-a)^{p-1}} \int_0^\infty t^{a(p-1)} f^p(t) \left(\frac{t}{g(t)} \right)^q \int_t^\infty x^{(1-a)(p-1)-q} dx dt$$
$$- (1-a) \int_0^\infty \int_t^\infty \frac{x^{a-1+p}}{g^q(x)} \left(\left| \left(\frac{1}{1-a} \right) \left(\frac{t}{x} \right)^a f(t) - \frac{1}{x} \int_0^x f(t)dt \right| \right)^p t^{-a} dx dt$$
$$= \frac{1}{[(a-1)(p-1)+q-1](1-a)^{p-1}} \int_0^\infty \frac{(tf(t))^p}{g^q(t)} dt$$
$$- (1-a) \int_0^\infty \int_t^\infty \left(\left| \left(\frac{1}{1-a} \right) \left(\frac{t}{x} \right)^a f(t) - \frac{1}{x} \int_0^x f(t)dt \right| \right)^p \frac{x^{a-1+p}}{g^q(x)} dx t^{-a} dt.$$

(b) The proof of (b) is similar to the proof of (a) except that the sign of the inequality is reversed. □

Example 3.3. If $q = k, g(x) = x$ and $a = 1 - \frac{k-1}{p}$, then for $p \geq 2$, Theorem 3.2 yields the following inequality for any $k > 1$:

$$\int_0^\infty x^{-k} \left(\int_0^x f(t)dt \right)^p dx$$
$$+ \frac{k-1}{p} \int_0^\infty \int_t^\infty \left| \frac{p}{k-1} \left(\frac{t}{x} \right)^{1-\frac{k-1}{p}} f(t) - \frac{1}{x} \int_0^x f(t)dt \right|^p x^{p-k-\frac{k-1}{p}} dx t^{\frac{k-1}{p}-1} dt$$
$$\leq \left(\frac{p}{k-1} \right)^p \int_0^\infty x^{p-k} f^p(x) dx. \tag{3.3}$$

For the case $1 < p \leq 2$, inequality (3.3) holds in the reversed direction so in particular for $p = 2$ we have equality.

Remark 3.4. Denote the three integrals in (3.3) by I_1, I_2 and I_3, respectively. Then Example 3.3 shows that Hardy's weighted inequality (1.2) i.e.

$$I_1 \leq I_3,$$

can be refined to

$$I_1 + I_2 \leq I_3$$

for $p \geq 2$ while for $1 < p \leq 2$ we even have the following two sided estimate

$$I_1 \leq I_3 \leq I_1 + I_3.$$

Hence, our Theorem 3.2 gives a further generalization of this refinement of Hardy's inequality (1.2).

Remark 3.5. Observe that inequality (3.3) coincides with inequality (1.6) obtained by Oguntuase and Persson in [15] when $b = \infty$.

Our next aim is to state a dual form of Theorem 3.2 but first we state the following variant of Lemma 3.1 which, for $p \geq 2$, refines the Hölder inequality

$$\Big(\int_l^\infty f(t)dt \Big)^p \leq \frac{l^{(p-1)(a-1)}}{(a-1)^{p-1}} \int_l^\infty t^{a(p-1)} f^p(t)dt,$$

where $a > 1, 0 < l < \infty$.

Lemma 3.6. *Let $p > 1, a > 1$ and $f \geq 0$. Then, for $p \geq 2$ and $0 < l < \infty$,*

$$\begin{aligned}&\Big(\int_l^\infty f(t)dt \Big)^p \\ &+ l^{a-1+p}(a-1) \int_l^\infty t^{-a} \Big(\Big| \Big(\frac{1}{a-1} \Big) \Big(\frac{t}{l} \Big)^a f(t) - \frac{1}{l} \int_l^\infty f(t)dt \Big| \Big)^p dt \\ &\leq \frac{l^{(p-1)(1-a)}}{(a-1)^{p-1}} \int_l^\infty t^{a(p-1)} f^p(t)dt. \qquad (3.4)\end{aligned}$$

If $1 < p \leq 2$, then the sign of (3.4) is reversed.

Proof. Let $p \geq 2$ and this time we define the probability measure $d\mu$ on (l, ∞) by

$$d\mu = (a-1)t^{-a} l^{a-1} dt.$$

By again using Remark 2.5 and by making similar calculations as in the proof of Lemma 3.1 we arrive at (3.4). The proof of the case $1 < p \leq 2$ is also the same (the only inequality sign is reversed). □

We are now ready to formulate the dual version of Theorem 3.2.

Theorem 3.7. (a) *Let $f \geq 0, g > 0, a > 1, p \geq 2$ and $q < p - a(p-1)$. If $\frac{x}{g(x)}$ is non-decreasing and $F_1(x) = \int_x^\infty f(t)dt$, then*

$$\begin{aligned}&\int_0^\infty \frac{F_1^p(x)}{g^q(x)} dx \\ &+ (a-1) \int_0^\infty \int_0^t \Big(\Big| \frac{1}{(a-1)} \Big(\frac{t}{x} \Big)^a f(t) - \frac{1}{x} \int_x^\infty f(t)dt \Big| \Big)^p \frac{x^{a-1+p}}{g^q(x)} dx t^{-a} dt \\ &\leq \frac{1}{[-q+1+(p-1)(1-a)](a-1)^{p-1}} \int_0^\infty \frac{\big(xf(x)\big)^p}{g^q(x)} dx. \qquad (3.5)\end{aligned}$$

(b) *If instead $\frac{x}{g(x)}$ is non-increasing and $1 < p \leq 2$, then (3.5) holds in the reversed direction.*

Proof. The proof is step by step the same as the proof of Theorem 3.2. Here we just use Lemma 3.6 instead of Lemma 3.1 and come to integrals of the type $\int_0^t$ instead of $\int_t^\infty$ so we must assume that $\frac{x}{g(x)}$ is non-decreasing (non-increasing) instead of non-increasing (non-decreasing). Moreover, the restriction $q > p - a(p-1)$ must be changed to $q < p - a(p-1)$. We omit the details. □

Example 3.8. Let $q = k, g(x) = x$ and $a = 1 - \frac{k-1}{p}$. Then we obtain for $p \geq 2$ the following dual form of (3.5): If $k < 1$, then

$$\begin{aligned} &\int_0^\infty x^{-k} \left(\int_x^\infty f(t)dt \right)^p dx \\ &+ (1-k) \int_0^x \int_0^t \left| \frac{p}{1-k} \left(\frac{t}{x} \right)^{1-\frac{k-1}{p}} f(t) - \frac{1}{x} \int_0^x f(t)dt \right|^p x^{p-k-\frac{k-1}{p}} dx t^{\frac{k-1}{p}-1} dt \\ &\leq \left(\frac{p}{k-1} \right)^p \int_0^\infty x^{p-k} f^p(x) dx. \end{aligned} \tag{3.6}$$

For the case $1 < p \leq 2$, inequality (3.6) holds in the reversed direction so in particular for $p = 2$ we have equality.

Remark 3.9. By using Example 3.8 we can state a similar refinement (for $p \geq 2$) and two sided estimate (for $1 < p \leq 2$) of Hardy's dual inequality (1.3) as we did in Remark 3.4 for (1.2).

We give one example more of a new variant refined Hardy type inequality which can be obtained by our technique.

Theorem 3.10. *Let $f \geq 0, \varphi \geq 0$ and φ superquadratic. If $a < 1$, $p \geq 2$ and $F(x) := \int_0^x f(t)dt$. Then*

$$\begin{aligned} &\int_0^\infty \varphi^p \left(\frac{F(x)}{x} \right) dx \\ &+ (1-a) \int_0^\infty \int_t^\infty \left| \left(\frac{1}{1-a} \right) \left(\frac{t}{x} \right)^a \varphi(f(t)) - \frac{1}{x} \int_0^x \varphi(f(t))dt \right|^p x^{a-1} dx t^{-a} dt \\ &\leq \frac{1}{(1-a)^{p-1}(p-1)a} \int_0^\infty \varphi^p(f(t))dt. \end{aligned} \tag{3.7}$$

Proof. By applying Lemmas 2.2 and 3.1, Jensen's inequality, Fubini's theorem and using the fact that $\frac{t}{\varphi(t)}$ is non-increasing, we find that

$$\int_0^\infty \varphi^p \left(\frac{F(x)}{x} \right) dx = \int_0^\infty \left(\varphi \left(\frac{1}{x} \int_0^x f(t)dt \right) \right)^p dx$$

$$\le \int_0^\infty \Big(\frac{1}{x}\int_0^x \varphi\Big(f(t)\Big)dt\Big)^p dx \;=\; \int_0^\infty x^{-p}\left(\int_0^x \varphi(f(t))dt\right)^p dx$$
$$= \int_0^\infty \frac{x^{a-ap-1}}{(1-a)^{p-1}}\int_0^x t^{a(p-1)}\varphi^p(f(t))dtdx$$
$$- \int_0^\infty x^{a-1}(1-a)\int_0^x t^{-a}\left(\left|\left(\frac{1}{1-a}\right)\left(\frac{t}{x}\right)^a \varphi(f(t)) - \frac{1}{x}\int_0^x \varphi(f(t))dt\right|\right)^p dtdx$$
$$\le \frac{1}{(1-a)^{p-1}}\int_0^\infty t^{a(p-1)}\varphi^p(f(t))\int_t^\infty x^{a-ap-1}dxdt$$
$$- \int_0^\infty\int_t^\infty x^{a-1}(1-a)\left|\left(\frac{1}{1-a}\right)\left(\frac{t}{x}\right)^a \varphi(f(t)) - \frac{1}{x}\int_0^x \varphi(f(t))dt\right|^p dxt^{-a}dt$$
$$\le \frac{1}{(1-a)^{p-1}(ap-a)}\int_0^\infty \varphi^p(f(t))dt$$
$$- (1-a)\int_0^\infty\int_t^\infty \left|\left(\frac{1}{1-a}\right)\left(\frac{t}{x}\right)^a \varphi(f(t)) - \frac{1}{x}\int_0^x \varphi(f(t))dt\right|^p x^{a-1}dxt^{-a}dt.$$

□

Corollary 3.11. *Let $f \ge 0$, $\varphi \ge 0$ and φ superquadratic. If $p \ge 2$ and $F(x) := \int_0^x f(t)dt$, then*

$$\int_0^\infty \varphi^p\left(\frac{F(x)}{x}\right)dx$$
$$+ \frac{p-1}{p}\int_0^\infty\int_t^\infty \left|\left(\frac{p}{p-1}\right)\left(\frac{t}{x}\right)^{1/p}\varphi(f(t)) - \frac{1}{x}\int_0^x \varphi(f(t))dt\right|^p x^{1/p-1}dxt^{-1/p}dt$$
$$\le \left(\frac{p}{p-1}\right)^p \int_0^\infty \varphi^p(f(t))dt. \tag{3.8}$$

Proof. This follows from Theorem 3.10 by setting $a = \frac{1}{p}$. □

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James Adedayo Oguntuase
Department of Mathematics
Federal University of Agriculture
P.M.B. 2240
Abeokuta, Ogun State
Nigeria

E-mail: oguntuase@yahoo.com

Olanrewaju Olabiyi Fabelurin
Department of Mathematics
Obafemi Awolowo University
Ile-Ife, Osun State
Nigeria

E-mail: fabelur@yahoo.com

Lars-Erik Persson
Department of Mathematics
Luleå University of Technology
SE-971 87 Luleå
Sweden

and

UiT The Arctic University of Norway
P.O. Box 385
N-8505 Narvik
Norway

E-mail: larserik@ltu.se

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