

Partial representation of partial twisted smash products

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Abstract. In this paper we first give the sufficient conditions under which a partial twisted smash product algebra and the usual tensor product coalgebra become a bialgebra. Furthermore, we introduce the notion of partial representation of partial twisted smash products and explore its relationship with partial actions of Hopf algebras. Finally, we give the conditions for the partial twisted smash products to be Frobenius.

1. Introduction

Partial group actions were considered first by EXEL in the context of operator algebras and they turned out to be a powerful tool in the study of C^* -algebras generated by partial isometries on a Hilbert space in [9]. A treatment from a purely algebraic point of view was given recently in [6], [7], [8]. In particular, the algebraic study of partial actions and partial representations was initiated in [7] and [8], motivating investigations in diverse directions. Now, the results are formulated in a purely algebraic way independent of the C^* -algebraic techniques which originated them.

The concepts of partial actions and partial coactions of Hopf algebras on algebras were introduced by CAENEPEEL and JANSSEN in [5], in which they put the Galois theory for partial group actions on rings into a broader context, namely, the partial entwining structures. In particular, partial actions of a group G determine partial actions of the group algebra kG in a natural way. Further developments in the theory of partial Hopf actions were done by LOMP in [11].

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ALVES and BATISTA extended several results from the theory of partial group actions to the Hopf algebra setting, they constructed a Morita context relating the fixed point subalgebra for partial actions of finite dimensional Hopf algebras, and constructed the partial smash product in [1]. Later, they constructed a Morita context between the partial smash product and the smash product related to the enveloping action, defined partial representations of Hopf algebras and showed some results relating partial actions and partial representations in [2]. Furthermore, they proved a dual version of the globalization theorem: every partial coaction of a Hopf algebra admits an enveloping coaction. They explored some consequences of globalization theorems in order to present versions of the duality theorems of Cohen–Montgomery and Blattner–Montgomery for partial Hopf actions in [3]. The authors generalized the above results to partial twisted smash products in [10]. Recently, they introduced partial representations of Hopf algebras and gave the paradigmatic examples of them, namely, the partial representation defined from a partial action and the partial representation related to the partial smash product in [4]. In this paper, we mainly discuss the partial representation, Frobenius properties of partial twisted smash products in the sense of [10]. The results in [4], [5] and [13] are slightly generalized and more properties are given.

This paper is organized as follows. In Section 2, we give the sufficient conditions under which a partial twisted smash product algebra and the usual tensor product coalgebra become a bialgebra. In Section 3, we introduce the notion of partial representation of partial twisted smash products and explore its relationship with partial actions of Hopf algebras. In Section 4, we give the conditions for the partial twisted smash products to be Frobenius.

2. Partial twisted smash products

Definition 2.1 ([5]). A left partial action of a Hopf algebra H on a unital algebra A is a linear map

$$\rightharpoonup: H \otimes A \rightarrow A, h \otimes a \mapsto h \rightharpoonup a$$

such that

$$(PLA1) \quad 1_H \rightharpoonup a = a,$$

$$(PLA2) \quad h \rightharpoonup (ab) = (h_{(1)} \rightharpoonup a)(h_{(2)} \rightharpoonup b),$$

$$(PLA3) \quad h \rightharpoonup (g \rightharpoonup a) = (h_{(1)} \rightharpoonup 1_A)(h_{(2)}g \rightharpoonup a).$$

The algebra A , on which H acts partially is called a partial left H module algebra.

Definition 2.2 ([10]). A right partial action of a Hopf algebra H on a unital algebra A is a linear map

$$\leftarrow: A \otimes H \rightarrow A, a \otimes h \mapsto a \leftarrow h$$

such that

$$(PRA1) \quad a \leftarrow 1_H = a,$$

$$(PRA2) \quad (ab) \leftarrow h = (a \leftarrow h_{(1)})(b \leftarrow h_{(2)}),$$

$$(PRA3) \quad (a \leftarrow g) \leftarrow h = (1_A \leftarrow h_{(1)})(a \leftarrow gh_{(2)}).$$

The algebra A , on which H acts partially is called a partial right H module algebra.

Definition 2.3 ([10]). Let H be a Hopf algebra and A an algebra. A is called a partial H -bimodule algebra if the following conditions hold:

- (i) A is not only a partial left H -module algebra with the partial left module action \rightarrow but also a partial right H -module algebra with the partial right module action \leftarrow .
- (ii) These two partial module structure maps satisfy the compatibility condition, i.e., $(h \rightarrow a) \leftarrow g = h \rightarrow (a \leftarrow g)$ for all $a \in A$ and $h, g \in H$.

Recall from [10] that to a Hopf algebra H with an antipode S and a partial H -bimodule algebra A one can associate a unital algebra, the partial twisted smash product of A by H . We first propose a multiplication on the vector space $A \otimes H$:

$$(a \otimes h)(b \otimes g) = a(h_{(1)} \rightarrow b \leftarrow S(h_{(3)})) \otimes h_{(2)}g,$$

for all $a, c \in A$ and $g, h \in H$. It is obvious that the multiplication is associative. In order to make it to be a unital algebra, we project onto the unital subalgebra

$$\underline{A\#H} = (A \otimes H)(1_A \otimes 1_H).$$

Then we can deduce directly the form and the properties of typical elements of this algebra

$$\underline{a\#h} = (a\#h)(1_A \otimes 1_H) = a(h_{(1)} \rightarrow 1_A \leftarrow S(h_{(3)})) \otimes h_{(2)},$$

and finally verify that the product among typical elements satisfy

$$(\underline{a\#h})(\underline{b\#g}) = \underline{a(h_{(1)} \rightarrow b \leftarrow S(h_{(3)}))\#h_{(2)}g}, \quad (2.1)$$

for all $h, g \in H$ and $a, b \in A$.

From the above definitions, we have

Proposition 2.4. *With the notations as above, $A\#H$ is an associative algebra with a multiplication given by (2.1) and with the unit $\underline{1_A\#1_H}$, and call it by a partial twisted smash product, where 1_A is the unit of A .*

PROOF. Similar to [1]. \square

Definition 2.5. Let H and A be Hopf algebras. A skew pair is a triple (A, H, σ) endowed with a k -linear map $\sigma : A \otimes H \rightarrow k$ such that the following conditions are satisfied.

- (1) $\sigma(ab, h) = \sigma(a, h_{(1)})\sigma(b, h_{(2)})$,
- (2) $\sigma(a_{(1)}, h)\sigma(a_{(2)}, g) = \sigma(1_A, g_{(1)})\sigma(a, g_{(2)}h) = \sigma(1_A, h_{(1)})\sigma(a, gh_{(2)})$,
- (3) $\sigma(a, 1) = \varepsilon(a)$,

for all $h, g \in H$ and $a, b \in A$.

Example 2.6. Let H be a Hopf algebra with a bijective antipode S and A a Hopf algebra. Suppose that (A, H, σ) is a skew pair, then we can define two actions of H and A : for any $h \in H, b \in A$,

$$\begin{aligned} h \rightharpoonup b &= b_{(2)}\sigma(b_{(1)}, h), \\ b \leftharpoonup h &= b_{(1)}\sigma(b_{(2)}, (S^{-1})^2(h)). \end{aligned}$$

It follows that

$$\begin{aligned} \underline{a\#h} &= a(h_{(1)} \rightharpoonup 1_A \leftharpoonup S(h_{(3)})) \otimes h_{(2)} \\ &= \sigma(1_A, h_{(1)})a \otimes h_{(2)}\sigma(1_A, S^{-1}(h_{(3)})). \end{aligned}$$

It is not hard to verify that $(A, \rightharpoonup, \leftharpoonup)$ is a partial H -bimodule algebra and the multiplication of $A\#H$ is

$$(\underline{a\#h})(\underline{b\#k}) = \sigma(b_{(1)}, h_{(1)})\underline{ab_{(2)}\#h_{(2)}k}\sigma(b_{(3)}, S^{-1}(h_{(3)})),$$

for all $h, k \in H$ and $a, b \in A$. Then $A\#H$ is a partial twisted smash product.

Example 2.7. Recall the definition of H_4 . As a k -algebra, H_4 is generated by two symbols c and x , which satisfies the relations $c^2 = 1$, $x^2 = 0$ and $xc + cx = 0$. The coalgebra structure on H_4 is determined by

$$\Delta(c) = c \otimes c, \quad \Delta(x) = x \otimes 1 + c \otimes x, \quad \varepsilon(c) = 1, \quad \varepsilon(x) = 0.$$

Consequently, H_4 has the basis 1 (identity), c, x, cx , we now consider the dual H_4^* of H_4 . We have $H_4 \cong H_4^*$ (as Hopf algebras) via

$$1 \mapsto 1^* + c^*, \quad c \mapsto 1^* + c^*, \quad x \mapsto x^* + (cx)^*, \quad cx \mapsto x^* - (cx)^*,$$

here $\{1^*, c^*, x^*, (cx)^*\}$ denote the dual basis of $\{1, c, x, cx\}$, then we let $T = 1^* + c^*$, $P = x^* + (cx)^*$, $TP = x^* - (cx)^*$, we get another basis $\{1, T, P, TP\}$ of H_4^* . Recall from [5] that let A be the subalgebra $k[x]$ of H_4 , it is shown that A is a right partial H_4 -comodule algebra with the coaction $\rho^r(1) = \frac{1}{2}(1 \otimes 1 + 1 \otimes c + 1 \otimes cx)$, $\rho^r(x) = \frac{1}{2}(x \otimes 1 + x \otimes c + x \otimes cx)$. By similar way we can define A as a left partial H_4 -comodule algebra with the coaction $\rho^l(1) = \frac{1}{2}(1 \otimes 1 + c \otimes 1 + cx \otimes 1)$, $\rho^l(x) = \frac{1}{2}(1 \otimes x + c \otimes x + cx \otimes x)$, and it can be easily checked that A is a partial H_4 -bicomodule algebra, then A is a partial H_4^* -bimodule algebra via $f \rightharpoonup a = \sum \langle f, a_{[1]} \rangle a_{[0]}$ and $a \leftharpoonup g = \langle g, a_{[-1]} \rangle a_{[0]}$, for $a \in A, f, g \in H^*$. Then $A \# H_4^*$ is a partial twisted smash product.

Theorem 2.8. *Let H be a Hopf algebra with an antipode S , A be a bialgebra and a partial H -bimodule algebra.*

(1) *The partial twisted smash product algebra $A \# H$ equipped with the tensor product coalgebra structure makes $A \# H$ into a bialgebra, if the following conditions hold:*

- (a) $\varepsilon_A(h_{(1)} \rightharpoonup a \leftharpoonup S(h_{(2)})) = \varepsilon_A(a)\varepsilon_H(h)$,
- (b) $\Delta_A(h_{(1)} \rightharpoonup a \leftharpoonup S(h_{(2)})) = (h_{(1)} \rightharpoonup a_{(1)} \leftharpoonup S(h_{(2)})) \otimes (h_{(3)} \rightharpoonup a_{(2)} \leftharpoonup S(h_{(4)}))$,
- (c) $(h_{(1)} \rightharpoonup a) \otimes h_{(2)} = (h_{(2)} \rightharpoonup a) \otimes h_{(1)}$,
- (d) $(a \leftharpoonup S(h_{(1)})) \otimes h_{(2)} = (a \leftharpoonup S(h_{(2)})) \otimes h_{(1)}$.

(2) *Furthermore, if A is a Hopf algebra, and we assume that the following formula holds:*

$$h_{(1)} \rightharpoonup 1_A \leftharpoonup S(h_{(2)}) = \varepsilon_H(h)1_A. \quad (2.2)$$

then $A \# H$ is also a Hopf algebra with antipode $S_{A \# H}$ defined by:

$$S_{A \# H}(a \# h) = (1 \# S(h))(S_A(a) \# 1).$$

PROOF. (1) First we verify $\Delta_{A \# H}$ is an algebra morphism with respect to the multiplication on $A \# H$ and the tensor product coalgebra structure on $A \# H$,

$$\begin{aligned} & \Delta_{A \# H}((a \# h)(b \# l)) \\ &= \Delta_{A \# H}(a_{(1)}(h_{(1)} \rightharpoonup b \leftharpoonup S h_{(3)}) \# h_{(2)} l) \\ &= a_{(1)}(h_{(1)} \rightharpoonup b \leftharpoonup S h_{(4)})_{(1)} \# h_{(2)} l_{(1)} \otimes a_{(2)}(h_{(1)} \rightharpoonup b \leftharpoonup S h_{(4)})_{(2)} \# h_{(3)} l_{(2)} \\ &\stackrel{(d)}{=} a_{(1)}(h_{(1)} \rightharpoonup b \leftharpoonup S h_{(3)})_{(1)} \# h_{(2)} l_{(1)} \otimes a_{(2)}(h_{(1)} \rightharpoonup b \leftharpoonup S h_{(3)})_{(2)} \# h_{(4)} l_{(2)} \\ &\stackrel{(d)}{=} a_{(1)}(h_{(1)} \rightharpoonup b \leftharpoonup S h_{(2)})_{(1)} \# h_{(3)} l_{(1)} \otimes a_{(2)}(h_{(1)} \rightharpoonup b \leftharpoonup S h_{(2)})_{(2)} \# h_{(4)} l_{(2)} \end{aligned}$$

$$\begin{aligned}
&\stackrel{(b)}{=} \underline{a_1(h_{(1)} \rightarrow b \leftarrow Sh_{(2)})\#h_{(5)}l_{(1)}} \otimes \underline{a_{(2)}(h_{(3)} \rightarrow b \leftarrow Sh_{(4)})\#h_{(6)}l_{(2)}} \\
&\stackrel{(d)}{=} \underline{a_1(h_{(1)} \rightarrow b \leftarrow Sh_{(2)})\#h_{(4)}l_{(1)}} \otimes \underline{a_{(2)}(h_{(3)} \rightarrow b \leftarrow Sh_{(6)})\#h_{(5)}l_{(2)}} \\
&\stackrel{(c)}{=} \underline{a_1(h_{(1)} \rightarrow b \leftarrow Sh_{(2)})\#h_{(3)}l_{(1)}} \otimes \underline{a_{(2)}(h_{(4)} \rightarrow b \leftarrow Sh_{(6)})\#h_{(5)}l_{(2)}} \\
&\stackrel{(d)}{=} \underline{a_1(h_{(1)} \rightarrow b \leftarrow Sh_{(3)})\#h_{(2)}l_{(1)}} \otimes \underline{a_{(2)}(h_{(4)} \rightarrow b \leftarrow Sh_{(6)})\#h_{(5)}l_{(2)}} \\
&= \Delta(\underline{a\#h})\Delta(\underline{b\#l}).
\end{aligned}$$

Next, we verify $\varepsilon_{\underline{A\#H}}$ is also an algebra morphism. It is easy to verify

$$\varepsilon_{\underline{A\#H}}(\underline{a\#h}) = \varepsilon_A(a)\varepsilon_H(h).$$

In fact,

$$\begin{aligned}
\varepsilon_{\underline{A\#H}}(\underline{a\#h}) &= \varepsilon_{\underline{A\#H}}(\underline{a(h_{(1)} \rightarrow 1_A \leftarrow S(h_{(3)})) \otimes h_{(2)}}) \\
&= \varepsilon_A(\underline{a(h_{(1)} \rightarrow 1_A \leftarrow S(h_{(3)}))})\varepsilon_H(h_{(2)}) \\
&\stackrel{(a)}{=} \varepsilon_A(a)\varepsilon_H(h), \\
\varepsilon_{\underline{A\#H}}(\underline{(a\#h)(b\#l)}) &= \varepsilon_{\underline{A\#H}}(\underline{a(h_{(1)} \rightarrow b \leftarrow S(h_{(3)}))\#h_{(2)}l}) \\
&= \varepsilon_A(\underline{a(h_{(1)} \rightarrow b \leftarrow S(h_{(3)}))})\varepsilon_H(h_{(2)}l) \\
&\stackrel{(a)}{=} \varepsilon_A(a)\varepsilon_H(h)\varepsilon_A(b)\varepsilon_H(l) \\
&= \varepsilon_{\underline{A\#H}}(\underline{a\#h})\varepsilon_{\underline{A\#H}}(\underline{b\#l}).
\end{aligned}$$

Hence, $\underline{A\#H}$ is a bialgebra.

(2) For any $a \in A$ and $h \in H$, we have

$$\begin{aligned}
(\underline{S_{A\#H}} * id)(\underline{a\#h}) &= (\underline{1_A\#Sh_{(1)}})(\underline{S_A(a)\#1})(\underline{a_{(2)}\#h_{(2)}}) \\
&= (\underline{1_A\#Sh_{(1)}})(\underline{S(a_1)a_2\#h_{(2)}}) \\
&= \varepsilon_A(a)(\underline{1_A\#Sh_{(1)}})(\underline{1_A\#h_{(2)}}) \\
&= \varepsilon_A(a)(\underline{Sh_{(3)} \rightarrow 1_A \leftarrow S(Sh_{(1)})\#(Sh_{(2)})h_4}) \\
&\stackrel{(c)}{=} \varepsilon_A(a)(\underline{Sh_{(2)} \rightarrow 1_A \leftarrow S(Sh_{(1)})\#(Sh_{(3)})h_{(4)}}) \\
&= \varepsilon_A(a)(\underline{(Sh)_1 \rightarrow 1_A \leftarrow S(Sh)_2})\#1_H \\
&\stackrel{(2.2)}{=} \varepsilon_A(a)\varepsilon_H(h)\underline{1_A\#1_H}.
\end{aligned}$$

In the similar way, one can check that

$$(id * \underline{S_{A\#H}})(\underline{a\#h}) = \varepsilon_A(a)\varepsilon_H(h)\underline{1_A\#1_H}.$$

Therefore, $\underline{A\#H}$ is a Hopf algebra with antipode

$$S_{\underline{A\#H}} = (\underline{1\#S(h)})(\underline{S_A(a)\#1}). \quad \square$$

Remark 2.9. In Theorem 2.8, the conditions (b), (c) and (d) of the item (1) can be easily verified for the case where H^* is cocommutative (therefore, H is commutative). If a Hopf algebra H^* satisfies these three conditions, then H^* is not necessarily cocommutative.

A concrete counterexample is presented as follows.

With the notations as above, we have shown that $A\#H_4^*$ is a partial twisted smash product. So we only consider the element P of H_4^* and check the condition (b) as follows:

$$\begin{aligned} \Delta_A\left(\sum P_1 \rightarrow x \leftarrow S^*(P_2)\right) &= \Delta_A(P \rightarrow x \leftarrow S^*(1) + T \rightarrow x \leftarrow S^*(P)) \\ &= \Delta_A\left(\langle P, \frac{1}{2}(1+c+cx) \rangle x \langle 1, \frac{1}{2}(1+c+cx) \rangle + \langle T, \frac{1}{2}(1+c+cx) \rangle \right. \\ &\quad \left. \langle P, \frac{1}{2}(1+c+cx) \rangle x\right) \\ &= \langle P, \frac{1}{2}(1+c+cx) \rangle (x \otimes 1 + 1 \otimes x) + \langle T, \frac{1}{2}(1+c+cx) \rangle \\ &\quad \langle P, \frac{1}{2}(1+c+cx) \rangle (x \otimes 1 + 1 \otimes x) \\ &= \langle P, \frac{1}{2}(1+c+cx) \rangle (x \otimes 1 + 1 \otimes x), \end{aligned}$$

and

$$\begin{aligned} &\sum (P_1 \rightarrow x_1 \leftarrow S^*(P_2)) \otimes (P_3 \rightarrow x_2 \leftarrow S^*(P_4)) \\ &= \sum (P_1 \rightarrow x \leftarrow S^*(P_2)) \otimes (P_3 \rightarrow 1 \leftarrow S^*(P_4)) \\ &\quad + \sum (P_1 \rightarrow 1 \leftarrow S^*(P_2)) \otimes (P_3 \rightarrow x \leftarrow S^*(P_4)) \\ &= \sum (P \rightarrow x \leftarrow S^*(1)) \otimes (1 \rightarrow 1 \leftarrow S^*(1)) \\ &\quad + \sum (P \rightarrow 1 \leftarrow S^*(1)) \otimes (1 \rightarrow x \leftarrow S^*(1)) \\ &\quad + \sum (T \rightarrow x \leftarrow S^*(T)) \otimes (P \rightarrow 1 \leftarrow S^*(1)) \\ &\quad + \sum (T \rightarrow 1 \leftarrow S^*(T)) \otimes (P \rightarrow x \leftarrow S^*(1)) \\ &\quad + \sum (T \rightarrow x \leftarrow S^*(T)) \otimes (T \rightarrow 1 \leftarrow S^*(P)) \end{aligned}$$

$$\begin{aligned}
& + \sum (T \rightarrow 1 \leftarrow S^*(T)) \otimes (T \rightarrow x \leftarrow S^*(P)) \\
& + \sum (T \rightarrow x \leftarrow S^*(P)) \otimes (1 \rightarrow 1 \leftarrow S^*(1)) \\
& + \sum (T \rightarrow 1 \leftarrow S^*(P)) \otimes (1 \rightarrow x \leftarrow S^*(1)) \\
& = \langle P, \frac{1}{2}(1 + c + cx) \rangle (x \otimes 1 + 1 \otimes x).
\end{aligned}$$

Direct computations show that conditions (a), (c) and (d) of Theorem 3.5 hold.

3. Partial representations

A first definition of partial representations of Hopf algebras was proposed in [2], requiring only axioms (PR1) and (PR2) below. This was mainly motivated by the constructions done in [5] for partial H -module algebras A , originated from partial entwining structures. In the case of partial representations of Hopf algebras, the authors have found a richer and more complex structure in [4], and we have introduced partial representations of the partial twisted smash product $A\#H$ in [10], so we would like to define a richer and more complex structure of partial representation of the partial twisted smash product $A\#H$. In order to carry out the work of section 3, we assume that

$$(a \leftarrow S(h_{(1)})) \otimes h_{(2)} = (a \leftarrow S(h_{(2)})) \otimes h_{(1)}, \quad (3.1)$$

$$(a \leftarrow S(h_{(1)})) \otimes h_{(2)} = (S(h_{(2)}) \rightarrow a) \otimes h_{(1)} \quad (3.2)$$

and the components $h_{(1)}$ and $h_{(2)}$ can be switched independently of the k 's that they are multiplying, for any $a \in A$ and $h, k \in H$.

Definition 3.1 ([4]). Let H be a Hopf algebra. A partial representation of H on a unital algebra B is a linear map

$$\pi : H \rightarrow B, \quad h \mapsto \pi(h)$$

such that

$$(PR1) \quad \pi(1_H) = 1_B;$$

$$(PR2) \quad \pi(h)\pi(k_{(1)})\pi(S(k_{(2)})) = \pi(hk_{(1)})\pi(S(k_{(2)}));$$

$$(PR3) \quad \pi(h_{(1)})\pi(S(h_{(2)}))\pi(k) = \pi(h_{(1)})\pi(S(h_{(2)})k);$$

$$(PR4) \quad \pi(h)\pi(S(k_{(1)}))\pi(k_{(2)}) = \pi(hS(k_{(1)}))\pi(k_{(2)});$$

$$(PR5) \quad \pi(S(h_{(1)}))\pi(h_{(2)})\pi(k) = \pi(S(h_{(1)}))\pi(h_{(2)}k).$$

If (B, π) and (B', π') are two partial representations of H , then we say that an algebra morphism $f : B \rightarrow B'$ is a morphism of partial representations if $\pi' = f \circ \pi$.

The category whose objects are partial representations of H and whose morphisms are morphisms of partial representations is denoted by ${}_{\mathbf{H}}\mathbf{ParRep}_{\mathbf{H}}$.

Lemma 3.2. *Let H be a Hopf algebra with antipode S and let A be a partial H -bimodule algebra. Then*

$$\begin{aligned} (1) \quad & h \rightarrow ab \leftarrow g = (h_{(1)} \rightarrow a \leftarrow g_{(1)})(h_{(2)} \rightarrow b \leftarrow g_{(2)}), \\ (2) \quad & k \rightarrow (h \rightarrow a \leftarrow g) \leftarrow l = (k_{(1)} \rightarrow 1_A \leftarrow l_{(1)})(k_{(2)} h \rightarrow a \leftarrow gl_{(2)}). \end{aligned}$$

PROOF. Straightforward. \square

We say that the partial H -bimodule structure is symmetric if, in addition, it satisfies

$$(PA4) \quad k \rightarrow (h \rightarrow a \leftarrow g) \leftarrow l = (k_{(1)} h \rightarrow a \leftarrow gl_{(1)})(k_{(2)} \rightarrow 1_A \leftarrow l_{(2)}),$$

for any $h, l, k, g \in H$ and $a \in A$.

If $(A, \rightarrow, \leftarrow)$ and $(B, \rightarrow, \leftarrow)$ are two partial H -bimodule algebras, then a morphism of partial H -module algebras is an algebra map $f : A \rightarrow B$ such that $f(h \rightarrow a \leftarrow g) = h \rightarrow f(a) \leftarrow g$. The category of all symmetric partial H -module algebras and the morphisms of partial H -module algebras between them is denoted as ${}_{\mathbf{H}}\mathbf{ParAct}_{\mathbf{H}}$.

Proposition 3.3. *Let A be a symmetric partial H -bimodule algebra, and let $B = \text{End}(A)$. Define*

$$\pi : H \rightarrow B, \quad h \mapsto \pi(h)$$

given by $\pi(h)(a) = h_{(1)} \rightarrow a \leftarrow S(h_{(2)})$. Then π satisfies the conditions (PR1)–(PR5).

PROOF. Since $1_H \rightarrow a \leftarrow 1_H = a$, for all $a \in A$, implies $\pi(1_H) = 1_B$, so (PR1) is satisfied. With respect to (PR2), and do the following calculation:

$$\begin{aligned} & \pi(h)\pi(k_{(1)})\pi(S(k_{(2)}))(a) \\ &= h_{(1)} \rightarrow [k_{(1)} \rightarrow (S(k_{(4)}) \rightarrow a \leftarrow S^2(k_{(3)})) \leftarrow S(k_{(2)})] \leftarrow S(h_{(2)}) \\ &= (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(4)}))[h_{(2)}k_{(1)} \rightarrow (S(k_{(4)}) \rightarrow a \leftarrow S^2(k_{(3)})) \leftarrow S(k_{(2)})S(h_{(3)})] \\ &= (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(6)}))(h_{(2)}k_{(1)} \rightarrow 1_A \leftarrow S(k_{(4)})S(h_{(5)})) \end{aligned}$$

$$\begin{aligned}
& [h_{(3)}k_{(2)}S(k_{(6)}) \rightarrow a \leftarrow S^2(k_{(5)})S(k_{(3)})S(h_{(4)})] \\
\stackrel{(3.1)}{=} & (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(6)}))(h_{(2)}k_{(1)} \rightarrow 1_A \leftarrow S(k_{(5)})S(h_{(5)})) \\
& [h_{(3)}k_{(2)}S(k_{(6)}) \rightarrow a \leftarrow S^2(k_{(4)})S(k_{(3)})S(h_{(4)})] \\
= & (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(6)}))(h_{(2)}k_{(1)} \rightarrow 1_A \leftarrow S(k_{(3)})S(h_{(5)})) \\
& [h_{(3)}k_{(2)}S(k_{(4)}) \rightarrow a \leftarrow S(h_{(4)})] \\
\stackrel{(3.2)}{=} & (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(6)}))(h_{(2)}k_{(1)} \rightarrow 1_A \leftarrow S(k_{(4)})S(h_{(5)})) \\
& [h_{(3)}k_{(2)}S(k_{(3)}) \rightarrow a \leftarrow S(h_{(4)})] \\
= & (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(6)}))(h_{(2)}k_{(1)} \rightarrow 1_A \leftarrow S(h_{(5)}k_{(2)}))[h_{(3)} \rightarrow a \leftarrow S(h_{(4)})] \\
= & h_{(1)} \rightarrow ([k_{(1)} \rightarrow 1_A \leftarrow S(k_{(2)})]a) \leftarrow S(h_{(2)}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& \pi(hk_{(1)})\pi(S(k_{(2)}))(a) \\
= & h_{(1)}k_{(1)} \rightarrow [S(k_{(4)}) \rightarrow a \leftarrow S^2(k_{(3)})] \leftarrow S(h_{(2)}k_{(2)}) \\
= & (h_{(1)}k_{(1)} \rightarrow 1_A \leftarrow S(h_{(4)}k_{(4)}))[h_{(2)}k_{(2)}S(k_{(6)}) \rightarrow a \leftarrow S^2(k_{(5)})S(h_{(3)}k_{(3)})] \\
\stackrel{(3.1)}{=} & (h_{(1)}k_{(1)} \rightarrow 1_A \leftarrow S(h_{(4)}k_{(5)}))[h_{(2)}k_{(2)}S(k_{(6)}) \rightarrow a \leftarrow S^2(k_{(4)})S(h_{(3)}k_{(3)})] \\
= & (h_{(1)}k_{(1)} \rightarrow 1_A \leftarrow S(h_{(4)}k_{(3)}))[h_{(2)}k_{(2)}S(k_{(4)}) \rightarrow a \leftarrow S(h_{(3)})] \\
\stackrel{(3.2)}{=} & (h_{(1)}k_{(1)} \rightarrow 1_A \leftarrow S(h_{(4)}k_{(4)}))[h_{(2)}k_{(2)}S(k_{(3)}) \rightarrow a \leftarrow S(h_{(3)})] \\
= & (h_{(1)}k_{(1)} \rightarrow 1_A \leftarrow S(h_{(4)}k_{(2)}))[h_{(2)} \rightarrow a \leftarrow S(h_{(3)})] \\
= & (h_{(1)}k_{(1)} \rightarrow 1_A \leftarrow S(h_{(6)}k_{(2)}))(h_{(2)} \rightarrow 1_A \leftarrow S(h_{(5)}))[h_{(3)} \rightarrow a \leftarrow S(h_{(4)})] \\
= & (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(6)}))(h_{(2)}k_{(1)} \rightarrow 1_A \leftarrow S(h_{(5)}k_{(2)}))[h_{(3)} \rightarrow a \leftarrow S(h_{(4)})] \\
= & h_{(1)} \rightarrow ([k_{(1)} \rightarrow 1_A \leftarrow S(k_{(2)})]a) \leftarrow S(h_{(2)}).
\end{aligned}$$

For equation (PR3), we have

$$\begin{aligned}
& \pi(h_{(1)})\pi(S(h_{(2)}))\pi(k)(a) \\
= & h_{(1)} \rightarrow [S(h_{(4)}) \rightarrow (k_{(1)} \rightarrow a \leftarrow S(k_{(2)})) \leftarrow S^2(h_{(3)})] \leftarrow S(h_{(2)}) \\
= & (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(4)}))[h_{(2)}S(h_{(6)}) \rightarrow (k_{(1)} \rightarrow a \leftarrow S(k_{(2)})) \leftarrow S^2(h_{(5)})S(h_{(3)})] \\
\stackrel{(3.1)}{=} & (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(5)}))[h_{(2)}S(h_{(6)}) \rightarrow (k_{(1)} \rightarrow a \leftarrow S(k_{(2)})) \leftarrow S^2(h_{(4)})S(h_{(3)})] \\
= & (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(3)}))[h_{(2)}S(h_{(4)}) \rightarrow (k_{(1)} \rightarrow a \leftarrow S(k_{(2)}))] \\
\stackrel{(3.2)}{=} & (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(4)}))[h_{(2)}S(h_{(3)}) \rightarrow (k_{(1)} \rightarrow a \leftarrow S(k_{(2)}))]
\end{aligned}$$

$$= (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(2)}))(k_{(1)} \rightarrow a \leftarrow S(k_{(2)}))$$

and

$$\begin{aligned} & \pi(h_{(1)})\pi(S(h_{(2)})k)(a) \\ &= h_{(1)} \rightarrow [S(h_{(4)})k_{(1)} \rightarrow a \leftarrow S(k_2)S^2(h_{(3)})] \leftarrow S(h_{(2)}) \\ &= (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(4)}))[h_{(2)}S(h_{(6)})k_{(1)} \rightarrow a \leftarrow S(k_2)S^2(h_{(5)})S(h_{(3)})] \\ &\stackrel{(3.1)}{=} (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(5)}))[h_{(2)}S(h_{(6)})k_{(1)} \rightarrow a \leftarrow S(k_2)S^2(h_{(4)})S(h_{(3)})] \\ &= (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(3)}))[h_{(2)}S(h_{(4)})k_{(1)} \rightarrow a \leftarrow S(k_2)] \\ &\stackrel{(3.2)}{=} (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(2)}))[h_{(3)}S(h_{(4)})k_{(1)} \rightarrow a \leftarrow S(k_2)] \\ &= (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(2)}))[k_{(1)} \rightarrow a \leftarrow S(k_2)]. \end{aligned}$$

For equation (PR4), we have

$$\begin{aligned} & \pi(h)\pi(S(k_{(1)}))\pi(k_{(2)})(a) \\ &= h_{(1)} \rightarrow [S(k_{(2)}) \rightarrow (k_{(3)} \rightarrow a \leftarrow S(k_{(4)})) \leftarrow S^2(k_{(1)})] \leftarrow S(h_{(2)}) \\ &= [h_{(1)}S(k_{(2)}) \rightarrow (k_{(3)} \rightarrow a \leftarrow S(k_{(4)})) \leftarrow S^2(k_{(1)})S(h_{(4)})](h_{(2)} \rightarrow 1_A \leftarrow S(h_{(3)})) \\ &= [h_{(1)}S(k_{(4)})k_{(5)} \rightarrow a \leftarrow S(k_{(6)})S^2(k_{(1)})S(h_{(6)})] \\ &\quad (h_{(2)}S(k_{(3)}) \rightarrow 1_A \leftarrow S^2(k_{(2)})S(h_{(5)}))(h_{(3)} \rightarrow 1_A \leftarrow S(h_{(4)})) \\ &= [h_{(1)} \rightarrow a \leftarrow S(h_{(3)})](h_{(2)}S(k_{(3)}) \rightarrow 1_A \leftarrow S(h_{(2)}S(k_{(2)}))) \end{aligned}$$

and

$$\begin{aligned} & \pi(hS(k_{(1)}))\pi(k_{(2)})(a) \\ &= h_{(1)}S(k_{(2)}) \rightarrow [k_{(3)} \rightarrow a \leftarrow S(k_{(4)})] \leftarrow S(h_{(2)}S(k_{(1)})) \\ &= [h_{(1)}S(k_{(4)})k_{(5)} \rightarrow a \leftarrow S(k_{(6)})S(h_{(3)})S(k_{(1)})](h_{(2)}S(k_{(3)}) \rightarrow 1_A \leftarrow S(h_{(2)}S(k_{(2)}))) \\ &= [h_{(1)} \rightarrow a \leftarrow S(k_{(4)})S(h_{(3)})S(k_{(1)})](h_{(2)}S(k_{(3)}) \rightarrow 1_A \leftarrow S(h_{(2)}S(k_{(2)}))) \\ &\stackrel{(3.1)}{=} [h_{(1)} \rightarrow a \leftarrow S(k_{(2)})S(h_{(3)})S(k_{(1)})](h_{(2)}S(k_{(3)}) \rightarrow 1_A \leftarrow S(h_{(2)}S(k_{(4)}))) \\ &= [h_{(1)} \rightarrow a \leftarrow S(h_{(3)})](h_{(2)}S(k_{(3)}) \rightarrow 1_A \leftarrow S(h_{(2)}S(k_{(2)}))). \end{aligned}$$

Finally, for equation (PR5), we have

$$\begin{aligned} & \pi(S(h_{(1)}))\pi(h_{(2)})\pi(k)(a) \\ &= S(h_{(2)}) \rightarrow [h_{(3)} \rightarrow [k_{(1)} \rightarrow a \leftarrow S(k_2)] \leftarrow S(h_{(4)})] \leftarrow S^2(h_{(1)}) \end{aligned}$$

$$\begin{aligned}
&= [S(h_{(4)})h_{(5)} \rightarrow [k_{(1)} \rightarrow a \leftarrow S(k_2)] \leftarrow S(h_{(6)})S^2(h_{(1)})](S(h_{(3)}) \rightarrow 1_A \leftarrow S^2(h_{(2)})) \\
&= [k_{(1)} \rightarrow a \leftarrow S(k_2)] \leftarrow S(h_{(4)})S^2(h_{(1)})(S(h_{(3)}) \rightarrow 1_A \leftarrow S^2(h_{(2)})) \\
&\stackrel{(3.1)(3.2)}{=} [k_{(1)} \rightarrow a \leftarrow S(k_2)] \leftarrow S(h_{(4)})S^2(h_{(3)})(S(h_{(2)}) \rightarrow 1_A \leftarrow S^2(h_{(1)})) \\
&= (k_{(1)} \rightarrow a \leftarrow S(k_2))(S(h_{(2)}) \rightarrow 1_A \leftarrow S^2(h_{(1)}))
\end{aligned}$$

and

$$\begin{aligned}
&\pi(S(h_{(1)}))\pi(h_{(2)}k)(a) \\
&= S(h_{(2)}) \rightarrow [h_{(3)}k_{(1)} \rightarrow a \leftarrow S(h_{(4)}k)] \leftarrow S^2(h_{(1)}) \\
&= [S(h_{(4)})h_{(5)}k_{(1)} \rightarrow a \leftarrow S(h_{(6)}k)S^2(h_{(2)})](S(h_{(3)}) \rightarrow 1_A \leftarrow S^2(h_{(1)})) \\
&= [k_{(1)} \rightarrow a \leftarrow S(h_{(4)}k)S^2(h_{(2)})](S(h_{(3)}) \rightarrow 1_A \leftarrow S^2(h_{(1)})) \\
&\stackrel{(3.2)}{=} [k_{(1)} \rightarrow a \leftarrow S(h_{(4)}k)S^2(h_{(3)})](S(h_{(2)}) \rightarrow 1_A \leftarrow S^2(h_{(1)})) \\
&= (k_{(1)} \rightarrow a \leftarrow S(k_2))(S(h_{(2)}) \rightarrow 1_A \leftarrow S^2(h_{(1)})).
\end{aligned}$$

Therefore, the partial action of H on A provides an example of a partial representation of H on $\text{End}_k(A)$. \square

Partial twisted smash products give another source of examples of partial representations of a Hopf algebra H .

Definition 3.4. Given a symmetric partial H -bimodule algebra A , the linear map $\pi_0 : H \rightarrow \underline{A\#H}$, given by $\pi_0(h) = 1_A\#h$ is a partial representation of H .

PROOF. First, the item (PR1) is easily obtained, for $\pi_0(1_H) = 1_A\#1_H = 1_{\underline{A\#H}}$. Now, verifying the item (PR2), we have

$$\begin{aligned}
\pi_0(h)\pi_0(k_{(1)})\pi_0(S(k_{(2)})) &= (1_A\#h)(1_A\#k_{(1)})(1_A\#S(k_{(2)})) \\
&= (1_A\#h)(k_{(1)} \rightarrow 1_A \leftarrow S(k_{(3)})\#k_{(2)}S(k_{(4)})) \\
&= (1_A\#h)(k_{(1)} \rightarrow 1_A \leftarrow S(k_{(2)})\#k_{(3)}S(k_{(4)})) \\
&= (1_A\#h)(k_{(1)} \rightarrow 1_A \leftarrow S(k_{(2)})\#1_H) \\
&= h_{(1)} \rightarrow (k_{(1)} \rightarrow 1_A \leftarrow S(k_{(2)})) \leftarrow S(h_{(3)})\#h_{(2)}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\pi_0(hk_{(1)})\pi_0(S(k_{(2)})) &= (1_A\#hk_{(1)})(1_A\#S(k_{(2)})) \\
&= h_{(1)}k_{(1)} \rightarrow 1_A \leftarrow S(h_{(3)}k_{(3)})\#h_{(2)}k_{(2)}S(k_{(4)})
\end{aligned}$$

$$\begin{aligned}
 & \stackrel{(3.1)}{=} h_{(1)}k_{(1)} \rightarrow 1_A \leftarrow S(h_{(3)}k_{(2)})\#h_{(2)}k_{(3)}S(k_{(4)}) \\
 & = h_{(1)}k_{(1)} \rightarrow 1_A \leftarrow S(h_{(3)}k_{(2)})\#h_{(2)} \\
 & = (h_{(1)}k_{(1)} \rightarrow 1_A \leftarrow S(h_{(5)}k_{(2)}))(h_{(2)} \rightarrow 1_A \leftarrow S(h_{(4)}))\#h_{(3)} \\
 & = h_{(1)} \rightarrow (k_{(1)} \rightarrow 1_A \leftarrow S(k_{(2)})) \leftarrow S(h_{(3)})\#h_{(2)}.
 \end{aligned}$$

For the equation (PR3), we have

$$\begin{aligned}
 \pi_0(h_{(1)})\pi_0(S(h_{(2)}))\pi_0(k) & = (1_A\#h_{(1)})(1_A\#S(h_{(2)}))(1_A\#k) \\
 & = (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(3)})\#h_{(2)}S(h_{(4)}))(1_A\#k) \\
 & = (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(2)})\#h_{(3)}S(h_{(4)}))(1_A\#k) \\
 & = (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(2)})\#1_H)(1_A\#k) \\
 & = h_{(1)} \rightarrow 1_A \leftarrow S(h_{(2)})\#k
 \end{aligned}$$

and

$$\begin{aligned}
 \pi_0(h_{(1)})\pi_0(S(h_{(2)}))k & = (1_A\#h_{(1)})(1_A\#S(h_{(2)}k)) \\
 & = h_{(1)} \rightarrow 1_A \leftarrow S(h_{(3)})\#h_{(2)}S(h_{(4)})k \\
 & = h_{(1)} \rightarrow 1_A \leftarrow S(h_{(2)})\#h_{(3)}S(h_{(4)})k \\
 & = h_{(1)} \rightarrow 1_A \leftarrow S(h_{(2)})\#k.
 \end{aligned}$$

For (PR4), we have

$$\begin{aligned}
 \pi_0(h)\pi(S(k_{(1)}))\pi_0(k_{(2)}) & = (1_A\#h)(1_A\#S(k_{(1)}))(1_A\#k_{(2)}) \\
 & = (1_A\#h)(S(k_{(3)}) \rightarrow 1_A \leftarrow S^2(k_{(1)})\#S(k_{(2)})k_{(4)}) \\
 & = h_{(1)} \rightarrow (S(k_{(3)}) \rightarrow 1_A \leftarrow S^2(k_{(1)})) \leftarrow S(h_{(3)})\#h_{(2)}S(k_{(2)})k_{(4)} \\
 & = (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(5)}))(h_{(2)}S(k_{(3)}) \rightarrow 1_A \leftarrow S^2(k_{(1)})S(h_{(4)}))\#h_{(3)}S(k_{(2)})k_{(4)}
 \end{aligned}$$

and

$$\begin{aligned}
 \pi_0(hS(k_{(1)}))\pi_0(k_{(2)}) & = (1_A\#hS(k_{(1)}))(1_A\#k_{(2)}) \\
 & = h_{(1)}S(k_{(3)}) \rightarrow 1_A \leftarrow S(h_{(3)}S(k_{(1)}))\#h_{(2)}S(k_{(2)})k_{(4)} \\
 & = (h_{(1)}S(k_{(5)}) \rightarrow 1_A \leftarrow S(h_{(5)}S(k_{(1)})))(h_{(2)}S(k_{(4)})k_{(6)} \rightarrow 1_A \leftarrow S(h_{(4)}S(k_{(2)})k_{(8)})) \\
 & \quad \#h_{(3)}S(k_{(3)})k_{(7)} \\
 & = (h_{(1)}S(k_{(3)}) \rightarrow (k_4 \rightarrow 1_A \leftarrow S(k_{(6)}))) \leftarrow S(h_{(3)}S(k_{(1)}))\#h_{(2)}S(k_{(2)})k_{(5)} \\
 & = (h_{(1)} \rightarrow 1_A \leftarrow S(h_{(5)}))(h_{(2)}S(k_{(3)}) \rightarrow 1_A \leftarrow S^2(k_{(1)})S(h_{(4)}))\#h_{(3)}S(k_{(2)})k_{(4)}.
 \end{aligned}$$

Finally, for (PR5), we have

$$\begin{aligned}
\pi_0(S(h_{(1)}))\pi_0(h_{(2)})\pi_0(k) &= (1_A \# S(h_{(1)}))(1_A \# h_{(2)})(1_A \# k) \\
&= (S(h_{(3)}) \rightharpoonup 1_A \leftarrow S^2(h_{(1)}) \# S(h_{(2)})h_{(4)})(1_A \# k) \\
&= (S(h_{(5)}) \rightharpoonup 1_A \leftarrow S^2(h_{(1)}))(S(h_{(4)})h_{(6)} \rightharpoonup 1_A \leftarrow S(S(h_{(2)})h_{(8)})) \# S(h_{(3)})h_{(7)}k \\
&= S(h_{(3)}) \rightharpoonup (h_{(4)} \rightharpoonup 1_A \leftarrow S(h_{(6)})) \leftarrow S^2(h_{(1)}) \# S(h_{(2)})h_{(5)}k \\
&= S(h_{(3)}) \rightharpoonup 1_A \leftarrow S^2(h_{(1)}) \# S(h_{(2)})h_{(4)}k \\
&= \pi(S(h_{(1)}))\pi(h_{(2)})k.
\end{aligned}$$

Therefore, π_0 is indeed a partial representation of H on the partial twisted smash product $\underline{A \# H}$. \square

In order to obtain the natural transformation of partial twisted smash products, we introduce the following definition and this definition is similar to [4].

Definition 3.5. Consider unital algebras A and B , and a Hopf algebra H that acts partially on A . A covariant pair associated to these data is a pair of maps (φ, π) where $\varphi : A \rightarrow B$ is an algebra morphism and $\pi : H \rightarrow B$ is a partial representation such that, for any $h \in H$ and $a \in A$,

$$\begin{aligned}
(\text{CP1}) \quad &\varphi(h \rightharpoonup a \leftarrow S(h_{(2)})) = \pi(h_{(1)})\varphi(a)\pi(S(h_{(2)})) \\
(\text{CP2}) \quad &\varphi(a)\pi(S(h_{(1)}))\pi(h_{(2)}) = \pi(S(h_{(1)}))\pi(h_{(2)})\varphi(a).
\end{aligned}$$

With this definition at hand, we can prove that the partial twisted smash product has the following universal property.

Theorem 3.6. *Let A and B be unital algebras and H a Hopf algebra with a symmetric partial action on A . Suppose that (φ, π) is a covariant pair associated to these data. Then there exists a unique algebra morphism $\Phi : \underline{A \# H} \rightarrow B$ such that $\varphi = \Phi \circ \varphi_0$ and $\pi = \Phi \circ \pi_0$, where the map $\varphi_0 : A \rightarrow \underline{A \# H}$ given by $\varphi_0(a) = a \# 1_H$ is an algebra morphism.*

PROOF. Define the linear map

$$\Phi : \underline{A \# H} \rightarrow B \quad a \# h \mapsto \varphi(a)\pi(h).$$

Let us verify that this is, in fact, an algebra morphism:

$$\begin{aligned}
\Phi((a \# h)(b \# g)) &= \Phi(a(h_{(1)} \rightharpoonup b \leftarrow S(h_{(3)})) \# h_{(2)}g) \\
&= \varphi(a(h_{(1)} \rightharpoonup b \leftarrow S(h_{(3)}))\pi(h_{(2)}g)
\end{aligned}$$

$$\begin{aligned}
 &= \varphi(a(h_{(1)} \rightharpoonup b \leftarrow S(h_{(2)}))\pi(h_{(3)}g) \\
 &= \varphi(a)\pi(h_{(1)})\varphi(b)\pi(S(h_{(2)}))\pi(h_{(3)}g) \\
 &= \varphi(a)\pi(h_{(1)})\varphi(b)\pi(S(h_{(2)}))\pi(h_{(3)})\pi(g) \\
 &= \varphi(a)\pi(h_{(1)})\pi(S(h_{(2)}))\pi(h_{(3)})\varphi(b)\pi(g) \\
 &= \varphi(a)\pi(h)\varphi(b)\pi(g) \\
 &= \Phi(a\#h)\Phi(b\#g).
 \end{aligned}$$

By construction, one can easily see that $\varphi = \Phi \circ \varphi_0$ and $\pi = \Phi \circ \pi_0$. Finally, for the uniqueness, suppose that there is another morphism $\Psi : \underline{A\#H} \rightarrow B$ factorizing both φ and π . Then we have

$$\begin{aligned}
 \Psi(a\#h) &= \Psi((a\#1_H)(1_A\#h)) = \Psi(a\#1_H)\Psi(1_A\#h) \\
 &= \Psi(\varphi_0(a))\Psi(\pi_0(h)) = \varphi(a)\pi(h) = \Phi(a\#h). \quad \square
 \end{aligned}$$

With the above theorem we have the following result.

Theorem 3.7. *Let H be a Hopf algebra, then there exist functors*

$$\Pi_0 : \mathbf{HParAct}_H \longrightarrow \mathbf{HParRep}_H, \quad \Pi_0(A, \cdot) = (\underline{A\#H}, \pi_0)$$

and

$$\Pi : \mathbf{HParAct}_H \longrightarrow \mathbf{HParRep}_H, \quad \Pi(A, \cdot) = (\text{End}_k(A), \pi).$$

and a natural transformation $\underline{\Phi} : \Pi_0 \rightarrow \Pi$.

PROOF. It is not hard to see that the constructions of the functors are indeed functorial. Moreover, if (A, \cdot) is a symmetric partial H -bimodule, $\Pi(A, \cdot) = (\text{End}_k(A), \pi)$ is the associated partial representation on $\text{End}_k(A)$ and $\varphi : A \rightarrow \text{End}(A)$ is the map $\varphi(a)(a') = aa'$, then (φ, π) is a covariant pair. Indeed, for any $h \in H$ and $a, a' \in A$, and let us check (CP1)

$$\begin{aligned}
 &\pi(h_{(1)})\varphi(a)\pi(S(h_{(2)}))(a') \\
 &= h_{(1)} \rightharpoonup [a(S(h_{(4)}) \rightharpoonup a' \leftarrow S^2(h_{(3)})) \leftarrow S(h_{(2)})] \\
 &= (h_{(1)} \rightharpoonup a \leftarrow S(h_{(4)}))[h_{(2)} \rightharpoonup (S(h_{(6)}) \rightharpoonup a' \leftarrow S^2(h_{(5)})) \leftarrow S(h_{(3)})] \\
 &= (h_{(1)} \rightharpoonup a \leftarrow S(h_{(6)}))(h_{(2)} \rightharpoonup 1_A \leftarrow S(h_{(5)}))[h_{(3)}S(h_{(8)}) \rightharpoonup a' \leftarrow S^2(h_{(7)})S(h_{(4)})] \\
 &= (h_{(1)} \rightharpoonup a \leftarrow S(h_{(4)}))[h_{(2)}S(h_{(6)}) \rightharpoonup a' \leftarrow S^2(h_{(5)})S(h_{(3)})] \\
 &\stackrel{(3.1)}{=} (h_{(1)} \rightharpoonup a \leftarrow S(h_{(3)}))[h_{(2)}S(h_{(6)}) \rightharpoonup a' \leftarrow S^2(h_{(5)})S(h_{(4)})]
 \end{aligned}$$

$$\begin{aligned}
&= (h_{(1)} \rightarrow a \leftarrow S(h_{(3)})) [h_{(2)} S(h_{(4)}) \rightarrow a'] \\
&\stackrel{c(3.2)}{=} (h_{(1)} \rightarrow a \leftarrow S(h_{(4)})) [h_{(2)} S(h_{(3)}) \rightarrow a'] \\
&= \varphi(h_{(1)} \rightarrow a \leftarrow S(h_{(2)}))(a').
\end{aligned}$$

Next, for (CP2), we find

$$\begin{aligned}
&\pi(S(h_{(1)}))\pi(h_{(2)})\varphi(a)(a') \\
&= S(h_{(2)}) \rightarrow [h_{(3)} \rightarrow aa' \leftarrow S(h_{(4)})] \leftarrow S^2(h_{(1)}) \\
&= S(h_{(2)}) \rightarrow [(h_{(3)} \rightarrow a \leftarrow S(h_{(5)}))(h_{(4)} \rightarrow a' \leftarrow S(h_{(6)}))] \leftarrow S^2(h_{(1)}) \\
&= \varphi(a)\pi(S(h_{(1)}))\pi(h_{(2)})(a').
\end{aligned}$$

Consequently, similar to [4], there exists an algebra morphism $\Phi : \underline{A\#H} \rightarrow \text{End}_k(A)$ such that $\pi = \Phi \circ \pi_0$, and hence Φ is a morphism of partial representations. \square

4. Frobenius properties

In this section, we shall discuss what conditions the algebra extension $\underline{A\#H}/A$ is Frobenius, generalizing the partial result in [5].

Let $i : R \rightarrow S$ be a ring homomorphism. Recall from [5] that i is called Frobenius (or we say that S/R is Frobenius) if there exists a Frobenius system (v, e) . This consists of an R -bimodule map $v : S \rightarrow R$ and an element $e = \sum e^1 \otimes_R e^2 \in S \otimes_R S$ such that $se = es$, for all $s \in S$, and $\sum v(e^1)e^2 = \sum v(e^2)e^1 = 1$.

A Hopf algebra H over a commutative ring k is Frobenius if and only if it is finitely generated projective, and the space of integrals is free of rank one. If H is Frobenius, then there exists a left integral $t \in H$ and a left integral $\varphi \in H^*$ such that $\langle \varphi, t \rangle = 1$. The Frobenius system is $(\varphi, t_{(2)} \otimes S^{-1}(t_{(1)}))$. In particular, we have

$$\langle \varphi, t_{(2)} \rangle S^{-1}(t_{(1)}) = t_{(2)} \langle \varphi, S^{-1}(t_{(1)}) \rangle = 1_H.$$

If $t \in H$ is a left integral, then it is easy to prove that

$$t_{(2)} \otimes S^{-1}(t_{(1)})h = ht_{(2)} \otimes S^{-1}(t_{(1)}),$$

for all $h \in H$.

Proposition 4.1. *Let H be a Frobenius Hopf algebra, let t and φ be as above, and take a partial H -module algebra A . Suppose that $(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(2)}))$ is central in A , for every $h \in H$, and that t satisfies the following cocommutativity property*

$$t_{(1)} \otimes t_{(2)} \otimes t_{(3)} \otimes t_{(4)} = t_{(1)} \otimes t_{(2)} \otimes t_{(4)} \otimes t_{(3)}.$$

Then $A\#H/A$ is Frobenius, with Frobenius system $(\underline{v} = (A\#\varphi) \circ \iota, \underline{e} = (1_A\#t_{(2)})1_A \otimes_A (1_A \otimes S^{-1}(t_{(1)}))1_A)$, where $\iota : \underline{A\#H} \rightarrow A\#H$ is the inclusion map.

PROOF. For all $a \in A$ and $h \in H$, we have

$$\begin{aligned} & (1_A \otimes t_{(2)})1_A \otimes_A (1_A \otimes S^{-1}(t_{(1)}))1_A(a\#h)1_A \\ &= (1_A \otimes t_{(2)}) \otimes_A (1_A \otimes S^{-1}(t_{(1)}))(a\#h)1_A \\ &= (1_A \otimes t_{(2)}) \otimes_A (1_A \otimes S^{-1}(t_{(1)}))(a(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}))\#h_{(2)})1_A \\ &= (1_A \otimes t_{(4)}) \otimes_A (S^{-1}(t_{(3)}) \rightharpoonup (a(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}))) \leftarrow t_{(1)})\#S^{-1}(t_{(2)})h_{(2)})1_A \\ &= (1_A \otimes t_{(4)})(S^{-1}(t_{(3)}) \rightharpoonup (a(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}))) \leftarrow t_{(1)}) \otimes_A (1_A\#S^{-1}(t_{(2)})h_{(2)})1_A \\ &= (t_{(4)} \rightharpoonup (S^{-1}(t_{(3)}) \rightharpoonup (a(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}))) \leftarrow t_{(1)}) \leftarrow S(t_{(6)})\#t_{(5)}) \\ &\quad \otimes_A (1_A\#S^{-1}(t_{(2)})h_{(2)})1_A \\ &= (t_{(4)} \rightharpoonup 1_A \leftarrow S(t_{(8)})(t_{(5)}S^{-1}(t_{(3)}) \rightharpoonup (a(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}))) \leftarrow t_{(1)})S(t_{(7)}))\#t_{(6)}) \\ &\quad \otimes_A (1_A\#S^{-1}(t_{(2)})h_{(2)})1_A \\ &= (t_{(5)} \rightharpoonup 1_A \leftarrow S(t_{(8)})(t_{(4)}S^{-1}(t_{(3)}) \rightharpoonup (a(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}))) \leftarrow t_{(1)})S(t_{(7)}))\#t_{(6)}) \\ &\quad \otimes_A (1_A\#S^{-1}(t_{(2)})h_{(2)})1_A \\ &= (t_{(4)} \rightharpoonup 1_A \leftarrow S(t_{(6)})(a(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}))) \leftarrow t_{(1)})S(t_{(2)})\#t_{(5)}) \\ &\quad \otimes_A (1_A\#S^{-1}(t_{(3)})h_{(2)})1_A \\ &= (t_{(2)} \rightharpoonup 1_A \leftarrow S(t_{(4)})(a(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)})))\#t_{(3)}) \otimes_A (1_A\#S^{-1}(t_{(1)})h_{(2)})1_A \\ &= ((a(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}))(t_{(2)} \rightharpoonup 1_A \leftarrow S(t_{(4)})\#t_{(3)}) \otimes_A (1_A\#S^{-1}(t_{(1)})h_{(2)})1_A \\ &= (((a(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}))(h_{(2)}t_{(2)} \rightharpoonup 1_A \leftarrow S(h_{(4)}t_{(4)}))\#h_{(3)}t_{(3)}) \otimes_A (1_A\#S^{-1}(t_{(1)}))1_A \\ &= (a(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}))\#h_{(2)}t_{(2)})1_A \otimes_A (1_A\#S^{-1}(t_{(1)}))1_A \\ &= (a\#h)(1_A\#t_{(2)}) \otimes_A (1_A\#S^{-1}(t_{(1)}))1_A \\ &= (a\#h)1_A(1_A\#t_{(2)})1_A \otimes_A (1_A\#S^{-1}(t_{(1)}))1_A. \end{aligned}$$

Using the fact that φ is a left integral, we easily find that

$$\underline{v}((a\#h)1_A) = \langle \varphi, h_{(2)} \rangle a(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)})) = \langle \varphi, h \rangle a.$$

The left A -linearity of \underline{v} is obvious, and the right A -linearity can be established as follows:

$$\begin{aligned} \underline{v}((a\#h)1_A b) &= \underline{v}((a\#h)b1_A) \\ &= \underline{v}((a(h_{(1)} \rightharpoonup b \leftarrow S(h_{(3)})\#h_{(2)})1_A) \\ &= \langle \varphi, h_{(2)} \rangle a(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)})) \\ &= \langle \varphi, h \rangle ab = \underline{v}((a\#h)1_A)b. \end{aligned}$$

Finally,

$$\begin{aligned} \underline{v}((1_A\#t_{(2)})1_A)((1_A\#S^{-1}(t_{(1)}))1_A) &= \langle \varphi, t_{(2)} \rangle 1_A\#S^{-1}(t_{(1)})1_A \\ &= 1_A\#1_H, \end{aligned}$$

and

$$\begin{aligned} ((1_A\#t_{(2)})1_A)\underline{v}((1_A\#S^{-1}(t_{(1)}))1_A) &= (1_A\#t_{(2)})\langle \varphi, S^{-1}(t_{(1)}) \rangle \\ &= 1_A\#1_H. \quad \square \end{aligned}$$

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