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On the variety of bands in completely regular semigroups

By MARIO PETRICH (Brač)

Abstract. Completely regular semigroups, enriched by the unary operation of inversion within their maximal subgroups, form a variety $\mathcal{CR}$ whose lattice of subvarieties is denoted by $\mathcal{L}(\mathcal{CR})$. Its subvariety $\mathcal{B}$ of all bands plays a seminal role in any study of the structure of $\mathcal{L}(\mathcal{CR})$. We present some new aspects of $\mathcal{B}$ relative to both $\mathcal{CR}$ and the variety $\mathcal{CS}$ of completely simple semigroups.

Since $\mathcal{B}$ is neutral in $\mathcal{L}(\mathcal{CR})$, the latter is a subdirect product of the lattice $(\mathcal{B}]$ of subvarieties of $\mathcal{B}$ and the lattice $[\mathcal{B})$ of supervarieties of $\mathcal{B}$. We determine the precise image of $\mathcal{L}(\mathcal{CR})$ in $(\mathcal{B}] \times [\mathcal{B})$.

For the relation $\mathbf{B}^\vee$ defined on $\mathcal{L}(\mathcal{CR})$ by $\mathcal{U}\mathbf{B}^\vee\mathcal{V}$ if $\mathcal{U} \vee \mathcal{B} = \mathcal{V} \vee \mathcal{B}$, we prove that each $\mathbf{B}^\vee$-class is embeddable into $(\mathcal{B}]$.

We establish several results concerning the variety $\mathcal{CS}$ in the context of the relation $\mathbf{B}^\vee$ and the structure of the lattice $\mathcal{L}(\mathcal{CS})$.

1. Introduction and summary

A *band* is a semigroup in which all elements are idempotent. One might say that it is the bands that distinguish semigroups from groups. For they are different from groups as they can be and are omnipresent in many studies of semigroups, in particular of completely regular semigroups.

The totality $\mathcal{B}$ of all bands forms a subvariety of the variety of all semigroups, and with the identity mapping as a unary operation, forms a subvariety of the variety $\mathcal{CR}$ of completely regular semigroups. For the latter are taken as

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unary semigroups with the operation of inversion within their maximal subgroups thereby forming a variety. It is this latter aspect that is seminal to our study.

The lattice of all subvarieties of $\mathcal{CR}$ is denoted by $\mathcal{L}(\mathcal{CR})$. Various relations on $\mathcal{L}(\mathcal{CR})$ are fundamental for the study of the structure of $\mathcal{L}(\mathcal{CR})$. We study only two of them as follows.

On $\mathcal{L}(\mathcal{CR})$ define the relations $\mathbf{B}^\wedge$ and $\mathbf{B}^\vee$ by

$$\begin{aligned} \mathcal{U}\mathbf{B}^\wedge\mathcal{V} \quad &\Longleftrightarrow \quad \mathcal{U}\cap\mathcal{B}=\mathcal{V}\cap\mathcal{B}, \\ \mathcal{U}\mathbf{B}^\vee\mathcal{V} \quad &\Longleftrightarrow \quad \mathcal{U}\vee\mathcal{B}=\mathcal{V}\vee\mathcal{B}. \end{aligned}$$

Since $\mathcal{B}$ has strong properties relative to $\mathcal{L}(\mathcal{CR})$, both of these relations are complete retractions, their intersection is the equality relation on $\mathcal{L}(\mathcal{CR})$, and all their classes are intervals. For any $\mathcal{V}\in\mathcal{L}(\mathcal{CR})$, we write its classes as

$$\mathcal{V}\mathbf{B}^\wedge=[\mathcal{V}_{B^\wedge},\mathcal{V}^{B^\wedge}] \quad \text{and} \quad \mathcal{V}\mathbf{B}^\vee=[\mathcal{V}_{B^\vee},\mathcal{V}^{B^\vee}].$$

In this way, we arrive at four operators on $\mathcal{L}(\mathcal{CR})$, namely

$$\mathcal{V}\to\mathcal{V}_{B^\wedge}, \quad \mathcal{V}\to\mathcal{V}^{B^\wedge}, \quad \mathcal{V}\to\mathcal{V}_{B^\vee}, \quad \text{and} \quad \mathcal{V}\to\mathcal{V}^{B^\vee}$$

which play a fundamental role in the present paper. Moreover, $\mathcal{V}_{B^\wedge}=\mathcal{V}\cap\mathcal{B}$, $\mathcal{V}^{B^\wedge}$ was determined in [4], $\mathcal{V}^{B^\vee}=\mathcal{V}\vee\mathcal{B}$, and $\mathcal{V}_{B^\vee}$ is known only sporadically.

The present paper represents a brief study of these concepts, as well as their role in $\mathcal{CR}$ and the variety $\mathcal{CS}$ of completely simple semigroups. Section 2 is a preamble mainly concerning references. In Section 3, we determine the image of $\mathcal{L}(\mathcal{CR})$ as a subdirect product of $(\mathcal{B}\,]$ and $[\mathcal{B}\,)$. We prove in Section 4 that every $\mathbf{B}^\vee$-class is embeddable into $(\mathcal{B}\,]$. In Section 5, we study the quotient $\mathcal{L}(\mathcal{CS})/\mathbf{B}^\vee$, where $\mathcal{L}(\mathcal{CS})$ is the lattice of varieties of completely simple semigroups. We wind up in Section 6 with an isomorphic copy of $\mathcal{L}(\mathcal{CS})$ in terms of triples.

2. Preamble

We adhere strictly to the terminology and notation of the book [9], and will not repeat them here. The strong properties of $\mathcal{B}$ will be our faithful companion, in particular that $\mathcal{B}$ is a neutral element in the lattice $\mathcal{L}(\mathcal{CR})$, as well as that the mappings $\mathcal{V}\to\mathcal{V}\cap\mathcal{B}$ and $\mathcal{V}\to\mathcal{V}\vee\mathcal{B}$ are complete retractions of $\mathcal{L}(\mathcal{CR})$ onto $(\mathcal{B}\,]$ and $[\mathcal{B}\,)$, respectively. For this, we refer to Fact 2.1 below. Recall that

$$(\mathcal{B}\,]=\{\mathcal{V}\in\mathcal{L}(\mathcal{CR})\mid\mathcal{V}\subseteq\mathcal{B}\}, \quad [\mathcal{B}\,)=\{\mathcal{V}\in\mathcal{L}(\mathcal{CR})\mid\mathcal{V}\supseteq\mathcal{B}\},$$

and $[\alpha, \beta]$ denotes an interval in any lattice.

We will take Polák's theorem on the construction of the join of varieties of completely regular semigroups for granted and refer the patient reader to the task of getting familiar with his papers [10], [11]; for even a minimal introduction to his theorem would take at least two dense pages. In this context, the ladders play a seminal role.

These are our principal references, but there are a few others. Papers [5], [6] are strongly related to the present work. We cite several results from the latter which are used in an auxiliary way and can be proved without great effort. With these qualifications, one might say that the present paper is relatively self-contained.

Fact 2.1. *The variety $\mathcal{B}$ of all bands is neutral and infinitely both $\cap$- and $\vee$-distributive in $\mathcal{L}(\mathfrak{CR})$.*

Proof. See [12, various remarks and Corollary 2.9]. $\square$

Corollary 2.2.

(i) $\mathbf{B}^{\wedge} \cap \mathbf{B}^{\vee} = \varepsilon$, *the equality relation on* $\mathcal{L}(\mathfrak{CR})$.

(ii) *The mapping*

$$\mathcal{V} \longrightarrow \mathcal{V}_{B^{\wedge}} \quad \big(\mathcal{V} \in \mathcal{L}(\mathfrak{CR})\big)$$

is a complete retraction of $\mathcal{L}(\mathfrak{CR})$ *onto* $(\mathcal{B}\,]$.

(iii) *The mapping*

$$\mathcal{V} \longrightarrow \mathcal{V}^{B^{\vee}} \quad \big(\mathcal{V} \in \mathcal{L}(\mathfrak{CR})\big)$$

is a complete retraction of $\mathcal{L}(\mathfrak{CR})$ *onto* $[\mathcal{B}\,)$.

(iv) $\mathcal{L}(\mathfrak{CR})/\mathbf{B}^{\wedge} \cong (\mathcal{B}\,], \quad \mathcal{L}(\mathfrak{CR})/\mathbf{B}^{\vee} \cong [\mathcal{B}\,)$.

Proof. (i)–(iii) This is a direct consequence of Fact 2.1.
(iv) This follows from parts (ii) and (iii). $\square$

This corollary gives a basic picture of the situation concerning the lattices $(\mathcal{B}\,]$ and $[\mathcal{B}\,)$. We will also need a few technical facts from the literature as follows.

Fact 2.3.

(i) $\mathcal{G} \vee \mathcal{B} = \mathfrak{OBG}$.

(ii) $\mathfrak{CS} \vee \mathcal{B} = L\mathfrak{OBG}$.

(iii) $\mathcal{T}^{B^{\wedge}} = \mathcal{G}$.

(iv) $\mathcal{G}_{B^{\vee}} = \mathcal{G}$.

(v) $\mathcal{B}_{B^{\vee}} = \mathcal{T}$.

PROOF. (i) See [6, Lemma 3.4(i)].
(ii) See [6, Lemma 3.4(ii)].
(iii) This follows from [4, Theorem 5.3].
(iv)(v) These follow from [6, Theorem 8.2(ii)]. □

We will often use some of the results in this section without reference, particularly the statement that $\mathcal{B}$ is neutral in $\mathcal{L}(\mathcal{CR})$.

3. A subdirect product representation of $\mathcal{L}(\mathcal{CR})$

By Corollary 2.2(i), the lattice $\mathcal{L}(\mathcal{CR})$ is a subdirect product of $(\mathcal{B}\,]$ and $[\mathcal{B}\,)$. We will now elaborate on this determining exactly the image of $\mathcal{L}(\mathcal{CR})$ in $(\mathcal{B}\,]\times[\mathcal{B}\,)$. The lattice $(\mathcal{B}\,]$ equals $\mathcal{L}(\mathcal{B})$, which is known, while the lattice $[\mathcal{B}\,)$ is the lattice of all varieties which contain all bands, which is not known.

In this and the next sections, we deduce the semigroup results from more general lattice theoretical results. To this end, we now introduce some notation.

Let a be a neutral element of a lattice L. On L define the relations $\mathbf{A}^{\wedge}$ and $\mathbf{A}^{\vee}$ by

$$x\,\mathbf{A}^{\wedge}\,y \iff x \wedge a = y \wedge a \quad \text{and} \quad x\,\mathbf{A}^{\vee}\,y \iff x \vee a = y \vee a,$$

respectively. For any $x \in L$, let

$$A_x = \{y \in L \mid y \wedge a = x \wedge a\},$$
$$B_x = \{y \in L \mid y \vee a = x \vee a\}.$$

For each $x \in L$, define

$$x_{a^{\wedge}} = \min A_x, \quad x^{a^{\wedge}} = \max A_x, \quad x_{a^{\vee}} = \min B_x, \quad x^{a^{\vee}} = \max B_x.$$

Clearly $x_{a^{\wedge}} = x \wedge a$ and $x^{a^{\vee}} = x \vee a$.

We are now ready for the principal result of this section.

Theorem 3.1. *Let a be a neutral element of a lattice L. The mappings*

$$\varphi : x \longrightarrow (x_{a^{\wedge}}, x^{a^{\vee}}), \qquad \psi : (y,z) \longrightarrow y \vee z_{a^{\vee}} = y^{a^{\wedge}} \wedge z$$

are mutually inverse isomorphisms between L and the lattice

$$\Gamma = \{(y,z) \in (a] \times [a) \mid y \vee z_{a^{\vee}} = y^{a^{\wedge}} \wedge z\}$$

which is a subdirect product $(a]$ and $[a)$.

PROOF. Neutrality of a in L implies that φ is a monomorphism of L into $(a] \times [a)$.

Let $x \in L$ and set $y = x \wedge a$ and $z = x \vee a$. Then $y\,\mathbf{A}^{\wedge}\,x$ and $z\,\mathbf{A}^{\vee}\,x$. Using [6, Lemma 4.2], we obtain

$$x = x_{a^{\wedge}} \vee x_{a^{\vee}} = y_{a^{\wedge}} \vee z_{a^{\vee}} = y \vee z_{a^{\vee}},$$
$$x = x^{a^{\wedge}} \wedge x^{a^{\vee}} = y^{a^{\wedge}} \wedge z^{a^{\vee}} = y^{a^{\wedge}} \wedge z,$$

so φ maps L into Γ. Trivially ψ maps Γ into L. Since

$$\begin{aligned} x\varphi\psi &= (x \wedge a, x \vee a)\psi = (x \wedge a) \vee (x \vee a)_{a^{\vee}} \\ &= x_{a^{\wedge}} \vee (x^{a^{\vee}})_{a^{\vee}} = x_{a^{\wedge}} \vee x_{a^{\vee}} = x, \\ (y,z)\psi\varphi &= (y^{a^{\wedge}} \wedge z)\varphi = (y \vee z_{a^{\vee}})\varphi \\ &= \big((y^{a^{\wedge}} \wedge z) \wedge a, (y \vee z_{a^{\vee}}) \vee a\big) \\ &= (y_{a^{\wedge}} \wedge z, y \vee z^{a^{\vee}}) = (y \wedge x \wedge z, y \vee a \vee z) \\ &= (y \wedge a, a \vee z) = (y,z), \end{aligned}$$

it follows that φ and ψ are mutually inverse bijections. Clearly each of them preserves inclusion, and thus they are isomorphisms.

If $y \in (a]$, then $y \vee a_{a^{\vee}} = y^{a^{\wedge}} \wedge a$ and thus $(y,a) \in \Gamma$. Analogously, if $z \in [a)$, then $a \vee z_{a^{\vee}} = a^{a^{\wedge}} \wedge z$ and $(a,z) \in \Gamma$. Therefore Γ is a subdirect product of $(a]$ and $[a)$. $\square$

For the case $L = \mathcal{L}(\mathcal{CR})$ and $a = \mathcal{B}$, the above theorem gives a representation of $\mathcal{L}(\mathcal{CR})$ as a subdirect product of $(\mathcal{B}\,]$ and $[\mathcal{B}\,)$ with the precise image of $\mathcal{L}(\mathcal{CR})$ in the direct product $(\mathcal{B}\,] \times [\mathcal{B}\,)$. We will encounter this result in Section 6.

4. Embedding $\mathbf{B}^{\vee}$-classes into $\mathcal{L}(\mathcal{B})$

We have seen in [4] that $\mathcal{L}(\mathcal{CR})/\mathbf{B}^{\wedge} \cong \mathcal{L}(\mathcal{B})$. Since the relation $\mathbf{B}^{\vee}$ looks as a kind of dual of $\mathbf{B}^{\wedge}$, we are led to expect some sort of duality between $\mathbf{B}^{\wedge}$ and $\mathbf{B}^{\vee}$.

In fact, we will deduce the statement in the heading of this section from a general theorem in lattice theory. The following result is not surprising.

Theorem 4.1. *Let* $\mathcal{P} \in \mathcal{L}(\mathcal{CR})$. *The mappings*

$$\varphi : \mathcal{U} \longrightarrow \mathcal{U} \cap \mathcal{B}, \quad \psi : \mathcal{V} \longrightarrow \mathcal{V} \vee \mathcal{P}$$

are mutually inverse isomorphisms between $[\mathcal{P}, \mathcal{P} \vee \mathcal{B}]$ *and* $[\mathcal{P} \cap \mathcal{B}, \mathcal{B}]$.

Proof. This is an immediate consequence of the result concerning a modular lattice L: the intervals $[a, a \vee b]$ and $[a \wedge b, b]$ are isomorphic, and mutually inverse isomorphisms are established by the mappings

$$x \longrightarrow x \vee a \ \ (x \in [a \wedge b, b]) \quad \text{and} \quad y \longrightarrow y \wedge b \ \ (y \in [a, a \vee b]);$$

see [1, Theorem 348]. The present case is $L = \mathcal{L}(\mathcal{CR})$ and $a = \mathcal{B}$, for it is well known that $\mathcal{L}(\mathcal{CR})$ is a modular lattice and $\mathcal{B}$ is a neutral element of $\mathcal{L}(\mathcal{CR})$. $\square$

As an immediate consequence, we have the following result.

Corollary 4.2. *For any $\mathcal{W} \in \mathcal{L}(\mathcal{CR})$, the mappings*

$$\varphi : \mathcal{U} \longrightarrow \mathcal{U} \cap \mathcal{B}, \quad \psi : \mathcal{V} \longrightarrow \mathcal{V} \vee \mathcal{W}$$

are mutually inverse isomorphisms between $\mathcal{W}\mathbf{B}^{\vee}$ and $[\mathcal{W}_{B^{\vee}B^{\wedge}}, \mathcal{B}]$. Consequently, every $\mathbf{B}^{\vee}$-class is embeddable into $(\mathcal{B}\,]$.

Proof. Apply Theorem 4.1 to the case $\mathcal{P} = \mathcal{W}_{B^{\vee}}$. $\square$

In the light of this corollary, we can visualize $\mathcal{L}(\mathcal{CR})$ as decomposed into the $\mathbf{B}^{\vee}$-classes, where each class can be identified with an interval of $\mathcal{L}(\mathcal{B})$ containing $\mathcal{B}$, and is thus determined by its lower end. As a consequence, we obtain a mapping $\mathcal{L}(\mathcal{CR})/\mathbf{B}^{\vee} \longrightarrow \mathcal{L}(\mathcal{B})$ given by $\mathcal{V}\mathbf{B}^{\vee} \longrightarrow \mathcal{V}_{B^{\vee}B^{\wedge}}$. A few interesting special cases follow.

Proposition 4.3. *Let $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$.*

(i) *$\mathcal{V}_{B^{\vee}B^{\wedge}} = \mathcal{T}$ if and only if $\mathcal{V} \subseteq \mathcal{OBG}$.*

(ii) *$\mathcal{V}_{B^{\vee}B^{\wedge}} = \mathcal{RB}$ if and only if $\mathcal{V} \in \mathcal{L}(L\mathcal{OBG}) \setminus \mathcal{L}(\mathcal{OBG})$.*

(iii) *$\mathcal{V}_{B^{\vee}B^{\wedge}} \in \{\mathcal{T}, \mathcal{RB}\}$ if and only if $\mathcal{V} \subseteq L\mathcal{OBG}$.*

(iv) *$\mathcal{V}_{B^{\vee}B^{\wedge}} = \mathcal{B}$ if and only if $\mathcal{V}_{B^{\vee}} \supseteq \mathcal{B}$.*

Proof. We will use Fact 2.3 freely.

(i) If $\mathcal{V}_{B^{\vee}B^{\wedge}} = \mathcal{T}$, then $(\mathcal{V}_{B^{\vee}B^{\wedge}})^{B^{\wedge}} = \mathcal{T}^{B^{\wedge}} = \mathcal{G}$, so that $(\mathcal{V}_{B^{\vee}})^{B^{\wedge}} = \mathcal{G}$ and thus

$$\mathcal{V} \subseteq \mathcal{V}^{B^{\vee}} = (\mathcal{V}_{B^{\vee}})^{B^{\vee}} \subseteq (\mathcal{V}_{B^{\vee}})^{B^{\wedge}B^{\vee}} = \mathcal{G}^{B^{\vee}} = \mathcal{G} \vee \mathcal{B} = \mathcal{OBG}.$$

Conversely, $\mathcal{V} \subseteq \mathcal{OBG}$ implies that

$$\mathcal{V}_{B^{\vee}} \subseteq \mathcal{OBG}_{B^{\vee}} = (\mathcal{G} \vee \mathcal{B})_{B^{\vee}} = \mathcal{G}_{B^{\vee}} \vee \mathcal{B}_{B^{\vee}} = \mathcal{G} \vee \mathcal{T} = \mathcal{G},$$

so that $\mathcal{V}_{B^{\vee}B^{\wedge}} \subseteq \mathcal{G}_{B^{\wedge}} = \mathcal{T}$.

(ii) Suppose that $\mathcal{V}_{B^\vee B^\wedge} = \mathcal{RB}$. Then establishing the inclusion $\mathcal{V} \subseteq L\mathcal{OBG}$ is analogous to establishing the inclusion $\mathcal{V} \subseteq \mathcal{OBG}$ in part (i). The exclusion $\mathcal{V} \nsubseteq \mathcal{OBG}$ follows from part (i). Hence $\mathcal{V} \in \mathcal{L}(L\mathcal{OBG}) \setminus \mathcal{L}(\mathcal{OBG})$.

The converse is also quite analogous to the proof of the converse of part (i).

(iii) This follows from parts (i) and (ii).

(iv) If $\mathcal{V}_{B^\vee B^\wedge} = \mathcal{B}$, then $\mathcal{V}_{B^\vee} \supseteq \mathcal{V}_{B^\vee B^\wedge} = \mathcal{B}$. Conversely, if $\mathcal{V}_{B^\vee} \supseteq \mathcal{B}$, then $\mathcal{V}_{B^\vee B^\wedge} \supseteq \mathcal{B}_{B^\wedge} = \mathcal{B}$; but $\mathcal{V}_{B^\vee B^\wedge} \subseteq \mathcal{B}$ always holds, so $\mathcal{V}_{B^\vee B^\wedge} = \mathcal{B}$. □

Proposition 4.4.

(i) *If* $\mathcal{V} \in \mathcal{L}(\mathcal{OBG})$, *then* $\mathcal{V}_{B^\vee B^\wedge} = \mathcal{V} \cap \mathcal{B}$.

(ii) *If* $\mathcal{V} \in \mathcal{L}(L\mathcal{OBG})\backslash\mathcal{L}(\mathcal{OBG})$, *then* $\mathcal{V}_{B^\vee B^\wedge} = \mathcal{V} \cap \mathcal{RB}$.

(iii) $\mathcal{CR}_{B^\vee B^\wedge} = \mathcal{B}$, $\mathcal{B}_{B^\vee B^\wedge} = \mathcal{T}$.

Proof. (i) Let $\mathcal{V} \in \mathcal{L}(\mathcal{OBG})$. By [6, Theorem 8.2], we have $\mathcal{V}_{B^\vee} = \mathcal{V} \cap \mathcal{G}$ whence $\mathcal{V}_{B^\vee B^\wedge} = (\mathcal{V} \cap \mathcal{G}) \cap \mathcal{B} = \mathcal{V} \cap \mathcal{B}$.

(ii) Let $\mathcal{V} \in \mathcal{L}(L\mathcal{OBG})\backslash\mathcal{L}(\mathcal{OBG})$. By [6, Theorem 9.3], we have $\mathcal{V}_{B^\vee} = \mathcal{V} \cap \mathcal{CS}$ whence $\mathcal{V}_{B^\vee B^\wedge} = (\mathcal{V} \cap \mathcal{CS}) \cap \mathcal{B} = \mathcal{V} \cap \mathcal{RB}$.

(iii) By [7, Theorem 5.7], $\mathcal{CR}$ is finitely join irreducible, which implies that $\mathcal{CR}_{B^\vee} = \mathcal{CR}$ whence $\mathcal{CR}_{B^\vee B^\wedge} = \mathcal{B}$. It is well known that $\mathcal{B}$ is finitely join irreducible, which implies that $\mathcal{B}_{B^\vee} = \mathcal{B}$ whence $\mathcal{B}_{B^\vee B^\wedge} = \mathcal{B}$. □

Remark 4.5. Consider the following special case. Take $\mathcal{W} \in \mathcal{L}(\mathcal{G})$ in Corollary 4.2. Then $\mathcal{W}_{B^\vee B^\wedge} \subseteq \mathcal{W}_{B^\wedge} = \mathcal{T}$ and thus $[\mathcal{W}_{B^\vee B^\wedge}, \mathcal{B}] = \mathcal{L}(\mathcal{B})$.

On the other hand, the mapping χ in [5, Proposition 11.7] of the form

$$\chi : \mathcal{V} \longrightarrow \mathcal{V} \vee \mathcal{W} \quad \big(\mathcal{V} \in \mathcal{L}(\mathcal{B})\big)$$

embeds $\mathcal{L}(\mathcal{B})$ into the kernel class $\mathcal{W}\mathbf{K}$ of $\mathcal{W}$. By [2, Theorem 11], $\mathbf{K}$ is a congruence on $\mathcal{L}(\mathcal{CR})$, and from [10, Theorem 2] we know that $[\mathcal{T}, \mathcal{B}]$ is a $\mathbf{K}$-class. Thus

$$\mathcal{W} = \mathcal{W} \vee \mathcal{T}\,\mathbf{K}\,\mathcal{W} \vee \mathcal{T}^K = \mathcal{W} \vee \mathcal{B}$$

so that $[\mathcal{W}, \mathcal{W} \vee \mathcal{B}] \subseteq \mathcal{W}\mathbf{K}$, and we get $\mathcal{W}\mathbf{B}^\vee \subseteq \mathcal{W}\mathbf{K}$. Therefore ψ maps $\mathcal{L}(\mathcal{B})$ onto $\mathcal{W}\mathbf{B}^\vee$, and χ maps $\mathcal{L}(\mathcal{B})$ into $\mathcal{W}\mathbf{K}$, and they have the same values. Hence we may say (not quite accurately) that $\psi = \chi$. This way, Corollary 4.2 implies [5, Proposition 11.7 and Corollary 11.8].

5. $\mathbf{B}^\vee$-relation on $\mathcal{L}(\mathcal{CS})$

Next to the variety $\mathcal{B}$ of bands, the variety $\mathcal{CS}$ of completely simple semigroups is probably the most important variety of completely regular semigroups.

The last two sections of the present work are dedicated to the various actions of $\mathcal{B}$ on $\mathcal{L}(\mathcal{CR})$ in a somewhat unexpected manner, crowned with a Rees-type representation of $\mathcal{L}(\mathcal{CS})$.

Here we describe the structure of $\mathcal{L}(\mathcal{CS})/\mathbf{B}^\vee$ where we now write $\mathbf{B}^\vee$ instead of $\mathbf{B}^\vee\big|_{\mathcal{L}(\mathcal{CS})}$. Toward this end, we first determine the restriction of $\mathbf{B}^\vee$ to $\mathcal{L}(\mathcal{CS})$ which will lead to a concrete description of the quotient $\mathcal{L}(\mathcal{CS})/\mathbf{B}^\vee$. We will need certain preliminaries.

Letting

$$\widetilde{K} = \mathcal{G} \cup (\mathcal{CS} \setminus \mathcal{R}e\mathcal{G}) \cup \big(\mathcal{D} \cap (\mathcal{CR} \setminus L\mathcal{O})\big),$$

the following result holds.

Fact 5.1 ([3, Theorem 1]). *For any* $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$, *we have* $\mathcal{U}\mathbf{K}\mathcal{V}$ *if and only if* $\mathcal{U} \cap \widetilde{K} = \mathcal{V} \cap \widetilde{K}$.

It looks like an unexplainable coincidence that the set below should look like a part $\widetilde{K}$, however not relative to $\mathcal{CR}$ but relative to $\mathcal{L}(\mathcal{CR})$.

The following parameter will turn out in our construction.

Definition 5.2. On the set

$$\Lambda = \mathcal{L}(\mathcal{G}) \cup \big(\mathcal{L}(\mathcal{CS}) \setminus \mathcal{L}(\mathcal{R}e\mathcal{G})\big)$$

define an operation of meet $\wedge$ by

$$\mathcal{U} \wedge \mathcal{V} = \begin{cases} \mathcal{U} \cap \mathcal{V} \cap \mathcal{G} & \text{if } \mathcal{U}, \mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G}) \text{ and } \mathcal{U} \cap \mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G}) \\ \mathcal{U} \cap \mathcal{V} & \text{otherwise} \end{cases}$$

and retain the join operation $\vee$ in $\mathcal{L}(\mathcal{CS})$.

We hasten to establish basic properties of the algebra Λ. The next lemma actually follows from Theorem 5.7, but it is instructive to see a direct proof.

Lemma 5.3. *With the given operations*, Λ *is a lattice.*

Proof. It is straightforward to check that Λ is closed under both operations. We already have the join $\vee$ which makes Λ an upper semilattice relative to inclusion. So it suffices to show that Λ is a lower semilattice under meet $\wedge$, again with respect to inclusion.

Let $\mathcal{U}, \mathcal{U}' \in \mathcal{L}(\mathcal{G})$ and $\mathcal{V}, \mathcal{V}' \in \mathcal{L}(\mathcal{CS}) \setminus \mathcal{L}(\mathcal{R}e\mathcal{G})$. We must show that each of the pairs $(\mathcal{U}, \mathcal{U}'), (\mathcal{U}, \mathcal{V}), (\mathcal{V}, \mathcal{V}')$ has a greatest lower bound. It suffices to check this for the pair $(\mathcal{V}, \mathcal{V}')$ when $\mathcal{V} \cap \mathcal{V}' \in \mathcal{L}(\mathcal{R}e\mathcal{G})$ since for other cases, we have $\mathcal{X} \cap \mathcal{Y}$ as in $\mathcal{L}(\mathcal{CS})$.

Assume that $\mathcal{V}, \mathcal{V}' \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$ but $\mathcal{V} \cap \mathcal{V}' \in \mathcal{L}(\mathcal{R}e\mathcal{G})$, and let $\mathcal{X} \in \Lambda$ satisfy $\mathcal{X} \subseteq \mathcal{V}$ and $\mathcal{X} \subseteq \mathcal{V}'$. In $\mathcal{L}(\mathcal{CS})$ this implies that $\mathcal{X} \subseteq \mathcal{V} \cap \mathcal{V}'$ so that $\mathcal{X} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$. Since $\mathcal{X} \in \Lambda$ and $\mathcal{V} \cap \mathcal{V}' \in \mathcal{L}(\mathcal{R}e\mathcal{G})$, we must have $\mathcal{X} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$, which by the definition of Λ yields that $\mathcal{X} \in \mathcal{L}(\mathcal{G})$. Hence $\mathcal{X} \subseteq \mathcal{V} \wedge \mathcal{V}'$ and $\mathcal{V} \wedge \mathcal{V}'$ is the greatest lower bound of $\mathcal{V}$ and $\mathcal{V}'$ in Λ. □

The support of the lattice Λ has an interesting property.

Lemma 5.4. *Every variety in Λ is the minimum in its **K**-class in $\mathcal{L}(\mathcal{CR})$.*

Proof. Let $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$ and $\mathcal{V} \in \Lambda$ be **K**-related. Then Fact 5.1 implies that $\mathcal{U} \cap \widetilde{K} = \mathcal{V} \cap \widetilde{K} = \mathcal{V}$ and thus $\mathcal{V} \subseteq \mathcal{U}$. Hence $\mathcal{V}$ is the least variety **K**-related to $\mathcal{U}$. □

We are now ready for the following result.

Proposition 5.5. *The intervals*

$$\{[\mathcal{V}, \mathcal{V} \vee \mathcal{B}] \mid \mathcal{V} \in \Lambda\} \tag{5.1}$$

constitute the complete set of $\mathbf{B}^{\vee}$-classes of varieties in $\mathcal{L}(\mathcal{CS})$.

Proof. By Lemma 5.4, for all $\mathcal{V} \in \Lambda$, we have $\mathcal{V} = \mathcal{V}_K$. By [6, Theorem 5.1], we have $\mathbf{B}^{\vee} \subseteq \mathbf{K}$ which implies that $\mathcal{V} = \mathcal{V}_{B^{\vee}}$. It follows that $\mathcal{V} = \mathcal{V}_{B^{\vee}}$ for all $\mathcal{V} \in \Lambda$, which yields that the intervals (5.1) are $\mathbf{B}^{\vee}$-classes.

Let $\mathcal{V} \in \mathcal{L}(\mathcal{CS}) \setminus \Lambda$. Then $\mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$ and

$$\mathcal{V}_{B^{\vee}} \vee \mathcal{B} = \mathcal{V} \vee \mathcal{B} = (\mathcal{V} \cap \mathcal{G}) \vee \mathcal{B}$$

which implies that $\mathcal{V}_{B^{\vee}} = \mathcal{V} \cap \mathcal{G} \in \mathcal{L}(\mathcal{G})$. Hence all varieties $\mathcal{V} = \mathcal{V}_{B^{\vee}} \in \mathcal{L}(\mathcal{CS})$ are contained in Λ. Now letting $\mathcal{V} \in \mathcal{L}(\mathcal{CS})$ be arbitrary, we obtain $\mathcal{V}_{B^{\vee}} \subseteq \mathcal{V} \subseteq (\mathcal{V}_{B^{\vee}})^{B^{\vee}}$, and $\mathcal{V}$ is contained in some interval in (5.1). Therefore the collection (5.1) covers all $\mathcal{L}(\mathcal{CS})$, proving the desired completeness. □

Corollary 5.6. *The intervals*

$$\{[\mathcal{V}, \mathcal{V} \vee \mathcal{RB}] \mid \mathcal{V} \in \Lambda\}$$

constitute the complete set of $\mathbf{B}^{\vee}|_{\mathcal{L}(\mathcal{CS})}$-classes.

Proof. In view of Proposition 5.5, it suffices to show that for any $\mathcal{V} \in \Lambda$, we have $(\mathcal{V} \vee \mathcal{B}) \cap \mathcal{CS} = \mathcal{V} \vee \mathcal{RB}$. By [12, Corollary 2.9], $\mathcal{CS}$ is neutral in $\mathcal{L}(\mathcal{CR})$, which for any $\mathcal{V} \in \Lambda$ yields

$$(\mathcal{V} \vee \mathcal{B}) \cap \mathcal{CS} = (\mathcal{V} \cap \mathcal{CS}) \vee (\mathcal{B} \cap \mathcal{CS}) = \mathcal{V} \vee \mathcal{RB}. \qquad \square$$

We are now ready for the main result of this section.

Theorem 5.7. *The mapping*

$$\lambda : \mathcal{V} \longrightarrow \mathcal{V}_{\mathcal{B}^\vee} = \begin{cases} \mathcal{V} \cap \mathcal{G} & \textit{if } \mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G}) \\ \mathcal{V} & \textit{if } \mathcal{V} \in \mathcal{L}(\mathcal{CS}) \setminus \mathcal{L}(\mathcal{R}e\mathcal{G}) \end{cases}$$

is a homomorphism of $\mathcal{L}(\mathcal{CS})$ *onto* Λ *which induces* $\mathbf{B}^\vee$ *on* $\mathcal{L}(\mathcal{CS})$.

Proof. We first justify the equality sign. Let $\mathcal{V} \in \mathcal{L}(\mathcal{CS})$. By Corollary 5.6, there exists $\mathcal{U} \in \Lambda$ such that $\mathcal{V} \in [\mathcal{U}, \mathcal{U} \vee \mathcal{R}\mathcal{B}]$ which is a $\mathcal{B}^\vee$-class. Hence $\mathcal{U} = \mathcal{V}_{\mathcal{B}^\vee}$ and $\mathcal{U} \vee \mathcal{R}\mathcal{B} = \mathcal{V}^{\mathcal{B}^\vee} = \mathcal{V} \vee \mathcal{B}$.

If $\mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$, then $\mathcal{U} \vee \mathcal{R}\mathcal{B} = \mathcal{V} \vee \mathcal{R}\mathcal{B} \subseteq \mathcal{O}$ whence $\mathcal{U} \in \mathcal{L}(\mathcal{G})$ so that $\mathcal{V}_{\mathcal{B}^\vee} = \mathcal{V} \cap \mathcal{B}$. If $\mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$, then $\mathcal{U} \vee \mathcal{R}\mathcal{B} \not\subseteq \mathcal{O}$, so $\mathcal{U} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$ and $\mathcal{U} \vee \mathcal{B} = \mathcal{V} \vee \mathcal{B}$ yields $\mathcal{U} = \mathcal{V}$.

Let $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CS})$. First the meets.

Case: $\mathcal{U} \cap \mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$. Then $\mathcal{U}, \mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$ and

$$(\mathcal{U} \cap \mathcal{V})\lambda = \mathcal{U} \cap \mathcal{V} = \mathcal{U}\lambda \cap \mathcal{V}\lambda.$$

Case: $\mathcal{U}, \mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$ and $\mathcal{U} \cap \mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$. Then

$$(\mathcal{U} \cap \mathcal{V})\lambda = \mathcal{U} \cap \mathcal{V} \cap \mathcal{G} = \mathcal{U}\lambda \wedge \mathcal{V}\lambda.$$

Case: $\mathcal{U} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$ and $\mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$. Then

$$(\mathcal{U} \cap \mathcal{V})\lambda = \mathcal{U} \cap \mathcal{V} \cap \mathcal{G} = \mathcal{U}\lambda \wedge \mathcal{V}\lambda.$$

Case: $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$. Then $\mathcal{U} \cap \mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$ and

$$(\mathcal{U} \cap \mathcal{V})\lambda = \mathcal{U} \cap \mathcal{V} \cap \mathcal{G} = (\mathcal{U} \cap \mathcal{G}) \cap (\mathcal{V} \cap \mathcal{G}) = \mathcal{U}\lambda \wedge \mathcal{V}\lambda.$$

Next the joins.

Case: $\mathcal{U}, \mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$. Then $\mathcal{U} \vee \mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$ and

$$(\mathcal{U} \vee \mathcal{V})\lambda = \mathcal{U} \vee \mathcal{V} = \mathcal{U}\lambda \vee \mathcal{V}\lambda.$$

Case: $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$. Then $\mathcal{U} \vee \mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$. By [12, Corollary 2.9], $\mathcal{G}$ is neutral in $\mathcal{L}(\mathcal{CR})$. Hence

$$(\mathcal{U} \vee \mathcal{V})\lambda = (\mathcal{U} \vee \mathcal{V}) \cap \mathcal{G} = (\mathcal{U} \cap \mathcal{G}) \vee (\mathcal{V} \cap \mathcal{G}) = \mathcal{U}\lambda \vee \mathcal{V}\lambda.$$

Case: $\mathcal{U} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$ and $\mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$. Then $\mathcal{U} \vee \mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$ and thus $(\mathcal{U} \vee \mathcal{V})\lambda = \mathcal{U} \vee \mathcal{V}$. We have $\mathcal{U} = \mathcal{U}' \vee \mathcal{U}''$ for some $\mathcal{U}' \in \mathcal{L}(\mathcal{RB})$ and $\mathcal{U}'' \in \mathcal{L}(\mathcal{G})$. Again by neutrality of $\mathcal{G}$, we obtain

$$\mathcal{U} \cap \mathcal{G} = (\mathcal{U}' \vee \mathcal{U}'') \cap \mathcal{G} = (\mathcal{U}' \cap \mathcal{G}) \vee (\mathcal{U}'' \cap \mathcal{G}) = \mathcal{U}''.$$

Since $\mathcal{V} \in \Lambda$, we get

$$\mathcal{U}\lambda \vee \mathcal{V}\lambda = (\mathcal{U} \cap \mathcal{G}) \vee \mathcal{V} = \mathcal{U}'' \vee \mathcal{V}.$$

On the other hand, $\mathcal{U} \vee \mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$ so that

$$(\mathcal{U} \vee \mathcal{V})\lambda = \mathcal{U} \vee \mathcal{V} = \mathcal{U}' \vee \mathcal{U}'' \vee \mathcal{V} = \mathcal{U}'' \vee \mathcal{V}$$

since $\mathcal{U}' \subseteq \mathcal{V}$.

Therefore λ is a homomorphism. It clearly maps $\mathcal{L}(\mathfrak{CS})$ onto Λ.

Note that $\mathcal{U}\lambda = \mathcal{V}\lambda$ if and only if

$$\left\{ \begin{array}{lll} \mathcal{U} \cap \mathcal{G} & \text{if} & \mathcal{U} \in \mathcal{L}(\mathcal{R}e\mathcal{G}) \\ \mathcal{U} & \text{if} & \mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G}) \end{array} \right\} = \left\{ \begin{array}{lll} \mathcal{V} \cap \mathcal{G} & \text{if} & \mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G}) \\ \mathcal{V} & \text{if} & \mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G}) \end{array} \right\}.$$

Assume that $\mathcal{U}\lambda = \mathcal{V}\lambda$.

Case: $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$. Again we have $\mathcal{U} = \mathcal{U}' \vee \mathcal{U}''$ and $\mathcal{V} = \mathcal{V}' \vee \mathcal{V}''$ for some $\mathcal{U}', \mathcal{V}' \in \mathcal{L}(\mathcal{RB})$ and $\mathcal{U}'', \mathcal{V}'' \in \mathcal{L}(\mathcal{G})$, so that

$$\mathcal{U} \vee \mathcal{B} = \mathcal{U}'' \vee \mathcal{B}, \quad \mathcal{V} \vee \mathcal{B} = \mathcal{V}'' \vee \mathcal{B}$$

where $\mathcal{U} \cap \mathcal{G} = \mathcal{U}''$ and $\mathcal{V} \cap \mathcal{G} = \mathcal{V}''$, which implies that $\mathcal{U}'' = \mathcal{V}''$ whence $\mathcal{U} \vee \mathcal{B} = \mathcal{V} \vee \mathcal{B}$, and finally $\mathcal{U}\mathbf{B}^{\vee}\mathcal{V}$.

Case: $\mathcal{U}, \mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$. Then $\mathcal{U} = \mathcal{V}$ and $\mathcal{U}\mathbf{B}^{\vee}\mathcal{V}$.

Case: $\mathcal{U} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$ and $\mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$. Then $\mathcal{U}\lambda = \mathcal{V} \cap \mathcal{G}$ and $\mathcal{V}\lambda = \mathcal{V}$, and hence $\mathcal{U} \cap \mathcal{G} = \mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$ which is impossible.

Therefore $\mathcal{U}\mathbf{B}^{\vee}\mathcal{V}$. Conversely, suppose that $\mathcal{U}\mathbf{B}^{\vee}\mathcal{V}$, so that $\mathcal{U} \vee \mathcal{B} = \mathcal{V} \vee \mathcal{B}$.

Case: $\mathcal{U} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$. Then $\mathcal{U} \vee \mathcal{B} \in \mathcal{L}(\mathcal{O})$ whence $\mathcal{V} \vee \mathcal{B} \in \mathcal{L}(\mathcal{O})$, so that $\mathcal{V} \in \mathcal{L}(\mathcal{O})$ and finally $\mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$. It follows that

$$\mathcal{U} \in \mathcal{L}(\mathcal{R}e\mathcal{G}) \iff \mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G}),$$

and thus $\mathcal{V} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$. Again we have $\mathcal{U} = \mathcal{U}' \vee \mathcal{U}''$ and $\mathcal{V} = \mathcal{V}' \vee \mathcal{V}''$ for some $\mathcal{U}', \mathcal{V}' \in \mathcal{L}(\mathcal{RB})$ and $\mathcal{U}'', \mathcal{V}'' \in \mathcal{L}(\mathcal{G})$. Now $\mathcal{U} \vee \mathcal{B} = \mathcal{V} \vee \mathcal{B}$ implies $\mathcal{U}'' \vee \mathcal{B} = \mathcal{V}'' \vee \mathcal{B}$. Again we have that $\mathcal{G}$ is neutral in $\mathcal{L}(\mathcal{CR})$ which yields

$$(\mathcal{U}'' \cap \mathcal{G}) \vee (\mathcal{B} \cap \mathcal{G}) = (\mathcal{V}'' \cap \mathcal{G}) \vee (\mathcal{B} \cap \mathcal{G})$$

and thus $\mathcal{U}'' \cap \mathcal{G} = \mathcal{V}'' \cap \mathcal{G}$ so that $\mathcal{U}'' = \mathcal{V}''$, and finally $\mathcal{U}\lambda = \mathcal{V}\lambda$.

Case: $\mathcal{U} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$. Then $\mathcal{V} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$ so that $\mathcal{U}\lambda = \mathcal{U}$ and $\mathcal{V}\lambda = \mathcal{V}$. As above, we have $\mathcal{U} \vee \mathcal{B} = \mathcal{V} \vee \mathcal{B}$. Using [9, Lemma I.2.2], we get $\mathcal{U}_K \vee \mathcal{B}_K = \mathcal{V}_K \vee \mathcal{B}_K$ whence $\mathcal{U}_K = \mathcal{V}_K$ and thus $\mathcal{U} = \mathcal{V}$ by Lemma 5.4. Then trivially $\mathcal{U}\lambda = \mathcal{V}\lambda$.

In all cases, we have $\mathcal{U}\lambda = \mathcal{V}\lambda$ and thus λ induces $\mathbf{B}^\vee$ on $\mathcal{L}(\mathfrak{CS})$. □

Corollary 5.8. *The lattice $\mathcal{L}(\mathfrak{CS})$ is a subdirect product of the lattices $\mathcal{L}(\mathcal{RB})$ and Λ.*

Proof. This follows from Theorem 5.7 and Corollary 2.2(i). □

6. Coordinatization of $\mathcal{L}(\mathfrak{CS})$

The purpose of this section is constructing an isomorphic copy of $\mathcal{L}(\mathfrak{CS})$ in terms of triples, in a vaguely similar way to the Rees construction of completely simple semigroups. We divide the argument into three steps. Step 1 contains a heuristic explanation of the origin of the parameters in our construction, in the spirit of the role of $\mathcal{B}$ in the structure of $\mathcal{L}(\mathfrak{CR})$. Step 2 contains the construction, and the main result. Step 3 consists of an alternative way of constructing the parameters.

Step 1. The application of Theorem 3.1 to the case $L = \mathcal{L}(\mathfrak{CR})$ and $a = \mathcal{B}$ provides an isomorphism

$$\varphi : \mathcal{W} \longrightarrow (\mathcal{W}_{\mathcal{B}^\vee}, \mathcal{W}^{\mathcal{B}^\vee}) \qquad (\mathcal{W} \in \mathcal{L}(\mathfrak{CR}))$$

of $\mathcal{L}(\mathfrak{CR})$ onto the lattice Γ adjusted to this particular case. Of interest here is the restriction

$$\varphi\big|_{\mathcal{L}(\mathfrak{CS})} : \mathcal{W} \longrightarrow (\mathcal{W} \cap \mathcal{B}, \mathcal{W} \vee \mathcal{B}) \qquad (\mathcal{W} \in \mathcal{L}(\mathfrak{CS})).$$

Let $\Gamma_{\mathfrak{CS}}$ denote its image. Since the elements of $\Gamma_{\mathfrak{CS}}$ are pairs, we may consider its projection into $(\mathcal{B}]$, which is evidently isomorphic to $\mathcal{L}(\mathcal{RB})$, and its projection onto $[\mathcal{B})$, which we denote by

$$\Psi = \{\mathcal{V} \vee \mathcal{B} \mid \mathcal{V} \in \mathcal{L}(\mathfrak{CS})\}.$$

Clearly $\mathcal{L}(\mathcal{RB})$ can be coordinatized by the mapping

$$\mathcal{T} \to (0,0), \quad \mathcal{LZ} \to (1,0), \quad \mathcal{RZ} \to (0,1), \quad \mathcal{RB} \to (1,1) \tag{6.1}$$

with the operation of minimum.

For the second projection, we have the following result.

Lemma 6.1. *The mapping*

$$\psi : \mathcal{V} \vee \mathcal{B} \longrightarrow \big((\mathcal{V} \vee \mathcal{B}) \cap \mathcal{CS}\big)_{\mathcal{B}^\vee} \qquad (\mathcal{V} \in \mathcal{L}(\mathcal{CS}))$$

is an isomorphism of Ψ *onto* Λ.

Proof. That ψ maps Ψ into Λ is a direct consequence of Theorem 5.7. We show next that ψ is injective. Hence let $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CS})$ satisfy $\mathcal{U}\psi = \mathcal{V}\psi$.

By Theorem 5.7, we have two cases.

Case: $(\mathcal{U} \vee \mathcal{B}) \cap \mathcal{CS}, (\mathcal{V} \vee \mathcal{B}) \cap \mathcal{CS} \in \mathcal{L}(\mathcal{R}e\mathcal{G})$. In this case

$$\big((\mathcal{U} \vee \mathcal{B}) \cap \mathcal{CS}\big) \cap \mathcal{G} = \big((\mathcal{V} \vee \mathcal{B}) \cap \mathcal{CS}\big) \cap \mathcal{G}$$

whence $(\mathcal{U} \vee \mathcal{B}) \cap \mathcal{G} = (\mathcal{V} \vee \mathcal{B}) \cap \mathcal{G}$. Since $\mathcal{G}$ is neutral in $\mathcal{L}(\mathcal{CR})$, we have

$$(\mathcal{U} \cap \mathcal{G}) \vee (\mathcal{B} \cap \mathcal{G}) = (\mathcal{V} \cap \mathcal{G}) \vee (\mathcal{B} \cap \mathcal{G})$$

so that $\mathcal{U} \cap \mathcal{G} = \mathcal{V} \cap \mathcal{G}$. In Λ, this yields $\mathcal{U} = \mathcal{V}$.

Case: $(\mathcal{U} \vee \mathcal{B}) \cap \mathcal{CS}, (\mathcal{V} \vee \mathcal{B}) \cap \mathcal{CS} \in \mathcal{L}(\mathcal{CS}) \backslash \mathcal{L}(\mathcal{R}e\mathcal{G})$, in which case $(\mathcal{U} \vee \mathcal{B}) \cap \mathcal{CS} = (\mathcal{V} \vee \mathcal{B}) \cap \mathcal{CS}$. Since $\mathcal{CS}$ is neutral in $\mathcal{L}(\mathcal{CR})$, we obtain

$$(\mathcal{U} \cap \mathcal{CS}) \vee (\mathcal{B} \cap \mathcal{CS}) = (\mathcal{V} \cap \mathcal{CS}) \vee (\mathcal{B} \cap \mathcal{CS})$$

and thus $\mathcal{U} \vee \mathcal{RB} = \mathcal{V} \vee \mathcal{RB}$. Since $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CS}) \backslash \mathcal{L}(\mathcal{R}e\mathcal{G})$, we must have $\mathcal{RB} \subseteq \mathcal{U} \cap \mathcal{V}$. Therefore $\mathcal{U} = \mathcal{V}$.

Clearly ψ maps Ψ onto Λ, which implies that ψ is a bijection between Ψ and Λ. Trivially ψ preserves inclusion. Conversely, assume that $(\mathcal{U} \vee \mathcal{B})\psi \subseteq (\mathcal{V} \vee \mathcal{B})\psi$. Then we have either the first or the second case above. In either case, it follows that $\mathcal{U} \vee \mathcal{B} \subseteq \mathcal{V} \vee \mathcal{B}$. Therefore ψ^{-1} also preserves inclusion. It follows that ψ is an isomorphism of Ψ onto Λ. □

We have arrived at the two projections: (6.1) for the first and Lemma 6.1 for the second. Hence to each variety $\mathcal{V} \in \mathcal{L}(\mathcal{CS})$, we may associate the pair (i, j) where $i, j \in \{0, 1\}$, and the element $(\mathcal{V} \vee \mathcal{B})\psi$, which we write together as $(i, \mathcal{W}, j)$ where $\mathcal{W} = (\mathcal{V} \vee \mathcal{B})\psi \in \Lambda$.

Finally, the meet and the join of two such triples follow the formulae (6.2) below, since the operations are componentwise.

Step 2.

Notation 6.2. Let

$$\begin{aligned}\Pi = &\big\{(i,\mathcal{U},j) \,\big|\, i,j \in \{0,1\},\, \mathcal{U} \in \mathcal{L}(\mathcal{G})\big\}\\ &\cup \big\{(1,\mathcal{V},1) \,\big|\, \mathcal{V} \in \mathcal{L}(\mathcal{CS}) \setminus \mathcal{L}(\mathcal{R}e\mathcal{G})\big\},\end{aligned}$$

with meet and join

$$\begin{aligned}(i,\mathcal{W},j) \wedge (i',\mathcal{W}',j') &= \big(\min\{i,i'\}, \mathcal{W} \wedge \mathcal{W}', \min\{j,j'\}\big),\\ (i,\mathcal{W},j) \vee (i',\mathcal{W}',j') &= \big(\max\{i,i'\}, \mathcal{W} \vee \mathcal{W}', \max\{j,j'\}\big).\end{aligned} \tag{6.2}$$

Define a mapping π on $\mathcal{L}(\mathcal{CS})$ by

$$\begin{aligned}\mathcal{U} &\longrightarrow (0,\mathcal{U},0),\\ \mathcal{LZ} \vee \mathcal{U} &\longrightarrow (1,\mathcal{U},0),\\ \mathcal{RZ} \vee \mathcal{U} &\longrightarrow (0,\mathcal{U},1),\\ \mathcal{RB} \vee \mathcal{U} &\longrightarrow (1,\mathcal{U},1),\\ \mathcal{V} &\longrightarrow (1,\mathcal{V},1)\end{aligned}$$

for all $\mathcal{U} \in \mathcal{L}(\mathcal{G})$ and $\mathcal{V} \in \mathcal{L}(\mathcal{CS}) \setminus \mathcal{L}(\mathcal{R}e\mathcal{G})$.

Theorem 6.3. *The mapping π is an isomorphism of $\mathcal{L}(\mathcal{CS})$ onto Π.*

Proof. It follows from [9, Lemma III.5.13] that

$$[\mathcal{T},\mathcal{G}],\;\; [\mathcal{LZ},\mathcal{LG}],\;\; [\mathcal{RZ},\mathcal{RG}],\;\; [\mathcal{RB},\mathcal{R}e\mathcal{G}],\;\; \mathcal{L}(\mathcal{CS}) \setminus \mathcal{L}(\mathcal{R}e\mathcal{G})$$

form a partition of $\mathcal{L}(\mathcal{CS})$. It is well known that each variety in $[\mathcal{LZ},\mathcal{LG}]$ is of the form $\mathcal{LZ} \vee \mathcal{U}$ for some $\mathcal{U} \in \mathcal{L}(\mathcal{G})$.

Let $\mathcal{U},\mathcal{U}' \in \mathcal{L}(\mathcal{G})$ and suppose that $\mathcal{LZ} \vee \mathcal{U} = \mathcal{LZ} \vee \mathcal{U}'$. By [12, Corollary 2.9], $\mathcal{G}$ is a neutral element in $\mathcal{L}(\mathcal{CR})$. Hence

$$(\mathcal{LZ} \vee \mathcal{U}) \cap \mathcal{G} = (\mathcal{LZ} \cap \mathcal{G}) \vee (\mathcal{U} \cap \mathcal{G}) = \mathcal{U}$$

and similarly $(\mathcal{LZ}\vee\mathcal{U}')\cap\mathcal{G} = \mathcal{U}'$. The hypothesis implies that $\mathcal{U} = \mathcal{U}'$. Therefore $\mathcal{U}$ is unique with this property. The same type of statement holds for the intervals $[\mathcal{RZ},\mathcal{RG}]$ and $[\mathcal{RB},\mathcal{R}e\mathcal{G}]$.

Therefore π is defined on all of $\mathcal{L}(\mathcal{CS})$ and is unambiguous. Next we establish the tables for meet and join for the varieties in $\mathcal{L}(\mathcal{CS})$ in this representation. Let

$\mathcal{U}, \mathcal{U}' \in \mathcal{L}(\mathcal{G})$ and $\mathcal{V}, \mathcal{V}' \in \mathcal{L}(\mathcal{CS}) \backslash \mathcal{L}(\mathcal{R}e\mathcal{G})$. Then by writing $\mathcal{U}^\dagger = \mathcal{U} \cap \mathcal{U}'$ for brevity, we have

$\cap$	$\mathcal{U}'$	$\mathcal{LZ} \vee \mathcal{U}'$	$\mathcal{RZ} \vee \mathcal{U}'$	$\mathcal{RB} \vee \mathcal{U}'$	$\mathcal{V}'$
$\mathcal{U}$	$\mathcal{U}^\dagger$	$\mathcal{U}^\dagger$	$\mathcal{U}^\dagger$	$\mathcal{U}^\dagger$	$\mathcal{U} \cap \mathcal{V}'$
$\mathcal{LZ} \vee \mathcal{U}$		$\mathcal{LZ} \vee \mathcal{U}^\dagger$	$\mathcal{U}^\dagger$	$\mathcal{LZ} \vee \mathcal{U}^\dagger$	$\mathcal{LZ} \vee (\mathcal{U} \cap \mathcal{V}')$
$\mathcal{RZ} \vee \mathcal{U}$			$\mathcal{RZ} \vee \mathcal{U}^\dagger$	$\mathcal{RZ} \vee \mathcal{U}^\dagger$	$\mathcal{RZ} \vee (\mathcal{U} \cap \mathcal{V}')$
$\mathcal{RB} \vee \mathcal{U}$				$\mathcal{RB} \vee \mathcal{U}^\dagger$	$\mathcal{RB} \vee (\mathcal{U} \cap \mathcal{V}')$
$\mathcal{V}$					$\mathcal{V} \cap \mathcal{V}'$

where the duals of the blanks already appear in the table.

Next we justify possibly questionable entries in this table. The above reference guarantees that $\mathcal{G}$ is neutral in $\mathcal{L}(\mathcal{CR})$, which we will use freely.

Case: $\mathcal{U} \cap (\mathcal{LZ} \vee \mathcal{U}') = \mathcal{X}$, say. Intersecting both sides by $\mathcal{G}$, we get

$$(\mathcal{U} \cap \mathcal{G}) \cap \big((\mathcal{LZ} \cap \mathcal{G}) \vee (\mathcal{U}' \cap \mathcal{G})\big) = \mathcal{X} \cap \mathcal{G}.$$

Since evidently $\mathcal{X} \in \mathcal{L}(\mathcal{G})$, we obtain $\mathcal{U} \cap \mathcal{U}' = \mathcal{X}$, as in the table.

Cases: $\mathcal{RZ} \vee \mathcal{U}$ and $\mathcal{RB} \vee \mathcal{U}$. These have essentially the same proof. Note that $\mathcal{U} \cap \mathcal{V}' = \mathcal{U} \cap (\mathcal{V}' \cap \mathcal{G})$, is again a meet in $\mathcal{L}(\mathcal{G})$.

By the above reference, also $\mathcal{LZ}$ is neutral in $\mathcal{L}(\mathcal{CR})$. Hence

$$(\mathcal{LZ} \vee \mathcal{U}) \cap (\mathcal{LZ} \vee \mathcal{U}') = \mathcal{LZ} \vee (\mathcal{U} \cap \mathcal{U}').$$

Case: $(\mathcal{LZ} \vee \mathcal{U}) \cap (\mathcal{RZ} \vee \mathcal{U}') = \mathcal{X}$, say. Then $\mathcal{X}$ is clearly contained in $\mathcal{L}(\mathcal{G})$. Now intersecting both sides by $\mathcal{G}$, we get

$$\big((\mathcal{LZ} \cap \mathcal{G}) \vee (\mathcal{U} \cap \mathcal{G})\big) \cap \big((\mathcal{RZ} \cap \mathcal{G}) \vee (\mathcal{U}' \cap \mathcal{G})\big) = \mathcal{X} \cap \mathcal{G}$$

whence $\mathcal{U} \cap \mathcal{U}' = \mathcal{X}$, as asserted.

The case $(\mathcal{LZ} \vee \mathcal{U}) \cap (\mathcal{RB} \vee \mathcal{U}')$ requires a very similar argument. Ditto for $(\mathcal{LZ} \vee \mathcal{U}) \cap \mathcal{V}'$. The remaining cases are either duals of those treated or require a very similar proof.

And now for the table of joins. Writing $\mathcal{U}^\ddagger = \mathcal{U} \vee \mathcal{U}'$ for brevity, we have

$\vee$	$\mathcal{U}'$	$\mathcal{LZ} \vee \mathcal{U}'$	$\mathcal{RZ} \vee \mathcal{U}'$	$\mathcal{RB} \vee \mathcal{U}'$	$\mathcal{V}'$
$\mathcal{U}$	$\mathcal{U}^\ddagger$	$\mathcal{LZ} \vee \mathcal{U}^\ddagger$	$\mathcal{RZ} \vee \mathcal{U}^\ddagger$	$\mathcal{RB} \vee \mathcal{U}^\ddagger$	$\mathcal{U} \vee \mathcal{V}'$
$\mathcal{LZ} \vee \mathcal{U}$		$\mathcal{LZ} \vee \mathcal{U}^\ddagger$	$\mathcal{RB} \vee \mathcal{U}^\ddagger$	$\mathcal{RB} \vee \mathcal{U}^\ddagger$	$\mathcal{U} \vee \mathcal{V}'$
$\mathcal{RZ} \vee \mathcal{U}$			$\mathcal{RZ} \vee \mathcal{U}^\ddagger$	$\mathcal{RB} \vee \mathcal{U}^\ddagger$	$\mathcal{U} \vee \mathcal{V}'$
$\mathcal{RB} \vee \mathcal{U}$				$\mathcal{RB} \vee \mathcal{U}^\ddagger$	$\mathcal{U} \vee \mathcal{V}'$
$\mathcal{V}$					$\mathcal{V} \vee \mathcal{V}'$

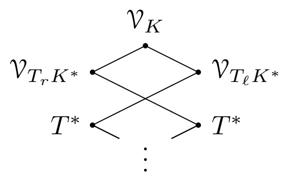
In the light of the form of representing the varieties in $\mathcal{L}(\mathcal{CS})$, the table for joins is easily shown to be correct.

We may conclude that π is a homomorphism and thus an isomorphism of $\mathcal{L}(\mathcal{CS})$ onto Π. $\square$

Step 3. There is another way for arriving at Π. First an outline. We know that $\mathcal{L}(\mathcal{CS}) \cong [\mathcal{S}, \mathcal{NBG}]$ by the mutually inverse isomorphisms

$$\mathcal{U} \longrightarrow \mathcal{U} \vee \mathcal{S}, \quad \mathcal{V} \longrightarrow \mathcal{V} \cap \mathcal{CS};$$

see [9, Corollary IV.1.11]. Each ladder of varieties in $[\mathcal{S}, \mathcal{NBG}]$ is of the form

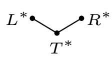


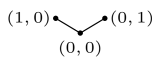
where $\mathcal{V}_K \in \mathcal{L}(\mathcal{G})$ or $\mathcal{V}_K \in \mathcal{L}(\mathcal{CS}) \setminus \mathcal{L}(\mathcal{R}e\mathcal{G})$. Hence we must have

$$\mathcal{V}_{T_r K^*} \in \{T^*, L^*\}, \quad \mathcal{V}_{T_\ell K^*} \in \{T^*, R^*\},$$

respectively. We may replace

$$L^* \longrightarrow (1,0), \quad T^* \longrightarrow (0,0), \quad R^* \longrightarrow (0,1)$$

which takes the partially ordered set 

onto 

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Combining these two transformations, we see that every variety in $\mathcal{L}(\mathcal{CS})$ can be given by three parameters

$$(i, \mathcal{U}, j),\ \mathcal{U} \in \mathcal{L}(\mathcal{G}); \quad (1, \mathcal{V}, 1),\ \mathcal{V} \in \mathcal{L}(\mathcal{CS}) \setminus \mathcal{L}(\mathcal{R}e\mathcal{G}) \quad \big(i, j \in \{0,1\}\big)$$

which is evidently reminiscent of the lattice Λ studied in Section 5. It coordinatizes $\mathcal{L}(\mathcal{CS})$. This is the heuristic idea; the numerous details follow.

Notation 6.4.

(a) Let φ denote the mapping

$$\mathcal{V} \longrightarrow \mathcal{V} \vee \mathcal{S} \quad \big(\mathcal{V} \in \mathcal{L}(\mathcal{CS})\big).$$

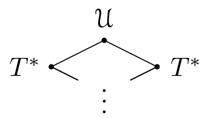
(b) Let $\ell ad\,[\mathcal{S}, \mathcal{NBG}]$ denote the set of ladders of varieties in $[\mathcal{S}, \mathcal{NBG}]$ with componentwise join.

(c) Let χ denote the mapping

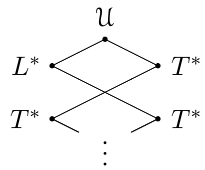
$$\mathcal{V} \longrightarrow \ell ad\,\mathcal{V} \quad \big(\mathcal{V} \in [\mathcal{S}, \mathcal{NBG}]\big)$$

where $\ell ad\,\mathcal{V}$ is the ladder of $\mathcal{V}$.

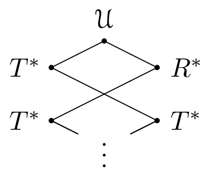
(d) Let ψ denote the mapping given by



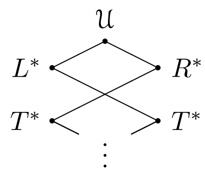
$\longrightarrow \quad (0, \mathcal{U}, 0)$



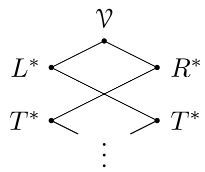
$\longrightarrow \quad (1, \mathcal{U}, 0)$



$\longrightarrow \quad (0, \mathcal{U}, 1)$



$\longrightarrow \quad (1, \mathcal{U}, 1)$



$\longrightarrow \quad (1, \mathcal{V}, 1)$

where $\mathcal{U} \in \mathcal{L}(\mathcal{G})$ and $\mathcal{V} \in \mathcal{L}(\mathcal{CS}) \setminus \mathcal{L}(\mathcal{R}e\mathcal{G})$.

We are now ready for properties and relationship of these functions.

Proposition 6.5.

(i) φ *is an isomorphism of* $\mathcal{L}(\mathcal{CS})$ *onto* $[\mathcal{S}, \mathcal{NBG}]$.

(ii) χ *is a* $\vee$*-isomorphism of* $[\mathcal{S}, \mathcal{NBG}]$ *onto* $\ell ad\,[\mathcal{S}, \mathcal{NBG}]$.

(iii) ψ *is a* $\vee$*-isomorphism of* $\ell ad\,[\mathcal{S}, \mathcal{NBG}]$ *onto* Π.

(iv) *The following diagram is commutative*:

$$\begin{tikzcd}
{[\mathcal{S}, \mathcal{NBG}]} \arrow[r, "\chi"] & \ell ad\,[\mathcal{S}, \mathcal{NBG}] \arrow[d, "\psi"] \\
\mathcal{L}(\mathcal{CS}) \arrow[u, "\varphi"] \arrow[r, "\pi"] & \Pi
\end{tikzcd}$$

Proof. (i) This follows immediately from the fact that $\mathcal{S}$ is neutral in $\mathcal{L}(\mathcal{CR})$; see [12, Corollary 2.9] and [9, Corollary IV.1.11].

(ii) This is the restriction of Polák's theorem [10], [11] to $[\mathcal{S}, \mathcal{NBG}]$.

(iii) It is straightforward to check that ψ transforms the join of ladders to the join of triples, and that ψ is a bijection.

(iv) Indeed, for $\mathcal{U} \in \mathcal{L}(\mathcal{G})$, we have

$$\mathcal{U} \xrightarrow{\varphi} \mathcal{U} \vee \mathcal{S} \xrightarrow{\chi} \ell ad\,(\mathcal{U} \vee \mathcal{S}) \xrightarrow{\psi} (0, \mathcal{U}, 0)$$

and the other cases are just as simple. $\square$

The key function is ψ which transforms ladders into triples.

We can easily extend π to a mapping on $\mathcal{L}(\mathcal{NBG})$, and extend Π accordingly adding one more parameter, again with the values in $\{0, 1\}$ with min and max, and to $\mathcal{L}(\mathcal{CS})$ assigning 0, and to $[\mathcal{S}, \mathcal{NBG}]$ assigning 1. All this depends on [9, Lemma IV.1.11].

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References

[1] G. Grätzer, Lattice Theory: Foundation, *Birkhäuser Verlag*, *Basel*, 2011.

[2] F. Pastijn, The lattice of completely regular semigroup varieties, *J. Aust. Math. Soc. Ser. A* **49** (1990), 24–42.

[3] M. Petrich, Some relations on the lattice of varieties of completely regular semigroups, *Boll. Unione Mat. Ital. B* **5** (2002), 265–278.

[4] M. Petrich, Canonical varieties of completely regular semigroups, *J. Aust. Math. Soc.* **83** (2007), 87–104.

[5] M. Petrich, Varieties of completely regular semigroups related to canonical varieties, *Semigroup Forum* **90** (2015), 53–99.

[6] M. Petrich, New operators for varieties of completely regular semigroups, *Semigroup Forum* **91** (2015), 415–449.

[7] M. Petrich and N. R. Reilly, Semigroups generated by certain operators on varieties of completely regular semigroups, *Pacific J. Math.* **132** (1988), 151–175.

[8] M. Petrich and N. R. Reilly, Operators related to E-disjunctive and fundamental completely regular semigroups, *J. Algebra* **134** (1990), 1–27.

[9] M. Petrich and N. R. Reilly, Completely Regular Semigroups, *Wiley*, *New York*, 1999.

[10] L. Polák, On varieties of completely regular semigroups I, *Semigroup Forum* **32** (1985), 97–123.

[11] L. Polák, On varieties of completely regular semigroups II, *Semigroup Forum* **36** (1987), 253–284.

[12] P. G. Trotter, Subdirect decompositions of the lattice of varieties of completely regular semigroups, *Bull. Aust. Math. Soc.* **39** (1989), 343–351.

MARIO PETRICH
21420 BOL
BRAČ
CROATIA

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