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Lie derivatives on a real hypersurface in complex two-plane Grassmannians

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Abstract. On a real hypersurface of a complex two-plane Grassmannian we have two connections: the Levi–Civita one, and for any nonnull k the k-th generalized Tanaka–Webster connection. Therefore we have the corresponding Lie derivatives. We classify such real hypersurfaces for which both Lie derivatives coincide when we apply them to the shape operator of the hypersurface.

1. Introduction

The generalized Tanaka–Webster connection (from now on, g-Tanaka–Webster connnection) for contact metric manifolds was introduced by Tanno [9] as a generalization of the connection defined by Tanaka in [8] and, independently, by Webster in [10]. This connection coincides with Tanaka–Webster connection if the associated CR-structure is integrable. The Tanaka–Webster connection is defined as a canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold. A real hypersurface M in a Kähler manifold has an (integrable) CR-structure associated with the almost contact structure (ϕ, ξ, η, g) induced on M by the Kähler structure, but, in general, this CR-structure is not guaranteed to be pseudo-Hermitian. Cho [4] and Tanno [9] defined the k-th g-Tanaka–Webster connection for a real hypersurface of a Kähler manifold by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.1}$$

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for any X, Y tangent to M, where ∇ denotes the Levi–Civita connection on M, A is the shape operator on M and k is a non-zero real number. In particular, if the real hypersurface satisfies $A\phi+\phi A = 2k\phi$, then the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka–Webster connection, see [4].

We define the k-th Cho operator on M associated to a tangent vector field X as $F_X^{(k)}Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, for any Y tangent to M.

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space has a remarkable geometric structure. It is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (see Berndt and Suh [2]). In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible Kähler, quaternionic Kähler manifold which is not a hyper-Kähler manifold.

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ and N a local normal unit vector field on M. Let also A be the shape operator of M associated to N. The almost contact structure vector field $\xi = -JN$ is said to be a Reeb vector field. Moreover, if $\{J_1, J_2, J_3\}$ is a local basis of $\mathfrak{J}$, we define $\xi_i = -J_iN$, $i = 1, 2, 3$. We will call $\mathbb{D}^{\perp} = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$. Its orthogonal complement in TM will be denoted by $\mathbb{D}$.

Berndt and Suh [2] proved that for a connected hypersurface M in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, both $\mathrm{Span}\{\xi\}$ and $\mathbb{D}^{\perp}$ are invariant under the shape operator A if and only if either (A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or (B) m is even, say $m = 2n$ and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$. Both types of real hypersurfaces have constant principal curvatures.

The Reeb vector field ξ is said to be Hopf if it is invariant under the shape operator A. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a Hopf foliation of M. We say that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. This is equivalent to the fact that the Reeb vector field is Hopf, see [3].

In [5] Jeong, Lee and Suh studied real hypersurfaces in complex two-plane Grassmannians such that $\hat{\nabla}^{(k)}A = \nabla A$. They obtained a non-existence result for such real hypersurfaces.

Let $\mathfrak{L}$ denote the Lie derivative of a real hypersurface M. Therefore $\mathfrak{L}_XY = \nabla_XY - \nabla_YX$ for any X, Y tangent to M. In [6] Jeong, Pak and Suh consider a so-called Lie derivative associated to the k-th g-Tanaka–Webster connection $\hat{\mathfrak{L}}^{(k)}$. Thus $\hat{\mathfrak{L}}_X^{(k)}Y = \hat{\nabla}_X^{(k)}Y - \hat{\nabla}_Y^{(k)}X$ for any X, Y tangent to M.

This paper is devoted to study real hypersurfaces in complex two-plane Grassmannians whose shape operator satisfies $\mathfrak{L}A = \hat{\mathfrak{L}}^{(k)}A$. Our results are

Theorem 1.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then $\mathfrak{L}_\xi A = \hat{\mathfrak{L}}_\xi^{(k)} A$ for some nonnull k if and only if M is locally congruent to a real hypersurface of type (A).*

Theorem 1.2. *There do not exist Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, such that $\mathfrak{L}_X A = \hat{\mathfrak{L}}_X^{(k)} A$ for any $X \in \mathbb{D}^\perp$ and some nonnull k.*

Theorem 1.3. *There do not exist Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, such that $\mathfrak{L}_X A = \hat{\mathfrak{L}}_X^{(k)} A$ for any $X \in \mathbb{D}$ and some nonnull k.*

From these Theorems we conclude

Corollary 1.4. *There do not exist real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, such that $\mathfrak{L} A = \hat{\mathfrak{L}}^{(k)} A$ for some nonnull k.*

2. Preliminaries

For the study of the Riemannian geometry of $G_2(\mathbb{C}^{m+2})$, see [1]. All the notations we will use from now on are the ones in [2] and [3]. We will suppose that the metric g of $G_2(\mathbb{C}^{m+2})$ is normalized for the maximal sectional curvature of the manifold to be eight.

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$. If $(\mathfrak{L}_X A)Y = (\hat{\mathfrak{L}}_X^{(k)} A)Y$ for any X, Y tangent to M, we get $F_X^{(k)} AY - F_{AY}^{(k)} X - AF_X^{(k)} Y + AF_Y^{(k)} X = 0$. That is

$$\begin{aligned} &g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY - g(\phi A^2 Y, X)\xi \\ &\quad + \eta(X)\phi A^2 Y + k\eta(AY)\phi X - g(\phi AX, Y)A\xi + \eta(Y)A\phi AX \\ &\quad + k\eta(X)A\phi Y + g(\phi AY, X)A\xi - \eta(X)A\phi AY - k\eta(Y)A\phi X = 0 \qquad (2.1) \end{aligned}$$

for any X, Y tangent to M.

To be used in the sequel we mention the following Propositions due to Berndt and Suh [2, Propositions 3 and 2].

Proposition 2.1. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathbb{D} \subset \mathbb{D}$, $A\xi = \alpha\xi$ and ξ is tangent to $\mathbb{D}^\perp$. Let $J_1 \in \mathfrak{J} = \mathrm{Span}\{J_1, J_2, J_3\}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r)\ ,\ \beta = \sqrt{2}\cot(\sqrt{2}r)\ ,\ \lambda = -\sqrt{2}\tan(\sqrt{2}r),\ \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. *The corresponding multiplicities are*

$$m(\alpha) = 1,\ m(\beta) = 2,\ m(\lambda) = 2m - 2 = m(\mu)$$

and as the corresponding eigenspaces we have

$$T_\alpha = \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1, \quad T_\beta = \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3,$$
$$T_\lambda = \{X | X \perp \mathbb{H}\xi, JX = J_1X\}, \quad T_\mu = \{X | X \perp \mathbb{H}\xi, JX = -J_1X\},$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ *and* $\mathbb{H}\xi$ *respectively denotes real, complex and quaternionic span of the structure vector* ξ *and* $\mathbb{C}^\perp\xi$ *denotes the orthogonal complement of* $\mathbb{C}\xi$ *in* $\mathbb{H}\xi$.

Proposition 2.2. *Let* M *be a connected real hypersurface of* $G_2(\mathbb{C}^{m+2})$. *Suppose that* $A\mathbb{D} \subset \mathbb{D}$, $A\xi = \alpha\xi$ *and* ξ *is tangent to* $\mathbb{D}$. *Then the quaternionic dimension* m *of* $G_2(\mathbb{C}^{m+2})$ *is even, say* $m = 2n$, *and* M *has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r)\ ,\ \beta = 2\cot(2r)\ \ \gamma = 0\ ,\ \lambda = \cot(r)\ ,\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. *The corresponding multiplicities are*

$$m(\alpha) = 1\ ,\ m(\beta) = 3 = m(\gamma)\ ,\ m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi\ ,\ T_\beta = \mathfrak{J}J\xi\ ,\ T_\gamma = \mathfrak{J}\xi\ ,\ T_\lambda\ ,\ T_\mu\ ,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp\ ,\ \mathfrak{J}T_\lambda = T_\lambda\ ,\ \mathfrak{J}T_\mu = T_\mu\ ,\ JT_\lambda = T_\mu\ .$$

Theorem 2.3 ([7]). *Let* M *be a connected orientable Hopf real hypersurface in* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. *Then the Reeb vector field* ξ *belongs to the distribution* $\mathbb{D}$ *if and only if* M *is locally congruent to an open part of a tube around a totally geodesic* $\mathbb{H}P^n$ *in* $G_2(\mathbb{C}^{m+2})$, *where* $m = 2n$.

3. Some lemmas

Lemma 3.1. *Let* M *be a real hypersurface in* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, *such that* $(\mathfrak{L}_\xi A)Y = (\hat{\mathfrak{L}}_\xi^{(k)} A)Y$ *for any* $Y \in TM$ *and some nonnull* k. *Then* M *is Hopf.*

Proof. Suppose that M is non-Hopf. Then we can write $A\xi = \alpha\xi + \beta U$, where U is a unit vector field orthogonal to ξ, α and β are functions on M and $\beta \neq 0$. From (2.1), taking $X = \xi$, we have

$$\begin{aligned}\beta g(\phi U, AY)\xi - \beta\eta(AY)\phi U - k\phi AY + \phi A^2 Y - \beta g(\phi U, Y)A\xi \\ + \beta\eta(Y)A\phi U + kA\phi Y - A\phi AY = 0 \qquad (3.1)\end{aligned}$$

for any Y tangent to M. If we take the scalar product of (3.1) with ξ we get

$$A\phi U = \frac{\alpha + k}{2}\phi U. \qquad (3.2)$$

Take $Y = \xi$ in (3.1). We obtain $\phi AU = k\phi U$. This gives

$$AU = \beta\xi + kU. \qquad (3.3)$$

If we take $Y = \phi U$ in (3.1), bearing in mind (3.2), we have

$$\frac{k-\alpha}{2}AU = \beta\left(\frac{k-\alpha}{2}\right)\xi + \left(\frac{k^2-\alpha^2}{4} - \beta^2\right)U. \qquad (3.4)$$

If $\alpha = k$, we get $\beta^2 U = 0$, which is impossible. Thus we obtain

$$AU = \beta\xi + \left(\frac{2}{k-\alpha}\right)\left(\frac{k^2-\alpha^2}{4} - \beta^2\right)U. \qquad (3.5)$$

From (3.3) and (3.5) $k(k-\alpha) = \frac{k^2-\alpha^2}{2} - 2\beta^2$. This yields $(k-\alpha)^2 = -4\beta^2$, which is impossible, finishing the proof. □

Suppose now that M is a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ and write $A\xi = \alpha\xi$. Then we have

Lemma 3.2. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, satisfying $(\mathfrak{L}_X A)Y = (\hat{\mathfrak{L}}^{(k)}_X A)Y$, for any X, Y tangent to M and some nonnull k. Then $\eta(X)(-k\phi AY + \phi A^2 Y + kA\phi Y - A\phi AY) = 0$.*

Proof. As we suppose M is Hopf, (2.1) yields

$$\begin{aligned}g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY - g(\phi A^2 Y, X)\xi \\ + \eta(X)\phi A^2 Y + k\eta(AY)\phi X - \alpha g(\phi AX, Y)\xi + \eta(Y)A\phi AX \\ + k\eta(X)A\phi Y + \alpha g(\phi AY, X)\xi - \eta(X)A\phi AY - k\eta(Y)A\phi X = 0 \qquad (3.6)\end{aligned}$$

for any X, Y tangent to M. Taking $Y = \xi$ in (3.6) we get

$$-\alpha\phi AX + k\alpha\phi X + A\phi AX - kA\phi X = 0 \tag{3.7}$$

for any X tangent to M. From (3.7), (3.6) becomes

$$\begin{aligned} &g(\phi AX, AY)\xi - k\eta(X)\phi AY - g(\phi A^2Y, X)\xi + \eta(X)\phi A^2Y \\ &\quad - \alpha g(\phi AX, Y)\xi + k\eta(X)A\phi Y + \alpha g(\phi AY, X)\xi - \eta(X)A\phi AY = 0 \end{aligned} \tag{3.8}$$

for any X, Y tangent to M.

The scalar product of (3.8) and ξ gives $-k\eta(X)\phi AY + \eta(X)\phi A^2Y + k\eta(X)A\phi Y - \eta(X)A\phi AY = 0$ and this finishes the proof. □

Finally we have the

Lemma 3.3. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then $(A\phi - \phi A)A = \gamma(A\phi - \phi A)$ for some non vanishing function γ on M if and only if M is locally congruent to a type (A) real hypersurface.*

Proof. $A\phi - \phi A$ is a symmetric operator on M. Moreover, for any X, Y tangent to M $g((A\phi - \phi A)AX, Y) = \gamma g((A\phi - \phi A)X, Y) = \gamma g(X, (A\phi - \phi A)Y) = g(X, (A\phi - \phi A)AY) = g(A(A\phi - \phi A)X, Y)$. This means that $(A\phi - \phi A)A = A(A\phi - \phi A)$. Thus we can find an orthonormal basis in TM that diagonalizes both $A\phi - \phi A$ and A. Let Y be a vector field in such a basis and suppose that $(A\phi - \phi A)Y = \lambda Y$ and $AY = \mu Y$. Therefore $A\phi Y - \mu\phi Y = \lambda Y$. The scalar product of this equation and Y yields $\lambda = 0$. Thus any eigenvalue of $A\phi - \phi A$ is null and $A\phi = \phi A$. From [2] M is locally congruent to a real hypersurface of type (A). The converse is trivial and we finish the proof. □

4. Proofs of the theorems

(1) Let us take $X = \xi$. From Lemma 3.1 M is Hopf. Taking $X = \xi$ in Lemma 3.2 we get $(A\phi - \phi A)AY = k(A\phi - \phi A)Y$ for any Y tangent to M. Then from Lemma 3.3 we prove Theorem 1.

(2) Suppose that M is Hopf and take $X \in \mathbb{D}^{\perp}$. From Lemma 3.2 we have two possibilities.

The first one is $\eta(X) = 0$ for any $X \in \mathbb{D}^{\perp}$. This yields $\xi \in \mathbb{D}$ and from Theorem 2.3 M is locally congruent to a real hypersurface of type (B).

If not, $k(A\phi - \phi A) = (A\phi - \phi A)A$ for some nonnull k and from Lemma 3.3 M is locally congruent to a real hypersurface of type (A).

Let us suppose that M is a real hypersurface of type (A). We write $\xi = \xi_1$. From (3.8) and the proof of Lemma 3.2 we have $A\phi AX{+}A^2\phi X{-}\alpha\phi AX{-}\alpha A\phi X = 0$ for any $X \in \mathbb{D}^{\perp}$. Take $X = \xi_2$ in this expression. From Proposition 2.1 we have $-\beta A\xi_3 - A^2\xi_3 + \alpha\beta\xi_3 + \alpha A\xi_3 = 0$. This yields $2\beta^2 = 2\alpha\beta$, that is, $\alpha = \beta$, which is impossible.

Suppose now that M is a real hypersurface of type (B). Taking $X = \xi_1$ in the expression of the above case and bearing in mind Proposition 2.2 as $A\phi\xi_1 = 0$ we obtain $-\alpha\beta\phi\xi_1 = 0$. As $\alpha\beta = -4$ we arrive to a contradiction, finishing the proof of Theorem 1.2.

(3) Suppose M is Hopf and $X \in \mathbb{D}$. From Lemma 3.2 we have two possibities again.

First, if $k(\phi A - A\phi) = (A\phi - \phi A)A$ for some nonnull k, from Lemma 3.3, M is locally congruent to a real hypersurface of type (A).

If not, $\eta(X) = 0$ for any $X \in \mathbb{D}$. That means $\xi \in \mathbb{D}^{\perp}$. We can write $\xi = \xi_1$. From [2] we have

$$2A\phi AX = \alpha A\phi X + \alpha\phi AX + 2\phi X + 2\phi_1 X \tag{4.1}$$

for any $X \in \mathbb{D}$. From (4.1) and (3.7) we get

$$\alpha(\phi A - A\phi)X + 2kA\phi X = (2k\alpha + 2)\phi X + 2\phi_1 X \tag{4.2}$$

for any $X \in \mathbb{D}$. The scalar product of (4.2) and ξ_2 yields

$$\alpha g(AX, \xi_3) + (2k - \alpha)g(A\phi X, \xi_2) = 0 \tag{4.3}$$

and the scalar product of (4.2) and ξ_3 gives

$$-\alpha g(AX, \xi_2) + (2k - \alpha)g(A\phi X, \xi_3) = 0 \tag{4.4}$$

for any $X \in \mathbb{D}$. Taking ϕX instead of X in (4.3) we have

$$-(2k - \alpha)g(AX, \xi_2) + \alpha g(A\phi X, \xi_3) = 0. \tag{4.5}$$

The determinant of the coefficient matrix of the linear system given by (4.4) and (4.5) is $-\alpha^2 + (2k - \alpha)^2$. It vanishes if and only if $\alpha = k$. If $\alpha \neq k$ we obtain $g(AX, \xi_2) = g(A\phi X, \xi_3) = 0$. Therefore $g(A\mathbb{D}, \mathbb{D}^{\perp}) = 0$ and M is locally congruent to a real hypersurface of type (A).

Let us suppose $\alpha = k$. In this case (4.2) becomes

$$k(\phi A + A\phi)X = (2k^2 + 2)\phi X + 2\phi_1 X \tag{4.6}$$

for any $X \in \mathbb{D}$. From (3.7) we also have

$$A(\phi A + A\phi)X = k(\phi A + A\phi)X \tag{4.7}$$

for any $X \in \mathbb{D}$. From (4.6) and (4.7) we get

$$\frac{1}{k}((2k^2+2)A\phi X + 2A\phi_1 X) = (2k^2+2)\phi X + 2\phi_1 X \tag{4.8}$$

for any $X \in \mathbb{D}$. The scalar product of (4.8) and ξ_ν, $\nu = 2,3$, gives

$$(2k^2+2)g(A\phi X, \xi_\nu) + 2g(A\phi_1 X, \xi_\nu) = 0 \tag{4.9}$$

for any $X \in \mathbb{D}$. Taking ϕX instead of X in (4.9) we get

$$-(2k^2+2)g(AX, \xi_\nu) + 2g(A\phi_1\phi X, \xi_\nu) = 0 \tag{4.10}$$

and taking $\phi_1 X$ instead of X in (4.9) we have

$$-2g(AX, \xi_\nu) + (2k^2+2)g(A\phi\phi_1 X, \xi_\nu) = 0 \tag{4.11}$$

for any $X \in \mathbb{D}$. For such an X $\phi_1\phi X = \phi\phi_1 X$. As $-(2k^2+2)^2+4 = 0$ if and only if $2k^2+2 = 2$, which is impossible, we obtain $g(AX, \xi_\nu) = 0$, for any $X \in \mathbb{D}$ and any $\nu = 2,3$ and again M is locally congruent to a real hypersurface of type (A).

For such a hypersurface, from Proposition 2.1, take $X \in T_\lambda$. Then $\phi X \in T_\lambda$. As $A\phi AX + A^2\phi X - \alpha\phi AX - \alpha A\phi X = 0$, we get $2\lambda^2 - 2\alpha\lambda = 0$. Thus either $\lambda = 0$ or $\lambda = \alpha$. Both situations are impossible and we finish the proof of Theorem 1.3.

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