

Existence of solutions for coupled integral boundary value problem at resonance

By YUJUN CUI (Qingdao)

Abstract. A coupled integral boundary value problem for a nonlinear differential system is considered in this article. An existence result is obtained with the use of the coincidence degree theory.

1. Introduction

This article deals with the following second order coupled integral boundary value problem

$$\begin{cases} -x''(t) = f_1(t, x(t), y(t), x'(t), y'(t)), & t \in (0, 1), \\ -y''(t) = f_2(t, x(t), y(t), x'(t), y'(t)), & t \in (0, 1), \\ x(0) = y(0) = 0, \quad x(1) = \alpha[y], \quad y(1) = \beta[x], \end{cases} \quad (1)$$

where f_1 and $f_2 : (0, 1) \times \mathbb{R}^4 \rightarrow \mathbb{R}$ are continuous and may be singular at $t = 0, 1$; $\alpha[x], \beta[x]$ are bounded linear functionals on $C[0, 1]$ given by

$$\alpha[x] = \int_0^1 x(t) dA(t), \quad \beta[x] = \int_0^1 x(t) dB(t),$$

Mathematics Subject Classification: 34B15, 34B10.

Key words and phrases: coincidence degree, coupled integral boundary conditions.

This paper is supported by NNSF of China (11371221, 11571207), the Specialized Research Foundation for the Doctoral Program of Higher Education of China (20123705110001), the Program for Scientific Research Innovation Team in Colleges, a project of Shandong province Higher Educational Science and Technology Program (J15LI16) and SDNSF (ZR2015AL002), the Tai'shan Scholar Engineering Construction Fund of Shandong Province of China.

involving Stieltjes integrals.

The coupled integral boundary value problem (1) happens to be at resonance in the sense that the associated linear homogeneous boundary value problem

$$\begin{cases} -x''(t) = 0, t \in (0, 1), \\ -y''(t) = 0, t \in (0, 1), \\ x(0) = y(0) = 0, \quad x(1) = \alpha[y], \quad y(1) = \beta[x] \end{cases}$$

has nontrivial solutions. Clearly, the resonance condition is $\kappa_1\kappa_2 = 1$, where

$$\kappa_1 = \int_0^1 t dA(t), \quad \kappa_2 = \int_0^1 t dB(t).$$

Coupled boundary conditions for ordinary differential systems arise in the study of reaction-diffusion equations and Sturm–Liouville problems, and have wide applications in various fields of sciences and engineering, for example mathematical biology and heat equation. Moreover, boundary value problems with Riemann–Stieltjes integral conditions constitute a very interesting and important class of problems. They include two, three, multi-point and integral boundary-value problems as special cases, see [10], [11], [16]. The existence and multiplicity of solutions for such problems have received a growing attention in the literature. We refer the reader to [1], [3], [4], [12], [13], [14], [16] for some recent results of integral boundary value problems at nonresonance and to [2], [4], [7], [8], [9], [15], [17], [18] at resonance. However, to our knowledge, the existence of solutions for a differential system with coupled integral boundary value problems at resonance has not been studied.

The purpose of this paper is to study the existence of solution for coupled integral boundary value problems (1) at resonance. Our method is based upon the coincidence degree theory of MAWHIN [5], [6].

The organization of this paper is as follows. In Section 2, we provide some necessary background. In particular, we shall introduce some lemmas and definitions associated with problem (1). In Section 3, the main results of problem (1) will be stated and proved. Finally, one example is also included to illustrate the main results.

Throughout this paper, we always suppose that

$$(H) \quad \kappa_1\kappa_2 = 1, \quad \kappa = \frac{\kappa_1}{2} \int_0^1 t(1-t)dB(t) + \frac{1}{2} \int_0^1 t(1-t)dA(t) \neq 0.$$

2. Preliminaries

In this section, to establish the existence of solutions in $C^1[0, 1] \times C^1[0, 1]$, we provide some background definitions and lemmas.

Definition 2.1. Let Y, Z be real Banach spaces, $L : \text{dom } L \subset Y \rightarrow Z$ be a linear operator. L is said to be the Fredholm operator of index zero provided that:

- (i) $\text{Im } L$ is a closed subset of Z ,
- (ii) $\dim \text{Ker } L = \text{codim } \text{Im } L < +\infty$.

Let Y, Z be real Banach spaces and $L : \text{dom } L \subset Y \rightarrow Z$ be a Fredholm operator of index zero. $P : Y \rightarrow Y$, $Q : Z \rightarrow Z$ are continuous projectors such that $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L$, $Y = \text{Ker } L \oplus \text{Ker } P$ and $Z = \text{Im } L \oplus \text{Im } Q$. It follows that $L|_{\text{dom } L \cap \text{Ker } P} : \text{dom } L \cap \text{Ker } P \rightarrow \text{Im } L$ is invertible. We denote the inverse of the operator by K_P (generalized inverse operator of L). If Ω is an open bounded subset of Y such that $\text{dom } L \cap \Omega \neq \emptyset$, the operator $N : Y \rightarrow Z$ will be called L -compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \rightarrow Y$ is compact.

The theorem we use is Theorem 2.4 of [5] or Theorem IV.13 [6].

Theorem 2.1. *Let L be a Fredholm operator of index zero and let N be L -compact on $\bar{\Omega}$. Assume that the following conditions are satisfied:*

- (i) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\text{dom } L \setminus \text{Ker } L) \cap \partial\Omega] \times (0, 1)$.
- (ii) $Nx \notin \text{Im } L$ for every $x \in \text{Ker } L \cap \partial\Omega$.
- (iii) $\deg(QN|_{\text{Ker } L}, \text{Ker } L \cap \Omega, 0) \neq 0$, where $Q : Z \rightarrow Z$ is a projector as above with $\text{Im } L = \text{Ker } Q$.

Then the equation $Lx = Nx$ has at least one solution in $\text{dom } L \cap \bar{\Omega}$.

We use the classical spaces $C[0, 1]$, $C^1[0, 1]$ and $L^1[0, 1]$. For $x \in C^1[0, 1]$, we use the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, where $\|x\|_\infty = \max_{t \in [0, 1]} |x(t)|$. And denote the norm in $L^1[0, 1]$ by $\|\cdot\|_1$. We also use the following three Banach spaces:

$$W^{2,1}(0, 1) = \{x : [0, 1] \rightarrow \mathbb{R} \mid x, x' \text{ are absolutely cont. on } [0, 1], x'' \in L^1[0, 1]\}$$

with its usual norm and $Y = C^1[0, 1] \times C^1[0, 1]$, with the norm

$$\|(x, y)\|_Y = \max\{\|x\|, \|y\|\} = \max\{\|(x, y)\|_\infty, \|(x', y')\|_\infty\}$$

and $Z = L^1[0, 1] \times L^1[0, 1]$, with the norm

$$\|(x, y)\|_Z = \max\{\|x\|_1, \|y\|_1\},$$

where

$$\|(x, y)\|_\infty = \max\{\|x\|_\infty, \|y\|_\infty\}.$$

Let the linear operator $L : \text{dom } L \subset Y \rightarrow Z$ with

$$\begin{aligned} \text{dom } L = \{(x, y) \in W^{2,1}(0, 1) \times W^{2,1}(0, 1) : x(0) = y(0) = 0, \\ x(1) = \alpha[y], y(1) = \beta[x]\} \end{aligned}$$

be defined by

$$L(x, y) = (-x'', -y'').$$

Let the nonlinear operator $N : Y \rightarrow Z$ be defined by

$$(N(x, y))(t) = (N_1(x, y)(t), N_2(x, y)(t)),$$

where $N_1, N_2 : Y \rightarrow L^1[0, 1]$ are defined by

$$\begin{aligned} N_1(x, y)(t) &= f_1(t, x(t), y(t), x'(t), y'(t)), \\ N_2(x, y)(t) &= f_2(t, x(t), y(t), x'(t), y'(t)). \end{aligned}$$

Then coupled integral boundary value problems (1) can be written as

$$L(x, y) = N(x, y).$$

Lemma 2.1. *Let L be the linear operator defined as above. Then*

$$\text{Ker } L = \{(x, y) \in \text{dom } L : (x, y) = c(\kappa_1 t, t), c \in \mathbb{R}, t \in [0, 1]\}$$

and

$$\begin{aligned} \text{Im } L = \left\{ (u, v) \in Z : \kappa_1 \int_0^1 \int_0^1 k(t, s)u(s)dsdB(t) \right. \\ \left. + \int_0^1 \int_0^1 k(t, s)v(s)dsdA(t) = 0 \right\}, \end{aligned}$$

where

$$k(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

PROOF. Let $(x(t), y(t)) = (\kappa_1 t, t)$. Considering $\kappa_1 \kappa_2 = 1$,

$$\alpha[y] = \alpha[t] = \kappa_1 = x(1), \quad \beta[x] = \beta[\kappa_1 t] = \kappa_1 \kappa_2 = y(1).$$

So $\{(x, y) \in \text{dom } L : (x, y) = c(\kappa_1 t, t), c \in \mathbb{R}, t \in [0, 1]\} \subset \text{Ker } L$. If $L(x, y) = (-x'', -y'') = (0, 0)$ and $(x(0), y(0)) = (0, 0)$, then $(x(t), y(t)) = (at, bt)$. Considering $x(1) = \alpha[y]$ and $y(1) = \beta[x]$, we can obtain that $a = \alpha[bt] = b\kappa_1$ and $b = \beta[at] = a\kappa_2$. It yields

$$\text{Ker } L \subset \{(x, y) \in \text{dom } L : (x, y) = c(\kappa_1 t, t), c \in \mathbb{R}, t \in [0, 1]\}.$$

We now show that

$$\text{Im } L = \left\{ (u, v) \in Z : \kappa_1 \int_0^1 \int_0^1 k(t, s)u(s)dsdB(t) + \int_0^1 \int_0^1 k(t, s)v(s)dsdA(t) = 0 \right\}.$$

If $(u, v) \in \text{Im } L$, then there exists $(x, y) \in \text{dom } L$ for which $-x''(t) = u(t)$ and $-y''(t) = v(t)$. Hence

$$x(t) = \int_0^1 k(t, s)u(s)ds + x(1)t. \quad (2)$$

and

$$y(t) = \int_0^1 k(t, s)v(s)ds + y(1)t. \quad (3)$$

Integrating (2) and (3) with respect to $dB(t)$ and $dA(t)$ respectively on $[0, 1]$ gives

$$\int_0^1 x(t)dB(t) = \int_0^1 \int_0^1 k(t, s)u(s)dsdB(t) + x(1)\kappa_2$$

and

$$\int_0^1 y(t)dA(t) = \int_0^1 \int_0^1 k(t, s)v(s)dsdA(t) + y(1)\kappa_1.$$

Therefore

$$\begin{bmatrix} -\kappa_2 & 1 \\ 1 & -\kappa_1 \end{bmatrix} \begin{bmatrix} x(1) \\ y(1) \end{bmatrix} = \begin{bmatrix} \int_0^1 \int_0^1 k(t, s)u(s)dsdB(t) \\ \int_0^1 \int_0^1 k(t, s)v(s)dsdA(t) \end{bmatrix}$$

and so

$$-\frac{\kappa_2}{1} = -\frac{1}{\kappa_1} = \frac{\int_0^1 \int_0^1 k(t, s)u(s)dsdB(t)}{\int_0^1 \int_0^1 k(t, s)v(s)dsdA(t)}.$$

It yields

$$\text{Im } L \subset \left\{ (u, v) \in Z : \kappa_1 \int_0^1 \int_0^1 k(t, s)u(s)dsdB(t) + \int_0^1 \int_0^1 k(t, s)v(s)dsdA(t) = 0 \right\}.$$

On the other hand, $(u, v) \in Z$ satisfies

$$\kappa_1 \int_0^1 \int_0^1 k(t, s)u(s)dsdB(t) + \int_0^1 \int_0^1 k(t, s)v(s)dsdA(t) = 0.$$

Let

$$\begin{aligned} x(t) &= \int_0^1 k(t, s)u(s)ds + t, \\ y(t) &= \int_0^1 k(t, s)v(s)ds + \kappa_2 t \left(1 - \int_0^1 \int_0^1 k(t, s)v(s)dsdA(t) \right), \end{aligned}$$

then $L(x, y) = (u, v)$, $x(0) = 0$, $y(0) = 0$, $x(1) = 1$ and

$$y(1) = \kappa_2 \left(1 - \int_0^1 \int_0^1 k(t, s)v(s)dsdA(t) \right).$$

Simple computations yield

$$\begin{aligned} \int_0^1 x(t)dB(t) &= \int_0^1 \int_0^1 k(t, s)u(s)dsdB(t) + \int_0^1 tdB(t) \\ &= -\kappa_2 \int_0^1 \int_0^1 k(t, s)v(s)dsdA(t) + \kappa_2 = y(1) \end{aligned}$$

and

$$\begin{aligned} \int_0^1 y(t)dA(t) &= \int_0^1 \int_0^1 k(t, s)v(s)dsdA(t) \\ &\quad + \kappa_2 \int_0^1 t dA(t) \left(1 - \int_0^1 \int_0^1 k(t, s)v(s)dsdA(t) \right) \\ &= \int_0^1 \int_0^1 k(t, s)v(s)dsdA(t) \\ &\quad + \kappa_2 \kappa_1 \left(1 - \int_0^1 \int_0^1 k(t, s)v(s)dsdA(t) \right) = 1 = x(1). \quad \square \end{aligned}$$

Lemma 2.2. *The operator L is a Fredholm operator of index zero and $\dim \text{Ker } L = \text{codim Im } L = 1$. Furthermore, the linear operator $K_p : \text{Im } L \rightarrow \text{dom } L \cap \text{Ker } P$ can be defined by*

$$(K_p(x, y))(t) = \left(\int_0^1 k(t, s)x(s)ds + t \int_0^1 \int_0^1 k(t, s)y(s)dsdA(t), \int_0^1 k(t, s)y(s)ds \right).$$

Also

$$\|K_p(x, y)\|_Y \leq \Delta \|(x, y)\|_Z, \quad \text{for all } (x, y) \in \text{Im } L,$$

where

$$\Delta = 1 + \int_0^1 t(1-t)d \left(\bigvee_0^t B \right)$$

and $\bigvee_0^t B$ denotes the variation of B on $[0, t]$.

PROOF. Firstly, we construct the following operator $Q : Z \rightarrow Z$ by

$$Q(x, y) = \frac{1}{\kappa} \left(\kappa_1 \int_0^1 \int_0^1 k(t, s)x(s)dsdB(t) + \int_0^1 \int_0^1 k(t, s)y(s)dsdA(t) \right) (1, 1).$$

Note that $\int_0^1 k(t, s)ds = \frac{1}{2}t(1-t)$, we have

$$Q^2(x, y) = Q(x, y).$$

Thus $Q : Z \rightarrow Z$ is a well-defined projector.

Now, it is obvious that $\text{Im } L = \text{Ker } Q$. Noting that Q is a linear projector, we have $Z = \text{Im } Q \oplus \text{Ker } Q$. Hence $Z = \text{Im } Q \oplus \text{Im } L$ and $\dim \text{Ker } L = \text{codim Im } L = 1$. This means that L is a Fredholm operator of index zero.

Taking $P : Y \rightarrow Y$ as

$$(P(x, y))(t) = y(1)(\kappa_1 t, t),$$

then the generalized inverse $K_p : \text{Im } L \rightarrow \text{dom } L \cap \text{Ker } P$ of L can be rewritten

$$(K_p(x, y))(t) = \left(\int_0^1 k(t, s)x(s)ds + t \int_0^1 \int_0^1 k(t, s)y(s)dsdA(t), \int_0^1 k(t, s)y(s)ds \right).$$

In fact, for $(x, y) \in \text{Im } L$, we have

$$(LK_p(x, y))(t) = \left(- \left(\int_0^1 k(t, s)x(s)ds + t \int_0^1 \int_0^1 k(t, s)y(s)dsdA(t) \right)'' - \left(\int_0^1 k(t, s)y(s)ds \right)'' \right) = (x(t), y(t))$$

and for $(x, y) \in \text{dom } L \cap \text{Ker } P$, we know

$$\begin{aligned} (K_p L(x, y))(t) &= \left(- \int_0^1 k(t, s)x''(s)ds - t \int_0^1 \int_0^1 k(t, s)y''(s)dsdA(t), \right. \\ &\quad \left. - \int_0^1 \int_0^1 k(t, s)y''(s)ds \right) \\ &= \left(x(t) - x(0)(1-t) - x(1)t + t \int_0^1 (y(t) - y(0)(1-t) \right. \\ &\quad \left. - y(1)t)dA(t), \quad y(t) - y(0)(1-t) - y(1)t \right). \end{aligned}$$

In view of $(x, y) \in \text{dom } L \cap \text{Ker } P$, $x(0) = y(0) = y(1) = 0$, $x(1) = \int_0^1 y(t)dA(t)$, thus

$$(K_p L(x, y))(t) = (x(t), y(t)).$$

This shows that $K_P = (L|_{\text{dom } L \cap \text{Ker } P})^{-1}$.

Since

$$\begin{aligned} \|K_p(x, y)\|_\infty &\leq \max \left\{ \int_0^1 |x(s)|ds + \int_0^1 \int_0^1 t(1-t)|y(s)|dsd \left(\bigvee_0^t B \right), \int_0^1 |y(s)|ds \right\} \\ &\leq \Delta \| (x, y) \|_Z \end{aligned}$$

and

$$\begin{aligned} \|(K_p(x, y))'\|_\infty &\leq \max \left\{ \int_0^1 |x(s)|ds + \int_0^1 \int_0^1 t(1-t)|y(s)|dsd \left(\bigvee_0^t B \right), \int_0^1 |y(s)|ds \right\} \\ &\leq \Delta \| (x, y) \|_Z. \end{aligned}$$

Thus $\|K_p(x, y)\|_Y \leq \Delta \| (x, y) \|_Z$. □

3. Main results

In this section, we will use Theorem 2.1 to prove the existence of solutions to BVP (1). To obtain our main theorem, we use the following assumptions.

(H1) There exist functions $a_i, b_i, c_i, d_i, e_i \in L^1[0, 1]$ ($i = 1, 2$), such that for all $(u, v, u', v') \in \mathbb{R}^4$ and $t \in [0, 1]$,

$$|f_1(t, u, v, u', v')| \leq a_1(t)|u| + b_1(t)|v| + c_1(t)|u'| + d_1(t)|v'| + e_1(t);$$

$$|f_2(t, u, v, u', v')| \leq a_2(t)|u| + b_2(t)|v| + c_2(t)|u'| + d_2(t)|v'| + e_2(t);$$

(H2) There exists a constant $C > 0$ such that for $(x, y) \in \text{dom } L$, if $|y'(t)| > C$ for all $t \in [0, 1]$, then

$$\begin{aligned} & \kappa_1 \int_0^1 \int_0^1 k(t, s) f_1(s, x(s), y(s), x'(s), y'(s)) ds dB(t) \\ & + \int_0^1 \int_0^1 k(t, s) f_2(s, x(s), y(s), x'(s), y'(s)) ds dA(t) \neq 0; \end{aligned}$$

(H3) There exists a constant $D > 0$ such that for $a \in \mathbb{R}$, if $|a| > D$, then either

$$\begin{aligned} & a\kappa_1 \int_0^1 \int_0^1 k(t, s) f_1(s, \kappa_1 a s, a s, \kappa_1 a, a) ds dB(t) \\ & + a \int_0^1 \int_0^1 k(t, s) f_2(s, \kappa_1 a s, a s, \kappa_1 a, a) ds dA(t) > 0; \end{aligned}$$

or

$$\begin{aligned} & a\kappa_1 \int_0^1 \int_0^1 k(t, s) f_1(s, \kappa_1 a s, a s, \kappa_1 a, a) ds dB(t) \\ & + a \int_0^1 \int_0^1 k(t, s) f_2(s, \kappa_1 a s, a s, \kappa_1 a, a) ds dA(t) < 0. \end{aligned}$$

Theorem 3.1. *Let (H1)–(H3) hold. Then (1) has at least one solution in $C^1[0, 1] \times C^1[0, 1]$ provided*

$$\max\{\Delta\alpha_1 + \delta\alpha_2, \alpha_2(\delta + \Delta)\} < 1,$$

where $\delta = \max\{|\kappa_1|, 1\}$, $\alpha_i = \|a_i\|_1 + \|b_i\|_1 + \|c_i\|_1 + \|d_i\|_1$ ($i = 1, 2$) and Δ is the same as Lemma 2.2.

PROOF. Set

$$\Omega_1 = \{(x, y) \in \text{dom } L \setminus \text{Ker } L : L(x, y) = \lambda N(x, y) \text{ for some } \lambda \in [0, 1]\}.$$

Then, for $(x, y) \in \Omega_1$, $L(x, y) = \lambda N(x, y)$, thus $\lambda \neq 0$, $N(x, y) \in \text{Im } L = \text{Ker } Q$, and hence

$$QN(x, y) = (0, 0), \quad \text{for all } t \in [0, 1].$$

By the definition of Q , we have

$$\begin{aligned} & \kappa_1 \int_0^1 \int_0^1 k(t, s) f_1(s, x(s), y(s), x'(s), y'(s)) ds dB(t) \\ & + \int_0^1 \int_0^1 k(t, s) f_2(s, x(s), y(s), x'(s), y'(s)) ds dA(t) = 0. \end{aligned}$$

Thus, from (H2), there exist $t_0 \in [0, 1]$ such that $|y'(t_0)| \leq C$. Since x, x', y, y' are absolutely continuous for all $t \in [0, 1]$,

$$\begin{aligned} |y'(t)| &= |y'(t_0) - \int_{t_0}^t y''(s) ds| \leq |y'(t_0)| + \|y''\|_1 \leq C + \|N_2(x, y)\|_1, \\ |y(t)| &= |y(0) - \int_0^t y'(s) ds| \leq t(C + \|N_2(x, y)\|_1) \leq C + \|N_2(x, y)\|_1. \end{aligned}$$

Thus

$$\begin{aligned} \|P(x, y)\|_Y &= \max\{\|P(x, y)\|_\infty, \|(P(x, y))'\|_\infty\} \\ &\leq \delta |y(1)| \leq \delta(C + \|N_2(x, y)\|_1). \end{aligned} \quad (4)$$

Also for $(x, y) \in \Omega_1$, $(x, y) \in \text{dom } L \setminus \text{Ker } L$, then

$$(I - P)(x, y) \in \text{dom } L \cap \text{Ker } P, \quad LP(x, y) = (0, 0),$$

thus from Lemma 2.2, we have

$$\begin{aligned} \|(I - P)(x, y)\|_Y &= \|K_P L(I - P)(x, y)\|_Y \leq \Delta \|L(I - P)(x, y)\|_Z \\ &= \Delta \|L(x, y)\|_Z \leq \Delta \|N(x, y)\|_Z. \end{aligned} \quad (5)$$

From (4) and (5), we obtain

$$\begin{aligned} \|(x, y)\|_Y &= \|P(x, y) + (I - P)(x, y)\|_Y \leq \|P(x, y)\|_Y + \|(I - P)(x, y)\|_Y \\ &\leq \delta C + \delta \|N_2(x, y)\|_1 + \Delta \|N(x, y)\|_Z \end{aligned}$$

$$= \max\{\delta C + \delta\|N_2(x, y)\|_1 + \Delta\|N_1(x, y)\|_1, \delta C + (\delta + \Delta)\|N_2(x, y)\|_1\}. \quad (6)$$

Writing $x(t) = \int_0^t x'(s)ds$, we obtain

$$\|x\|_\infty \leq \|x\|_1 \leq \|x'\|_\infty. \quad (7)$$

Similarly,

$$\|y\|_\infty \leq \|y\|_1 \leq \|y'\|_\infty. \quad (8)$$

From (6), we discuss various cases.

Case 1. $\|(x, y)\|_Y \leq \delta C + (\delta + \Delta)\|N_2(x, y)\|_1$.

From (H1), (7) and (8), we have

$$\|N_2(x, y)\|_1 \leq (\|a_2\|_1 + \|c_2\|_1)\|x'\|_\infty + (\|b_2\|_1 + \|d_2\|_1)\|y'\|_\infty + \|e_2\|_1.$$

Consequently, for

$$\|x'\|_\infty, \|y'\|_\infty \leq \|(x, y)\|_Y,$$

so,

$$\|x'\|_\infty \leq \frac{(\delta + \Delta)(\|b_2\|_1 + \|d_2\|_1)\|y'\|_\infty + \|e_2\|_1 + \delta C}{1 - (\delta + \Delta)(\|a_2\|_1 + \|c_2\|_1)},$$

$$\|y'\|_\infty \leq \frac{\|e_2\|_1(\delta + \Delta) + \delta C}{1 - (\delta + \Delta)(\|a_2\|_1 + \|c_2\|_1 + \|b_2\|_1 + \|d_2\|_1)}.$$

Therefore, there exists $M > 0$ such that

$$\|x'\|_\infty < M, \|y'\|_\infty < M,$$

so that $\|(x, y)\|_Y < M$. We have shown that Ω_1 is bounded.

Case 2. $\|(x, y)\|_Y \leq \delta C + \delta\|N_2(x, y)\|_1 + \Delta\|N_1(x, y)\|_1$.

From (H1), (7) and (8), we have

$$\|N_1(x, y)\|_1 \leq (\|a_1\|_1 + \|c_1\|_1)\|x'\|_\infty + (\|b_1\|_1 + \|d_1\|_1)\|y'\|_\infty + \|e_1\|_1,$$

$$\|N_2(x, y)\|_1 \leq (\|a_2\|_1 + \|c_2\|_1)\|x'\|_\infty + (\|b_2\|_1 + \|d_2\|_1)\|y'\|_\infty + \|e_2\|_1,$$

$$\|(x, y)\|_Y \leq \delta C + (\Delta(\|a_1\|_1 + \|c_1\|_1) + \delta(\|a_2\|_1 + \|c_2\|_1))\|x'\|_\infty$$

$$+ (\Delta(\|b_1\|_1 + \|d_1\|_1) + \delta(\|b_2\|_1 + \|d_2\|_1))\|y'\|_\infty + \Delta\|e_1\|_1 + \delta\|e_2\|_1,$$

$$\begin{aligned}\|x'\|_\infty &\leq \frac{\delta C + (\Delta(\|b_1\|_1 + \|d_1\|_1) + \delta(\|b_2\|_1 + \|d_2\|_1))\|y'\|_\infty + \Delta\|e_1\|_1 + \delta\|e_2\|_1}{1 - (\Delta(\|a_1\|_1 + \|c_1\|_1) + \delta(\|a_2\|_1 + \|e_2\|_1))}, \\ \|y'\|_\infty &\leq \frac{\delta C + \Delta\|e_1\|_1 + \delta\|e_2\|_1}{1 - \Delta\alpha_1 - \delta\alpha_2}.\end{aligned}$$

From the above inequality, there exists a constant $M > 0$ such that

$$\|x'\|_\infty < M, \quad \|y'\|_\infty < M.$$

Therefore Ω_1 is bounded.

Let

$$\Omega_2 = \{(x, y) \in \text{Ker } L : N(x, y) \in \text{Im } L\}.$$

For $(x, y) \in \Omega_2$, $(x, y) \in \text{Ker } L$ implies that (x, y) can be defined by $(x, y) = c(\kappa_1 t, t)$, $t \in [0, 1]$, $c \in \mathbb{R}$. By (H2), there exist $t_0 \in [0, 1]$ such that $|y'(t_0)| \leq C$, then

$$\|y'\|_\infty = |c| \leq C, \quad \|x'\|_\infty = |c\kappa_1| \leq \max\{\kappa_1, 1\}C.$$

Moreover,

$$\|x\|_\infty \leq \|x'\|_\infty, \quad \|y\|_\infty \leq \|y'\|_\infty.$$

So $\|(x, y)\|_Y \leq \max\{\kappa_1, 1\}C$. Thus, Ω_2 is bounded.

We define the isomorphism $J : \text{Ker } L \rightarrow \text{Im } Q$ by

$$J(a\kappa_1 t, at) = (a, a).$$

If the first part of (H3) is satisfied, and then let

$$\Omega_3 = \{(x, y) \in \text{Ker } L : \lambda J(x, y) + (1 - \lambda)QN(x, y) = (0, 0), \lambda \in [0, 1]\}.$$

For every $(x, y) = a(\kappa_1 t, t) \in \Omega_3$,

$$\begin{aligned}\lambda(a, a) &= -\frac{1 - \lambda}{\kappa} \left(\kappa_1 \int_0^1 \int_0^1 k(t, s) f_1(s, \kappa_1 a s, a s, \kappa_1 a, a) ds dB(t) \right. \\ &\quad \left. + \int_0^1 \int_0^1 k(t, s) f_2(s, \kappa_1 a s, a s, \kappa_1 a, a) ds dA(t) \right) (1, 1).\end{aligned}$$

If $\lambda = 1$, then $a = 0$, and if $|a| > D$, then by (H3)

$$\lambda(a^2, a^2) = -\frac{1 - \lambda}{\kappa} \left(a\kappa_1 \int_0^1 \int_0^1 k(t, s) f_1(s, \kappa_1 a s, a s, \kappa_1 a, a) ds dB(t) \right)$$

$$+ a \int_0^1 \int_0^1 k(t, s) f_2(s, \kappa_1 a s, a s, \kappa_1 a, a) ds dA(t) \Big) (1, 1) < (0, 0),$$

which, in either case, is a contradiction. If the other part of (H3) is satisfied, then we take

$$\Omega_3 = \{(x, y) \in \text{Ker } L : -\lambda J(x, y) + (1 - \lambda)QN(x, y) = (0, 0), \lambda \in [0, 1]\},$$

and, again, obtain a contradiction. Thus, in either case

$$\|(x, y)\|_Y = \max\{\|x\|, \|y\|\} = \max\{\kappa_1, 1\}|a| \leq \max\{\kappa_1, 1\}D$$

for all $(x, y) \in \Omega_3$, that is, Ω_3 is bounded. In the following, we shall prove that all the conditions of Theorem 2.1 are satisfied.

Set Ω be a bounded open subset of Y such that $\cup_{i=1}^3 \overline{\Omega}_i \subset \Omega$. By using the Ascoli–Arzela theorem, we can prove that $K_P(I - Q)N : \Omega \rightarrow Y$ is compact, thus N is L -compact on $\overline{\Omega}$. Then by the above argument we have

- (i) $Lx \neq \lambda Nx$, for every $(x, \lambda) \in [(\text{dom } L \setminus \text{Ker } L) \cap \partial\Omega] \times (0, 1)$,
- (ii) $Nx \notin \text{Im } L$ for $x \in \text{Ker } L \cap \partial\Omega$.

At last we will prove that (iii) of Theorem 2.1 is satisfied. Let $H((x, y), \lambda) = \pm\lambda J(x, y) + (1 - \lambda)QN(x, y)$. According to above argument, we know

$$H((x, y), \lambda) \neq 0 \quad \text{for } (x, y) \in \text{Ker } L \cap \partial\Omega,$$

thus, by the homotopy property of degree

$$\begin{aligned} \deg(QN|_{\text{Ker } L}, \text{Ker } L \cap \Omega, 0) &= \deg(H(\cdot, 0), \text{Ker } L \cap \Omega, 0) \\ &= \deg(H(\cdot, 1), \text{Ker } L \cap \Omega, 0) \\ &= \deg(\pm J, \text{Ker } L \cap \Omega, 0) \neq 0. \end{aligned}$$

Then by Theorem 2.1, $L(x, y) = N(x, y)$ has at least one solution in $\text{dom } L \cap \overline{\Omega}$, and so, the BVP (1) has at least one solution in the space Y . \square

4. Example

To illustrate how our main results can be used in practice, we present an example. Consider the couple boundary value problem

$$\begin{cases} -x''(t) = \frac{1}{12} \sin x(t) + \sin y^{\frac{1}{3}}(t) + e^t + \frac{1}{6}y'(t), & t \in (0, 1), \\ -y''(t) = 1 + \cos t + \cos y(t) \sin x'(t) + \frac{1}{8}(1+t)y'(t), & t \in (0, 1), \\ x(0) = y(0) = 0, \quad x(1) = \frac{4}{5} \int_0^1 y(s) ds, \quad y(1) = -x\left(\frac{1}{2}\right) + \frac{9}{2}x\left(\frac{2}{3}\right). \end{cases}$$

Let

$$\begin{aligned} f_1(t, x, y, x', y') &= \frac{1}{12} \sin x + \sin y^{\frac{1}{3}} + e^t + \frac{1}{6} y', \\ f_2(t, x, y, x', y') &= 1 + \cos t + \cos y \sin x' + \frac{1}{8} (1+t)y', \end{aligned}$$

$$A(t) = \frac{4}{5}t, \quad B(t) = \begin{cases} 0 & t \in \left[0, \frac{1}{2}\right), \\ -1 & t \in \left[\frac{1}{2}, \frac{2}{3}\right), \\ \frac{7}{2} & t \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

Then

$$|f_1(t, x, y, x', y')| \leq \frac{1}{12}|x| + \frac{1}{6}|y'| + 1 + e, \quad |f_2(t, x, y)| \leq \frac{1}{8}(1+t)|y'| + 3,$$

$$\kappa_1 = \frac{2}{5}, \quad \kappa_2 = \frac{5}{2}, \quad \kappa = \frac{13}{60}, \quad \delta = 1, \quad \Delta = \frac{9}{4}.$$

Again taking $b_1 = c_1 = a_2 = b_2 = c_2 = 0$, $a_1 = \frac{1}{12}$, $d_1 = \frac{1}{6}$ and $d_2 = \frac{1}{8}(1+t)$, we have

$$\begin{aligned} &\max\{\Delta(\|a_1\|_1 + \|b_1\|_1 + \|c_1\|_1 + \|d_1\|_1) + \delta(\|a_2\|_1 + \|b_2\|_1 + \|c_2\|_1 + \|d_2\|_1), \\ &\quad (\delta + \Delta)(\|a_2\|_1 + \|b_2\|_1 + \|c_2\|_1 + \|d_2\|_1)\} = \frac{39}{64} < 1. \end{aligned}$$

Finally taking $C = 30$, for any $y \in W^{2,1}(0,1)$, assume $|y'(t)| > C$ hold for any $t \in (0,1)$, since the continuity of y' , then either $y'(t) > C$ or $y'(t) < -C$ hold for any $t \in (0,1)$.

If $y'(t) > C$ hold for any $t \in (0,1)$, then

$$f_1(t, x(t), y(t), x'(t), y'(t)) > 0, \quad f_2(t, x(t), y(t), x'(t), y'(t)) > 0.$$

Therefore,

$$\begin{aligned} &\kappa_1 \int_0^1 \int_0^1 k(t, s) f_1(s, x(s), y(s), x'(s), y'(s)) ds dB(t) \\ &\quad + \int_0^1 \int_0^1 k(t, s) f_2(s, x(s), y(s), x'(s), y'(s)) ds dA(t) \\ &> \kappa_1 \int_0^1 \int_0^1 k(t, s) f_1(s, x(s), x'(s)) ds dB(t) \end{aligned}$$

$$\begin{aligned}
&= \kappa_1 \left(- \int_0^1 k\left(\frac{1}{2}, s\right) f_1(s, x(s), x'(s)) ds + \frac{9}{2} \int_0^1 k\left(\frac{2}{3}, s\right) f_1(s, x(s), x'(s)) ds \right) \\
&\geq \kappa_1 \left(\frac{9}{2} \int_0^1 \frac{2}{3} \left(1 - \frac{2}{3}\right) s(1-s) f_1(s, x(s), x'(s)) ds \right. \\
&\quad \left. - \int_0^1 s(1-s) f_1(s, x(s), x'(s)) ds \right) = 0.
\end{aligned}$$

If $y'(t) < -C$ hold for any $t \in (0, 1)$, then

$$\begin{aligned}
&\kappa_1 \int_0^1 \int_0^1 k(t, s) f_1(s, x(s), y(s), x'(s), y'(s)) ds dB(t) \\
&\quad + \int_0^1 \int_0^1 k(t, s) f_2(s, x(s), y(s), x'(s), y'(s)) ds dA(t) < 0.
\end{aligned}$$

Thus condition (H2) holds. Again taking $D = 30$, for any $a \in \mathbb{R}$, when $|a| > D$, we have

$$\begin{aligned}
&a \kappa_1 \int_0^1 \int_0^1 k(t, s) f_1(s, \kappa_1 a s, a s, \kappa_1 a, a) ds dB(t) \\
&\quad + a \int_0^1 \int_0^1 k(t, s) f_2(s, \kappa_1 a s, a s, \kappa_1 a, a) ds dA(t) > 0.
\end{aligned}$$

So condition (H3) holds. Hence from Theorem 3.1, BVP (1) has at least one solution $(x, y) \in C^1[0, 1] \times C^1[0, 1]$.

ACKNOWLEDGEMENTS. The authors are grateful to the referees for the valuable comments and corrections.

References

- [1] N. A. ASIF and R. A. KHAN, Positive solutions to singular system with four-point coupled boundary conditions, *J. Math. Anal. Appl.* **386** (2012), 848–861.
- [2] Z. BAI and Y. ZHANG, Solvability of fractional three-point boundary value problems with nonlinear growth, *Appl. Math. Comput.* **218** (2011), 1719–1725.
- [3] Y. CUI and J. SUN, On existence of positive solutions of coupled integral boundary value problems for a nonlinear singular superlinear differential system, *Electron. J. Qual. Theory Differ. Equ.* **41** (2012), 1–13.
- [4] Y. CUI, Solvability of second-order boundary-value problems at resonance involving integral conditions, *Electron. J. Differ. Eq.* **45** (2012), 1–9.
- [5] J. MAWHIN, Topological degree and boundary value problems for nonlinear differential equations, In: *Topological Methods for Ordinary Differential Equations*, Lecture Notes in Math., Vol. **1537**, Springer-Verlag, New York – Berlin, 1993.

- [6] J. MAWHIN, Topological degree methods in nonlinear boundary value problem, In: NS-FCBMS Regional Conference Series in Mathematics, *American Mathematical Society, Providence, RI*, 1979.
- [7] Z. HU and W. LIU, Solvability for fractional order boundary value problems at resonance, *Bound. Value Prob.* **20** (2011), 1–10.
- [8] W. JIANG, The existence of solutions for boundary value problems of fractional differential equations at resonance, *Nonlinear Anal.* **74** (2011), 1987–1994.
- [9] F. WANG, Y. CUI and F. ZHANG, Existence of nonnegative solutions for second order m -point boundary value problems at resonance, *Appl. Math. Comput.* **217** (2011), 4849–4855.
- [10] J. R. L. WEBB and G. INFANTE, Positive solutions of nonlocal boundary value problems involving integral conditions, *NoDEA Nonlinear Differential Equations Appl.* **15** (2008), 45–67.
- [11] J. R. L. WEBB and G. INFANTE, Positive solutions of nonlocal boundary value problems: a unified approach, *J. London Math. Soc.* **74** (2006), 673–693.
- [12] J. R. L. WEBB, Remarks on nonlocal boundary value problems at resonance, *Appl. Math. Comput.* **216** (2010), 497–500.
- [13] J. R. L. WEBB and M. ZIMA, Multiple positive solutions of resonance and non-resonance nonlocal fourth-order boundary value problems, *Glasgow Math. J.* **54** (2012), 225–240.
- [14] C. YUAN, D. JIANG, D. O'REGAN and R. P. AGARWAL, Multiple positive solutions to systems of nonlinear semipositone fractional differential equations with coupled boundary conditions, *Electron. J. Qual. Theory Differ. Equ.* **13** (2012), 1–17.
- [15] X. ZHANG, M. FENG and W. GE, Existence result of second-order differential equations with integral boundary conditions at resonance, *J. Math. Anal. Appl.* **353** (2009), 311–319.
- [16] X. ZHANG and J. SUN, On multiple sign-changing solutions for some second-order integral boundary value problems, *Electron. J. Qual. Theory Differ. Equ.* **44** (2010), 1–15.
- [17] Y. ZOU, L. LIU and Y. CUI, The existence of solutions for four-point coupled boundary value problems of fractional differential equations at resonance, *Abstr. Appl. Anal.* (2014), Art. ID 314083, 1–8.
- [18] Y. ZOU and Y. CUI, Existence results for a functional boundary value problem of fractional differential equations, *Adv. Difference Equ.* **2013** (2013), 1–25.

YUJUN CUI
 STATE KEY LABORATORY OF MINING
 DISASTER PREVENTION AND CONTROL
 CO-FOUNDED BY SHANDONG PROVINCE AND
 THE MINISTRY OF SCIENCE AND TECHNOLOGY
 SHANDONG UNIVERSITY OF SCIENCE
 AND TECHNOLOGY
 QINGDAO, 266590
 CHINA
 AND
 DEPARTMENT OF MATHEMATICS
 SHANDONG UNIVERSITY OF SCIENCE
 AND TECHNOLOGY
 QINGDAO, 266590
 P. R. CHINA

E-mail: cyj720201@163.com

(Received March 9, 2015; revised December 6, 2015)