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Approximately Jensen-convex functions

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Abstract. In this paper we show that if a function satisfies the Jensen-inequality (or the inequality describing $\mathbb{Q}$-convexity) with an appropriate error term, then the function is Jensen-convex (without error) as well.

First we consider a function f, which is defined on an open interval I of $\mathbb{R}$. We prove that if $f : I \to \mathbb{R}$ satisfies the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \psi(|x-y|)$$

for every $x, y \in I$, where $\lim\limits_{t\to 0+} \frac{\psi(t)}{t^2} = 0$, then f is Jensen-convex.

We also prove that if a real function f, which is defined on an $\mathbb{F}$-algebraically open and $\mathbb{F}$-convex subset D of a vector space X over $\mathbb{F}$ (where $\mathbb{F}$ is a subfield of $\mathbb{R}$), satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) + c\,[\lambda(1-\lambda)\,|x-y|]^p$$

for every $x, y \in D$ and $\lambda \in [0,1] \cap \mathbb{F}$, with a fixed non-negative real number c and a fixed exponent $p > 1$, then it has to be $\mathbb{F}$-convex, i.e., f satisfies the above inequality with $c = 0$ as well. Considering $\mathbb{F} = \mathbb{Q}$, we obtain another characterization of Jensen-convex functions.

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1. Introduction

The first paper devoted to approximately convex functions is due to Hyers and Ulam [6]. Their result motivated further investigations of approximate convexity (see, for instance, [5], [8], [9], [11], [12], [13], [17] and the references therein).

Rolewicz [14], [15], [16] investigated continuous real functions f satisfying the functional inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + Ct(1-t)\alpha(|x-y|) \tag{1}$$

for every $x, y \in \mathbb{R}$, $t \in [0,1]$, with a non-negative constant C and a non-decreasing function $\alpha : [0,+\infty[\to [0,+\infty[$ fulfilling $\lim\limits_{t\to 0+} \alpha(t)/t = 0$. In particular, he proved that under the additional assumption

$$\lim_{t\to 0+} \frac{\alpha(t)}{t^2} = 0,$$

every continuous solution f of inequality (1) is convex. Motivated by this result we deal with an analogue of (1) for Jensen-convexity (i.e., when $t = 1/2$) on an open interval without any regularity assumption.

In [1], Boros and the present author considered the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + c\left[\lambda(1-\lambda)\,|x-y|\right]^p. \tag{2}$$

It was supposed that the function f was defined on a convex, open subset D of a linear normed space, c was a fixed non-negative real number, $p > 1$ was a fixed exponent, and the inequality (2) was satisfied by every $x, y \in D$ and $\lambda \in [0,1]$. The properties of $\mathbb{F}$-differentiability and $\mathbb{F}$-convexity and their connection were described by Boros and Páles in [3]. Based on these results we can show for any function f, which satisfies (2) under the additional restriction $\lambda \in \mathbb{F}$, that f is $\mathbb{F}$-convex.

2. Rolewicz theorem for approximate Jensen-convexity

For the proof of our first theorem we also have to define the difference operator Δ_h^2 by the following recursion. If I is an open interval and $f : I \to \mathbb{R}$, let

$$\Delta_h^1 f(x) = f(x+h) - f(x) \quad (x \in I, h \in \mathbb{R} : x+h \in I),$$
$$\Delta_h^2 f(x) = \Delta_h^1 \Delta_h^1 f(x)$$
$$= f(x+2h) - 2f(x+h) + f(x) \quad (x \in I,\ h \in \mathbb{R} : x+2h \in I),$$

and we define the second order lower Dinghas interval derivative of $f : I \to \mathbb{R}$ at $\xi \in I$ as

$$\underline{D}^2 f(\xi) := \liminf_{(x,h)\to(\xi,0),\, x\le\xi\le x+2h} \frac{\Delta_h^2 f(x)}{h^2}.$$

The following statement is a particular case of a result proved by Gilányi and Páles [4, Corollary 1] (see also the details in the paragraphs preceding Proposition 2 in [2]):

Proposition 2.1. *A function $f : I \to \mathbb{R}$ is Jensen-convex if and only if $\underline{D}^2 f(\xi) \ge 0$ for every $\xi \in I$.*

The following result claims that approximate Jensen-convexity implies Jensen-convexity if the error function ψ is sufficiently small in the vicinity of zero.

Theorem 2.2. *Let $I \subset \mathbb{R}$ be an open interval, d_I be the length of the interval I, and $J_I = [0, d_I[$. Let the function $\psi : J_I \to [0, +\infty[$ satisfy*

$$\lim_{t\to 0+} \frac{\psi(t)}{t^2} = 0.$$

If a function $f : I \to \mathbb{R}$ satisfies

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} + \psi(|x-y|)$$

for all $x, y \in I$, then f is Jensen-convex.

Proof. Let us consider $x \in I$ and a positive real number h such that $y = x + 2h \in I$. Then $\frac{x+y}{2} = x + h,\ |x-y| = 2h$, and we have

$$f(x+2h) - 2f(x+h) + f(x) \ge -2\psi(2h). \tag{3}$$

Dividing by h^2, inequality (3) can be rewritten as

$$\frac{\Delta_h^2 f(x)}{h^2} \ge -2\frac{\psi(2h)}{h^2}. \tag{4}$$

Now let $\xi \in I$ be arbitrary and let us take the $\liminf$ on both sides of (4) as h tends to 0 and x tends to ξ such that $x \le \xi \le x+2h$. We obtain that

$$\underline{D}^2 f(\xi) \ge 0.$$

Applying Proposition 2.1, we get that the function f is Jensen-convex. □

A similar result was established in [10, Theorem 5 and Corollary 6]. In fact, [10, Theorem 5] seems to be more general. However, its proof is more complicated as well. The short proof of our theorem allows generalizations for higher order Jensen-convexity as it appears in [2, Section 3].

3. Approximate convexity with respect to a subfield

Throughout this section, let $\mathbb{F}$ be a subfield of $\mathbb{R}$, X be a vector space over $\mathbb{F}$ and $\mathbb{F}_+ = \mathbb{F} \cap]0, +\infty[$.

In [3] Boros and Páles defined the notions of $\mathbb{F}$-algebraically openness and $\mathbb{F}$-convexity:

Definition 3.1. A subset D of the space X is called $\mathbb{F}$-algebraically open if, for every $x \in D$ and $u \in X$, there exists $\delta > 0$ such that $x + ru \in D$ whenever $r \in \mathbb{F}\cap] - \delta, \delta[$.

We say that D is $\mathbb{F}$-convex if $rx + (1 - r)y \in D$ for every $x, y \in D$ and $r \in [0, 1] \cap \mathbb{F}$.

Let D be an $\mathbb{F}$-algebraically open and $\mathbb{F}$-convex subset of X, $c \geq 0$, and $p > 1$. We use specific differences and difference ratios in order to reformulate the assumption that a function $f : D \to \mathbb{R}$ fulfils inequality (2) for every $x, y \in D$ and $\lambda \in [0, 1] \cap \mathbb{F}$. Our first observation is that the inequality (2) is obviously satisfied if $\lambda = 0$, $\lambda = 1$, or $x = y$. It is therefore sufficient to investigate functions $f : D \to \mathbb{R}$ that fulfil inequality (2) for every $x, y \in D$ and $\lambda \in \mathbb{F}\cap]0, 1[$ such that $x \neq y$.

For our convenience, let us substitute z in the place of x in (2). Clearly, if $y, z \in D$, $y \neq z$, and $\lambda \in \mathbb{F}\cap]0, 1[$, $x = \lambda z + (1 - \lambda)y$, $u = y - z$, $s = \lambda$, and $q = 1 - \lambda$, then $s, q \in \mathbb{F}_+$, $u \in X$, $z = x - qu$, and $y = x + su$. Conversely, if $x \in D$, $u \in X$, and $q, s \in \mathbb{F}_+$ such that $z = x - qu \in X$ and $y = x + su \in X$, then $\lambda = \frac{s}{q+s} \in \mathbb{F}\cap]0, 1[$ and $x = \lambda z + (1 - \lambda)y$. Applying these substitutions, we can formulate the following proposition:

Proposition 3.2. *Let $D \subset X$ be an $\mathbb{F}$-algebraically open and $\mathbb{F}$-convex set, $c \geq 0$, $p > 1$. A function $f : D \to \mathbb{R}$ fulfils inequality (2) for every $x, y \in D$ and $\lambda \in [0, 1] \cap \mathbb{F}$ if and only if f satisfies the inequality*

$$f(x) \leq \frac{s}{q+s} f(x - qu) + \frac{q}{q+s} f(x + su) + c \left[\frac{qs}{q+s} \right]^p |u|^p \tag{5}$$

for every $x \in D$, $s, q \in \mathbb{F}_+$, and $u \in X$ such that $x - qu$, $x + su \in D$.

The proof of the following lemma is simple calculation, so it is left to the reader. We assume that D, c, p and f satisfy the assumptions of the previous proposition.

Lemma 3.3. *Suppose that $x \in D$, $s, q \in \mathbb{F}_+$, and $u \in X$ such that $x - qu$, $x + su \in D$. Then the following two inequalities are equivalent to inequality (5):*

$$\frac{f(x) - f(x - qu)}{q} \leq \frac{f(x + su) - f(x)}{s} + c\left[\frac{qs}{q + s}\right]^{p-1} |u|^p, \tag{6}$$

$$\frac{f(x) - f(x - qu)}{q} \leq \frac{f(x + su) - f(x - qu)}{q + s} + c\left[\frac{s}{q + s}\right]^{p} q^{p-1}|u|^p. \tag{7}$$

If we substitute a in the place of $x - qu$ in (7), we get

$$\frac{f(a + qu) - f(a)}{q} \leq \frac{f(a + (q + s)u) - f(a)}{q + s} + c\left[\frac{s}{q + s}\right]^{p} q^{p-1}|u|^p. \tag{8}$$

We can therefore formulate the following statement.

Lemma 3.4. *Inequality (5) holds for all $x \in D$, $s, q \in \mathbb{F}_+$ and $u \in X$ with $x - qu$, x, $x + su \in D$ if and only if inequality (8) holds for all $a \in D$, $u \in X$, q, $s \in \mathbb{F}_+$ with $a + (q + s)u \in D$.*

With the aid of the above lemmas, we can establish the main result of this section.

Theorem 3.5. *Let $D \subset X$ be an $\mathbb{F}$-algebraically open and $\mathbb{F}$-convex set, $c \geq 0$, $p > 1$ and $f : D \to \mathbb{R}$ such that f satisfies (2) for every $x, y \in D$ and $\lambda \in [0, 1] \cap \mathbb{F}$. Then f satisfies (2) with $c = 0$ as well, thus f is $\mathbb{F}$-convex.*

Proof. Let $x \in D$ and $u \in X$. We define the set $S_{\mathbb{F}} f(x, u)$ as

$$S_{\mathbb{F}} f(x, u) := \left\{\frac{f(x + su) - f(x)}{s} : s \in \mathbb{F}_+ \text{ such that } x + su \in D\right\}$$

and we show that $S_{\mathbb{F}} f(x, u)$ is bounded from below. Let $s, q \in \mathbb{F}_+$ such that $x + su$, $x - qu \in D$. From inequality (6) we get that

$$\begin{aligned}\frac{f(x + su) - f(x)}{s} &\geq \frac{f(x) - f(x - qu)}{q} - c\left[\frac{qs}{q + s}\right]^{p-1} |u|^p \\ &\geq \frac{f(x) - f(x - qu)}{q} - cq^{p-1}|u|^p,\end{aligned}$$

which verifies the boundedness of $S_{\mathbb{F}}f(x,u)$ from below. Denote by $d_{\mathbb{F}}f(x,u)$ the infimum of $S_{\mathbb{F}}f(x,u)$, i.e., $d_{\mathbb{F}}f(x,u) := \inf S_{\mathbb{F}}f(x,u) \in \mathbb{R}$ and let $\varepsilon > 0$. Since

$$\lim_{d\to 0+} c|u|^p d^{p-1} = 0,$$

there exists $\delta > 0$ such that

$$c|u|^p \delta^{p-1} < \frac{\varepsilon}{2}.$$

Moreover, there exists $r \in \mathbb{F}_+$ such that $x + ru \in D$ and

$$\frac{f(x+ru) - f(x)}{r} < d_{\mathbb{F}}f(x,u) + \frac{\varepsilon}{2}.$$

Let $\overline{\delta} := \min\{\delta, r\} > 0$. If for $t \in \mathbb{F}$ we have that $0 < t < \overline{\delta}$, then $0 < t < r$ and writing t in the place of q, $r - t$ in the place of s and x in the place of a in inequality (8), we get

$$\begin{aligned}\frac{f(x+tu) - f(x)}{t} &\le \frac{f(x+ru) - f(x)}{r} + c\left[\frac{r-t}{r}\right]^p t^{p-1}|u|^p \\ &\le \frac{f(x+ru) - f(x)}{r} + c|u|^p t^{p-1} \\ &< d_{\mathbb{F}}f(x,u) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = d_{\mathbb{F}}f(x,u) + \varepsilon.\end{aligned}$$

Hence, we have

$$d_{\mathbb{F}}f(x,u) = \lim_{s\to 0, s\in\mathbb{F}_+} \frac{f(x+su) - f(x)}{s}.$$

Applying inequality (6) for $q, s \in \mathbb{F}_+$ fulfilling $x + su$, $x - qu \in D$, we get

$$\begin{aligned}-\frac{f(x+q(-u)) - f(x)}{q} &= \frac{f(x) - f(x-qu)}{q} \\ &\le \frac{f(x+su) - f(x)}{s} + c\left[\frac{qs}{q+s}\right]^{p-1}|u|^p,\end{aligned}$$

consequently

$$-\lim_{q\to 0, q\in\mathbb{F}_+} \frac{f(x+q(-u)) - f(x)}{q} \le \frac{f(x+su) - f(x)}{s} + cs^{p-1}|u|^p,$$

which yields

$$-d_{\mathbb{F}}f(x,-u) \le \lim_{s\to 0, s\in\mathbb{F}_+} \left[\frac{f(x+su) - f(x)}{s} + cs^{p-1}|u|^p\right]$$

and thus

$$-d_{\mathbb{F}}f(x,-u) \le d_{\mathbb{F}}f(x,u). \tag{9}$$

From inequality (9), for every $q, s \in \mathbb{F}_+$, $u \in X$ and $x \in D$, where $x-qu$, $x+su \in D$ we get the following :

$$\begin{aligned}\frac{f(x)-f(x-qu)}{q} &= -\frac{f(x+q(-u))-f(x)}{q}\\ &\le -d_{\mathbb{F}}f(x,-u) \le d_{\mathbb{F}}f(x,u)\\ &\le \frac{f(x+su)-f(x)}{s}.\end{aligned}$$

We have thus proved that f satisfies the inequality (5) with $c=0$ (i.e., without error term) as well. Applying Proposition 3.2 also with $c=0$, we obtain that f satisfies the inequality (2) with $c=0$, as stated. □

Remark 3.6. Jensen [7] proved (see also [8]) that every Jensen-convex function is $\mathbb{Q}$-convex. Hence, considering the case $\mathbb{F}=\mathbb{Q}$, our last theorem says that approximately Jensen-convex functions in the sense of (2), with $\lambda \in \mathbb{Q}$, are, in fact, Jensen-convex.

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