Publ. Math. Debrecen
89/1-2 (2016), 105–124
DOI: 10.5486/PMD.2016.7391

Some geometrical properties of four-dimensional Lorentzian Damek–Ricci spaces

By JU TAN (Tianjin) and SHAOQIANG DENG (Tianjin)

Abstract. In this paper, we investigate some geometrical properties of four-dimensional Lorentzian Damek–Ricci spaces, including some problems related to Ricci solitons, harmonicity of invariant vector fields and curvature properties. We show that these spaces does not even admit a left-invariant Ricci soliton, although all Riemannian Damek–Ricci spaces are Einstein manifolds. Besides, we determine all the vector fields which are critical points for the energy functional restricted to vector fields of the same length. We also prove that there does not exist any invariant harmonic vector field or invariant vector field which defines a harmonic map. Finally, we determine all the invariant unit time-like vector fields which are spatially harmonic.

1. Introduction

The notion of Damek–Ricci spaces is the one-dimensional extension of generalized Heisenberg groups. These spaces were studied systematically in [4], where they are endowed with a left-invariant Riemannian metric. They are closely related to many special Riemannian manifolds such as symmetric spaces, naturally reductive spaces, Riemannian g.o. spaces, weakly symmetric spaces, harmonic spaces and commutative spaces (see [4]). Recently, the authors in [15] introduced the notion of 4-dimensional Lorentzian Damek–Ricci spaces. These spaces are endowed with left-invariant Lorentzian metrics. In this paper, we shall show that

Mathematics Subject Classification: 53C50, 53C43, 53C25, 53C30.
Key words and phrases: Damek–Ricci spaces, left-invariant Ricci solitons, harmonicity of invariant vector fields, spatially harmonic.
Supported by NSFC (no. 11271198, no. 51535008) and SRFDP of China.

a four-dimensional Lorentzian Damek–Ricci space does not even admit any left-invariant Ricci soliton. In particular, it can not be an Einstein manifold. By contrast, every Damek–Ricci space is an Einstein manifold in the Riemannian case (see [4, p. 85]).

The notion of Ricci solitons is introduced by HAMILTON in [18], which is a natural generalization of Einstein metrics. A Ricci soliton is a pseudo-Riemannian metric g on a smooth manifold M such that there exists a smooth vector field X on M satisfying the following equation:

$$L_X g + \rho = \lambda g, \tag{1.1}$$

where L_X is the Lie derivative in the direction of X, ρ is the Ricci tensor and λ is a real number. A Ricci soliton is said to be shrinking, steady or expanding, according as $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

In the special case that M is a Lie group and g is a left-invariant metric, we say that g is a left-invariant Ricci soliton on M if the above equation (1.1) holds with respect to a left-invariant vector field X.

A homogeneous Ricci soliton on a homogeneous space $M = G/H$ is a G-invariant metric g for which the above equation (1.1) holds [12]. Although there exist three-dimensional Riemannian homogeneous Ricci solitons [2], [21], there are no left-invariant Ricci solitons on three-dimensional Riemannian Lie groups [16] (see also [19], [23]). Left-invariant Ricci solitons on three-dimensional Lorentzian Lie groups were classified in [5], and four-dimensional Ricci solitons on non-reductive homogeneous pseudo-Riemannian manifolds were classified [13] (see also [14]). Recently, the authors in [12] classified homogeneous Ricci solitons on four-dimensional homogeneous pseudo-Riemannian manifolds with non-trivial isotropy.

On the other hand, parallel vector fields are the only ones which define harmonic maps from a compact Riemannian manifold (M, g) to (TM, g^s), where g^s denotes the Sasaki metric on the tangent bundle TM (see [20], [22]). In [17], GIL-MEDRANO showed that critical points of the restricted energy functional $E|_{\mathfrak{X}(M)}$ are again parallel vector fields. However, if g is Lorentzian, then vector fields satisfying some harmonicity properties need not be parallel (see [9], [10]). Since a Riemannian manifold admitting a parallel vector field is locally reducible, and it is also true for a pseudo-Riemannian manifold admitting a parallel vector field which is either space-like or time-like, it is worthwhile to consider non-parallel vector fields which satisfy some harmonicity properties.

In this paper, we investigate some properties of four-dimensional Lorentzian Damek–Ricci spaces. The paper is organized as follows: in Section 2, we present some preliminaries. In Section 3, we consider left-invariant Ricci solitons on four-dimensional Lorentzian Damek–Ricci spaces. We show that no four-dimensional Lorentzian Damek–Ricci space admits a left-invariant Ricci soliton. In particular, such a space cannot be an Einstein manifold. In Section 4, we study some curvature properties of four-dimensional Lorentzian Damek–Ricci spaces. We prove that these spaces are not conformally flat and these Ricci tensors are not Codazzi tensors but Killing tensors. In Section 5, we investigate the harmonicity of invariant vector fields on four-dimensional Lorentzian Damek–Ricci spaces. We show that there does not exist any invariant harmonic vector fields and invariant vector fields defining harmonic maps. We also determine all the invariant unit time-like vector fields which are spatially harmonic.

2. Preliminaries

Let (M, g) be a compact connected and oriented n-dimensional pseudo-Riemannian manifold. The tangent bundle TM of M can be equipped with the Sasaki metric g^s (see [10]). Given a smooth vector field V on M, the energy of a smooth vector field $V : (M, g) \to (TM, g^s)$ on M is defined by:

$$E(V) = \frac{n}{2}\mathrm{vol}(M, g) + \frac{1}{2}\int_M \|\nabla V\|^2 dv. \tag{2.1}$$

(In the non-compact case, one works over relatively compact domains, see [9]). V is said to define a harmonic map if $V : (M, g) \to (TM, g^s)$ is a critical point for the above energy functional. The Euler–Lagrange equations characterize vector fields V defining harmonic maps as the ones whose tension field $\tau(V) = \mathrm{tr}(\nabla^2 V)$ vanishes. Consequently, V defines a harmonic map from (M, g) to (TM, g^s) if and only if

$$\begin{cases} \nabla^*\nabla V = 0, \\ \mathrm{tr}[R(\nabla_. V, V).)] = 0, \end{cases} \tag{2.2}$$

where with respect to a pseudo-orthonormal local frame $\{e_1, \dots, e_n\}$ on (M, g), with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices i, one has

$$\nabla^*\nabla V = \sum_i \varepsilon_i (\nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V).$$

A smooth vector field V is said to be a harmonic section if it is a critical point of the vertical energy E^v, here $E^v(V) = \frac{1}{2}\int_M \|\nabla V\|^2 dv$. The corresponding Euler–Lagrange equations are given by

$$\nabla^*\nabla V = 0. \tag{2.3}$$

In the non-compact case, equation (2.3) (respectively, (2.2)) is used as the definition of harmonic vector fields (respectively, of vector fields defining harmonic maps).

Let ρ be a non-zero real number, and denote $\mathfrak{X}^\rho(M)=\{V\in\mathfrak{X}(M):\|V\|^2=\rho\}$. We consider the vector fields $V\in\mathfrak{X}^\rho(M)$ which are critical points for the energy functional $E|_{\mathfrak{X}^\rho(M)}$, restricted to vector fields of the same length. The Euler–Lagrange equations of this variational condition show that V is a harmonic vector field if and only if

$$\nabla^*\nabla V \text{ is collinear to } V. \tag{2.4}$$

This characterization is well known in the Riemannian case [1], [24], [25]. If V is not light-like, the same argument applies to the pseudo-Riemannian settings [10]. Even if V is a light-like vector field, (2.4) is still a sufficient condition for V to be a critical point for the energy functional $E|_{\mathfrak{X}^0(M)}$, restricted to light-like vector fields (see [10, Theorem 26]). Usually, condition (2.4) is used as a definition of critical points for the energy functional $E|_{\mathfrak{X}^\rho(M)}$ in the non-compact case.

Let V be a unit time-like vector field on a Lorentzian manifold (M,g). The space-like energy of V is defined to be the integral of the square norm of the restriction of ∇V to the distribution $V^\perp$. We say that V is spatially harmonic if it is a critical point of the space-like energy. The Euler–Lagrange equations then imply that V is spatially harmonic if and only if

$$\widehat{X}_v := -\nabla^*\nabla V - \nabla_V\nabla_V V - \operatorname{div}(V)\nabla_V V + (\nabla V)^t(\nabla_V V) \text{ is collinear to } V. \tag{2.5}$$

It is easy to see that conditions (2.4) and (2.5) coincide for geodesic vector fields.

Next, we recall the structures of the Lorentzian Damek–Ricci spaces from [15].

A generalized Riemannian Heisenberg algebra is a two-step nilpotent Lie algebra $\mathfrak{n}$ with an positive inner product $\langle,\rangle$ such that if $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $\mathfrak{p}=\mathfrak{z}^\perp$, then the map $J_Z:\mathfrak{p}\to\mathfrak{p}$ given by

$$\langle J_Z X, Y\rangle = \langle [X,Y], Z\rangle$$

for $X,Y\in\mathfrak{p}$ and $Z\in\mathfrak{z}$, satisfies the identity $J_Z^2=-|Z|^2 I$ for every $Z\in\mathfrak{z}$. The associated simple connected Lie group, endowed with the induced left-invariant Riemannian metric, is called a generalized Riemannian Heisenberg group.

A generalized Lorentzian Heisenberg algebra is introduced in [15]. It is a two-step nilpotent Lie algebra $\mathfrak{n}$ with a Lorentzian inner product $\langle,\rangle$ which is Lorentzian restricted to the center $\mathfrak{z}$ of $\mathfrak{n}$, and positive definite restricted to $\mathfrak{p}=\mathfrak{z}^\perp$. Moreover, the map $J_Z:\mathfrak{p}\to\mathfrak{p}$ given by

$$\langle J_Z X, Y\rangle = \langle [X,Y], Z\rangle$$

for $X, Y \in \mathfrak{p}$ and $Z \in \mathfrak{z}$, satisfies the condition

$$\begin{cases} J_Z^2 = -|Z|^2 I, & \text{if } Z \text{ is space-like,} \\ J_Z^2 = |Z|^2 I, & \text{if } Z \text{ is time-like.} \end{cases}$$

The associated simple connected Lie group with the induced left-invariant Lorentzian metric, is called a generalized Lorentzian Heisenberg group.

From [15], we know that there exist two kinds of Lorentzian Damek–Ricci spaces. A Lie algebra $\mathfrak{s}$ of the first kind of $(n+1)$-dimensional Lorentzian Damek–Ricci spaces is a direct sum of an n-dimensional generalized Riemannian Heisenberg algebra $\mathfrak{n}$ and a one-dimensional vector space $\mathfrak{a}$. Each vector in $\mathfrak{s}$ can be uniquely written as $U + X + sA$, where $U \in \mathfrak{p}$, $X \in \mathfrak{z}$, $s \in \mathbb{R}$ and A is a non-zero vector in $\mathfrak{a}$. We will always use the symbols U, V for vectors in $\mathfrak{p}$, X, Y for vectors in $\mathfrak{z}$ and r, s for real numbers. In [15], the inner product $\langle .,. \rangle$ and Lie brackets $[.,.]$ on $\mathfrak{s}$ are defined by

$$\langle U + X + rA, V + Y + sA \rangle = \langle U + X, V + Y \rangle_{\mathfrak{n}} - rs,$$

and

$$[U + X + rA, V + Y + sA] = [U, V]_{\mathfrak{n}} + \frac{1}{2} rV - \frac{1}{2} sU + rY - sX.$$

With respect to these brackets, $\mathfrak{s}$ becomes a Lie algebra with a Lorentzian metric. The corresponding connected simply connected Lie group attached to $\mathfrak{s}$, endowed with the induced left-invariant Lorentzian metric, is called a Lorentzian Damek–Ricci space of the first kind, denoted as $\mathbb{S}^1_{n+1}$. The Levi–Civita connection ∇ of $\mathbb{S}^1_{n+1}$ is given by

$$\begin{aligned} &\nabla_{V+Y+sA}(U + X + rA) \\ &\quad = -\frac{1}{2}\{J_Y U + J_X V + rV + [U, V] + 2rY + \langle U, V \rangle A + 2\langle X, Y \rangle A\}. \end{aligned}$$

On the other hand, from [15], we know that a Lie algebra $\mathfrak{s}'$ of the second kind of $(n+1)$-dimensional Lorentzian Damek–Ricci spaces is a direct sum of an n-dimensional generalized Lorentzian Heisenberg algebra $\mathfrak{n}$ and a one-dimensional vector space $\mathfrak{a}$. The vector decomposition and the Lie brackets in $\mathfrak{s}'$ are given in the same way as above, but the metric is given by

$$\langle U + X + rA, V + Y + sA \rangle = \langle U + X, V + Y \rangle_{\mathfrak{n}} + rs.$$

The corresponding connected simply connected Lie group attached to $\mathfrak{s}'$, endowed with the induced left-invariant Lorentzian metric, is called a Lorentzian Damek–Ricci space of the second kind, denoted as $\mathbb{S}^n_{n+1}$. The Levi–Civita connection ∇ of $\mathbb{S}^n_{n+1}$ is given by

$$\begin{aligned}&\nabla_{V+Y+sA}(U+X+rA)\\&\quad=-\frac{1}{2}\{J_YU+J_XV+rV+[U,V]+2rY-\langle U,V\rangle A-2\langle X,Y\rangle A\}.\end{aligned}$$

3. Left-invariant Ricci solitons on 4-dimensional Lorentzian Damek–Ricci spaces

In this section, we study left-invariant Ricci solitons on 4-dimensional Lorentzian Damek–Ricci spaces. This will be completed through a case by case consideration.

3.1. The $\mathbb{S}^1_4$ case. In [15], the left-invariant Lorentzian metric g on the 4-dimensional space $\mathbb{S}^1_4$ is given by

$$g=e^{-t}dx^2+e^{-t}dy^2+e^{-2t}\left(dz+\frac{c}{2}ydx-\frac{c}{2}xdy\right)^2-dt^2,$$

where $c\in\mathbb{R}$. The Lie algebra $\mathfrak{s}_4$ of $\mathbb{S}^1_4$ has an orthonormal basis

$$e_1=e^{\frac{t}{2}}\left(\frac{\partial}{\partial x}-\frac{cy}{2}\frac{\partial}{\partial z}\right),\quad e_2=e^{\frac{t}{2}}\left(\frac{\partial}{\partial y}+\frac{cx}{2}\frac{\partial}{\partial z}\right),\quad e_3=e^t\frac{\partial}{\partial z},\ e_4=\frac{\partial}{\partial t},$$

where e_1,e_2,e_3 are space-like and e_4 is time-like. The Lie brackets are given by

$$\begin{array}{lll}[e_1,e_2]=ce_3, & [e_1,e_3]=0, & [e_1,e_4]=-\frac{1}{2}e_1,\\ [e_2,e_3]=0, & [e_2,e_4]=-\frac{1}{2}e_2, & [e_3,e_4]=-e_3.\end{array}\tag{3.1}$$

By the definition of the map J_Z, we have $c^2=1$. The well-known Koszul formula can be used to determine the Levi–Civita connection ∇ of g. Set $\Lambda_i=\nabla_{e_i}$. Then, with respect to the pseudo-orthonormal basis $\{e_1,e_2,e_3,e_4\}$, where e_4 is time-like, we have

$$\Lambda_{e_1}=\begin{pmatrix}0&0&0&-\frac{1}{2}\\0&0&-\frac{c}{2}&0\\0&\frac{c}{2}&0&0\\-\frac{1}{2}&0&0&0\end{pmatrix},\quad \Lambda_{e_2}=\begin{pmatrix}0&0&\frac{c}{2}&0\\0&0&0&-\frac{1}{2}\\-\frac{c}{2}&0&0&0\\0&-\frac{1}{2}&0&0\end{pmatrix},$$

$$\Lambda_{e_3} = \begin{pmatrix} 0 & \frac{c}{2} & 0 & 0 \\ -\frac{c}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \Lambda_{e_4} = \mathbb{O}_{4\times 4}. \tag{3.2}$$

Using the identities $R(e_i, e_j) = \nabla_{[e_i,e_j]} - \Lambda_i\Lambda_j + \Lambda_j\Lambda_i$, we can determine the curvature as follows

$$\begin{array}{lll} R(e_1,e_2)e_1 = -\frac{1}{2}e_2, & R(e_1,e_2)e_2 = -\frac{1}{2}e_1, & R(e_1,e_2)e_3 = -\frac{c}{2}e_4, \\ R(e_1,e_2)e_4 = -\frac{c}{2}e_3, & R(e_1,e_3)e_1 = \frac{3}{4}e_3, & R(e_1,e_3)e_2 = -\frac{c}{4}e_4, \\ R(e_1,e_3)e_3 = -\frac{3}{4}e_1, & R(e_1,e_3)e_4 = -\frac{c}{4}e_2, & R(e_1,e_4)e_1 = \frac{1}{4}e_4, \\ R(e_1,e_4)e_2 = -\frac{c}{4}e_3, & R(e_1,e_4)e_3 = \frac{c}{4}e_2, & R(e_1,e_4)e_4 = \frac{1}{4}e_1, \\ R(e_2,e_3)e_1 = \frac{c}{4}e_4, & R(e_2,e_3)e_2 = \frac{3}{4}e_3, & R(e_2,e_3)e_3 = -\frac{3}{4}e_2, \\ R(e_2,e_3)e_4 = \frac{c}{4}e_1, & R(e_2,e_4)e_1 = \frac{c}{4}e_3, & R(e_2,e_4)e_2 = \frac{1}{4}e_4, \\ R(e_2,e_4)e_3 = -\frac{c}{4}e_1, & R(e_2,e_4)e_4 = \frac{1}{4}e_2, & R(e_3,e_4)e_1 = \frac{c}{2}e_2, \\ R(e_3,e_4)e_2 = -\frac{c}{2}e_1, & R(e_3,e_4)e_3 = e_4, & R(e_3,e_4)e_4 = e_3. \end{array} \tag{3.3}$$

Applying the Ricci tensor formula $\rho(X,Y) = \sum_{i=1}^{4} \varepsilon_i g(R(X,e_i)Y,e_i)$, we get

$$(\rho)_{ij} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{5}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}$$

On the other hand, for an arbitrary left-invariant vector field $X = \sum_{i=1}^{4} K_i e_i$ on $\mathbb{S}^1_4$, we have

$$\begin{aligned} \nabla_{e_1} X &= -\frac{1}{2}K_1e_4 + \frac{c}{2}K_2e_3 - \frac{c}{2}K_3e_2 - \frac{1}{2}K_4e_1, \\ \nabla_{e_2} X &= -\frac{c}{2}K_1e_3 - \frac{1}{2}K_2e_4 + \frac{c}{2}K_3e_1 - \frac{1}{2}K_4e_2, \\ \nabla_{e_3} X &= -\frac{c}{2}K_1e_2 + \frac{c}{2}K_2e_1 - K_3e_4 - K_4e_3, \\ \nabla_{e_4} X &= 0. \end{aligned} \tag{3.4}$$

By the identity $(L_Xg)(Y,Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)$, we have

$$L_Xg = \begin{pmatrix} -K_4 & 0 & cK_2 & \frac{1}{2}K_1 \\ 0 & -K_4 & -cK_1 & \frac{1}{2}K_2 \\ cK_2 & -cK_1 & -2K_4 & K_3 \\ \frac{1}{2}K_1 & \frac{1}{2}K_2 & K_3 & 0 \end{pmatrix} \tag{3.5}$$

3.2. The $\mathbb{S}_4^3$ case. In [15], the left-invariant Lorentzian metric g on the 4-dimensional space $\mathbb{S}_4^3$ is given by:

$$g = e^{-t}dx^2 + e^{-t}dy^2 - e^{-2t}\left(dz + \frac{c}{2}ydx - \frac{c}{2}xdy\right)^2 + dt^2,$$

where $c \in \mathbb{R}$. The Lie algebra $\mathfrak{s}_4^{'}$ of $\mathbb{S}_4^3$ has an orthonormal basis

$$e_1 = e^{\frac{t}{2}}\left(\frac{\partial}{\partial x} - \frac{cy}{2}\frac{\partial}{\partial z}\right), \quad e_2 = e^{\frac{t}{2}}\left(\frac{\partial}{\partial y} + \frac{cx}{2}\frac{\partial}{\partial z}\right), \quad e_3 = e^{t}\frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t},$$

where e_1, e_2, e_4 are space-like and e_3 is time-like. The Lie brackets are given by

$$\begin{array}{lll} [e_1, e_2] = ce_3, & [e_1, e_3] = 0, & [e_1, e_4] = -\frac{1}{2}e_1, \\ [e_2, e_3] = 0, & [e_2, e_4] = -\frac{1}{2}e_2, & [e_3, e_4] = -e_3. \end{array} \tag{3.6}$$

One can also prove that $c^2 = 1$ by the definition of the map J_Z. The well-known Koszul formula can be used to determine the Levi–Civita connection ∇ of g. Set $\Lambda_i = \nabla_{e_i}$. Then, with respect to the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$, where e_3 is time-like, we have:

$$\Lambda_{e_1} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{c}{2} & 0 \\ 0 & \frac{c}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_{e_2} = \begin{pmatrix} 0 & 0 & -\frac{c}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ -\frac{c}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix},$$

$$\Lambda_{e_3} = \begin{pmatrix} 0 & -\frac{c}{2} & 0 & 0 \\ \frac{c}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \Lambda_{e_4} = \mathbb{O}_{4\times 4}. \tag{3.7}$$

where $\mathbb{O}_{4\times 4}$ denotes the 4×4 matrix whose entries are all zero.

Using the identities $R(e_i, e_j) = \nabla_{[e_i,e_j]} - \Lambda_i\Lambda_j + \Lambda_j\Lambda_i$, we can determine the curvature as the following:

$$\begin{array}{lll} R(e_1, e_2)e_1 = \frac{1}{2}e_2, & R(e_1, e_2)e_2 = -\frac{1}{2}e_1, & R(e_1, e_2)e_3 = -\frac{c}{2}e_4, \\ R(e_1, e_2)e_4 = -\frac{c}{2}e_3, & R(e_1, e_3)e_1 = -\frac{3}{4}e_3, & R(e_1, e_3)e_2 = -\frac{c}{4}e_4, \\ R(e_1, e_3)e_3 = -\frac{3}{4}e_1, & R(e_1, e_3)e_4 = \frac{c}{4}e_2, & R(e_1, e_4)e_1 = -\frac{1}{4}e_4, \\ R(e_1, e_4)e_2 = -\frac{c}{4}e_3, & R(e_1, e_4)e_3 = -\frac{c}{4}e_2, & R(e_1, e_4)e_4 = \frac{1}{4}e_1, \\ R(e_2, e_3)e_1 = \frac{c}{4}e_4, & R(e_2, e_3)e_2 = -\frac{3}{4}e_3, & R(e_2, e_3)e_3 = -\frac{3}{4}e_2, \\ R(e_2, e_3)e_4 = -\frac{c}{4}e_1, & R(e_2, e_4)e_1 = \frac{c}{4}e_3, & R(e_2, e_4)e_2 = -\frac{1}{4}e_4, \\ R(e_2, e_4)e_3 = \frac{c}{4}e_1, & R(e_2, e_4)e_4 = \frac{1}{4}e_2, & R(e_3, e_4)e_1 = -\frac{c}{2}e_2, \\ R(e_3, e_4)e_2 = \frac{c}{2}e_1, & R(e_3, e_4)e_3 = e_4, & R(e_3, e_4)e_4 = e_3. \end{array} \tag{3.8}$$

Applying the Ricci tensor formula $\rho(X,Y) = \sum\limits_{i=1}^{4} \varepsilon_i g(R(X,e_i)Y,e_i)$, we get

$$(\rho)_{ij} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{5}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}.$$

Now, for an arbitrary left-invariant vector field $X = \sum\limits_{i=1}^{4} K_i e_i$ on $\mathbb{S}_4^3$, we have

$$\begin{aligned} \nabla_{e_1} X &= \frac{1}{2}K_1 e_4 + \frac{c}{2}K_2 e_3 + \frac{c}{2}K_3 e_2 - \frac{1}{2}K_4 e_1, \\ \nabla_{e_2} X &= -\frac{c}{2}K_1 e_3 + \frac{1}{2}K_2 e_4 - \frac{c}{2}K_3 e_1 - \frac{1}{2}K_4 e_2, \\ \nabla_{e_3} X &= \frac{c}{2}K_1 e_2 - \frac{c}{2}K_2 e_1 - K_3 e_4 - K_4 e_3, \\ \nabla_{e_4} X &= 0. \end{aligned} \tag{3.9}$$

By the left invariance, we have $(L_X g)(Y,Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)$. This implies that

$$L_X g = \begin{pmatrix} -K_4 & 0 & -cK_2 & \frac{1}{2}K_1 \\ 0 & -K_4 & cK_1 & \frac{1}{2}K_2 \\ -cK_2 & cK_1 & 2K_4 & -K_3 \\ \frac{1}{2}K_1 & \frac{1}{2}K_2 & -K_3 & 0 \end{pmatrix}. \tag{3.10}$$

Now we can prove

Proposition 3.1. *A 4-dimensional Lorentzian Damek–Ricci space does not admit any left-invariant Ricci soliton.*

Proof. We first consider the $\mathbb{S}_4^1$ case. If it admits a left-invariant Ricci soliton, then by the Ricci soliton formula (1.1), we get the following system of equations:

$$\begin{cases} \lambda = \frac{3}{2}, \\ K_1 = K_2 = K_3 = 0, \\ -K_4 + \frac{1}{2} = \lambda, \\ -2K_4 + \frac{5}{2} = \lambda. \end{cases}$$

From this, we get $K_4 = 0$. So $-c^2 = 1$, which is a contradiction. The proof for the $\mathbb{S}_4^3$ case is similar. □

4. Curvature of 4-dimensional Lorentzian Damek–Ricci spaces

The properties of curvature of 4-dimensional generalized symmetric spaces were investigated in [11]. In this section, we mainly study the curvature properties of 4-dimensional Lorentzian Damek–Ricci spaces.

A pseudo-Riemannian manifold (M, g) is said to be in class $\mathcal{A}$ if the Ricci tensor is cyclic-parallel, i.e., $\nabla_X\rho(Y,Z)+\nabla_Y\rho(Z,X)+\nabla_Z\rho(X,Y)=0$, or equivalently, it is a Killing tensor, i.e., $\nabla_X\rho(X,X)=0$. It is said to be in class $\mathcal{B}$ if its Ricci tensor is a Codazzi tensor, i.e., $\nabla_X\rho(Y,Z)=\nabla_Y\rho(X,Z)$, where

$$\nabla_i\rho_{jk} = -\sum_t(\varepsilon_j B_{ijt}\rho_{tk}+\varepsilon_k B_{ikt}\rho_{tj}), \tag{4.1}$$

here the B_{ijk} components are determined by $\nabla_{e_i}e_j=\sum_k \varepsilon_j B_{ijk}e_k$, and ρ_{tk} are the tensor Ricci components. Note that $B_{ikj}=-B_{ijk}$, for all i,j,k. In particular, $B_{ijj}=0$ for all indices i,j. For more detail, see [7], [8].

Proposition 4.1. *Every 4-dimensional Lorentzian Damek–Ricci space belongs to class $\mathcal{A}$ but not to class $\mathcal{B}$.*

Proof. In the $\mathbb{S}_4^1$ case, from (4.1), it is easily seen that $\nabla_i\rho_{ii}=0, i=1,2,3,4$. So it belongs to class $\mathcal{A}$. On the other hand, by (3.2), we have

$$\begin{aligned}
&B_{123}=\frac{c}{2},\quad B_{132}=-\frac{c}{2},\quad B_{213}=-\frac{c}{2},\quad B_{231}=\frac{c}{2}.\\
&\nabla_1\rho_{23}=-B_{123}\rho_{33}-B_{132}\rho_{22}=-c,\\
&\nabla_2\rho_{13}=-B_{213}\rho_{33}-B_{231}\rho_{11}=c.
\end{aligned}$$

Notice that $c\neq 0$. Thus it does not belong to class $\mathcal{B}$. The proof for the $\mathbb{S}_4^3$ case is similar. □

Now we recall the following theorem from [3].

Theorem 4.2. *A pseudo-Riemannian manifold (M^n, g) of dimension $n\geq 4$, is conformally flat if and only if its Weyl curvature tensor vanishes, that is,*

$$\begin{aligned}
R(X,Y,Z,W) = {}&\frac{1}{n-2}(g(X,Z)\rho(Y,W)+g(Y,W)\rho(X,Z)\\
&-g(X,W)\rho(Y,Z)-g(Y,Z)\rho(X,W))\\
&-\frac{\tau}{(n-1)(n-2)}(g(X,Z)g(Y,W)-g(Y,Z)g(X,W)),
\end{aligned} \tag{4.2}$$

where X,Y,Z,W are vector fields and τ is the scalar curvature.

Now we can prove

Proposition 4.3. *Every 4-dimensional Lorentzian Damek–Ricci space is not conformally flat.*

Proof. We first consider the $\mathbb{S}_4^1$ case. Since $R(X,Y,Z,W)$=$g(R(X,Y)Z,W)$, it follows form (3.3) that $R_{1234} = \frac{c}{2}$. By (4.2), we have $R_{1234} = 0$. So $\mathbb{S}_4^1$ is not conformally flat. Now, we consider $\mathbb{S}_4^3$. By (3.8) we have $R_{1234} = -\frac{c}{2}$. On the other hand, by (4.2), we also have $R_{1234} = 0$. Hence $\mathbb{S}_4^3$ is also not conformally flat. □

A vector field V is called a geodesic vector field if $\nabla_V V = 0$, and it is called a Killing vector field if $L_V g = 0$, where L denotes the Lie derivative. It is easily seen that X is Killing vector field if and only if $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for all $Y, Z \in \mathfrak{X}(M)$. A vector field V is called a parallel vector field if $\nabla_X V = 0$ for all $X \in \mathfrak{X}(M)$. It is obvious that parallel vector fields are both geodesic vector fields and Killing vector fields. From (3.4), (3.5) and (3.9), (3.10), we have:

Proposition 4.4. *On a 4-dimensional Lorentzian Damek–Ricci space, a left-invariant vector field is neither a parallel vector field, nor a Killing vector field.*

By (3.4) and (3.9) and some direct calculations, we get the following result.

Proposition 4.5. *Let V be a left-invariant vector field on a 4-dimensional Lorentzian Damek–Ricci space, then V is geodesic if and only if $V = ae_4, a \in \mathbb{R}$.*

A r-dimensional distribution $\mathcal{D}$ on a manifold is said to be parallel if $\nabla_X \mathcal{D} \subset \mathcal{D}$, i.e., if $\nabla_X Y \in \mathcal{D}$ for all $Y \in \mathcal{D}$ and any $X \in \mathfrak{X}(M)$. A Walker manifold is a pseudo-Riemannian manifold (M, g) which admits a parallel null distribution $\mathcal{D}$. Such structures possess many interesting properties with no Riemannian counterpart. For more detail, see [6]. Now we prove

Proposition 4.6. *A 4-dimensional Lorentzian Damek–Ricci space does not admit any 1-dimensional parallel null distribution.*

Proof. We first consider the $\mathbb{S}_4^1$ case. Set $X = K_1e_1 + K_2e_2 + K_3e_3 + K_4e_4$, and suppose $\mathcal{D} = \text{span}(X)$ is an invariant null parallel line field. Then there exist parameters w_1, w_2, w_3, w_4 satisfying the following equations:

$$\nabla_{e_1} X = w_1 X, \quad \nabla_{e_2} X = w_2 X, \quad \nabla_{e_3} X = w_3 X, \quad \nabla_{e_4} X = w_4 X,$$

From the first equation, we have

$$\begin{cases} -\frac{1}{2}K_4 & = w_1 K_1, \\ -\frac{c}{2}K_3 & = w_1 K_2, \\ \frac{c}{2}K_2 & = w_1 K_3, \\ -\frac{1}{2}K_1 & = w_1 K_4. \end{cases}$$

Hence $K_2 = K_3 = 0$. Since $\nabla_{e_2} X = w_2 X$, we have $K_1 = K_4 = 0$. So a non-trivial solution can not occur. The proof for the $\mathbb{S}_4^3$ case is similar. □

5. Harmonicity of invariant vector fields

In this section, we investigate the harmonicity of invariant vector fields on the 4-dimensional Lorentzian Damek–Ricci spaces. As in the previous sections, we will study the problem case by case.

5.1. The $\mathbb{S}_4^1$ case. A left-invariant vector field V on the Lorentzian Damek–Ricci space $\mathbb{S}_4^1$ is uniquely determined by its components with respect to the pseudo-orthonormal basis $\{e_i\}$ for which (3.1) holds. Thus it can be written as $V = K_1e_1 + K_2e_2 + K_3e_3 + K_4e_4$, for some real constants K_1, K_2, K_3, K_4. Notice that the constant norm of V is given by $\|V\|^2 = K_1^2 + K_2^2 + K_3^2 - K_4^2$.

Applying the equations (3.2) and (3.4) to the calculation of $\nabla_{e_i}\nabla_{e_i}V$ and $\nabla_{\nabla_{e_i}e_i}V$ for $i = 1, 2, 3, 4$, we get

$$\begin{aligned} \nabla_{e_1}\nabla_{e_1}V &= \frac{1}{4}K_1e_1 - \frac{1}{4}K_2e_2 - \frac{1}{4}K_3e_3 + \frac{1}{4}K_4e_4, \\ \nabla_{e_2}\nabla_{e_2}V &= -\frac{1}{4}K_1e_1 + \frac{1}{4}K_2e_2 - \frac{1}{4}K_3e_3 + \frac{1}{4}K_4e_4, \\ \nabla_{e_3}\nabla_{e_3}V &= -\frac{1}{4}K_1e_1 - \frac{1}{4}K_2e_2 + K_3e_3 + K_4e_4. \end{aligned}$$

Note that $\nabla_{e_4}\nabla_{e_4}V = 0$ and $\nabla_{\nabla_{e_i}e_i}V = 0$ for $i = 1, 2, 3, 4$. By the equation $\nabla^*\nabla V = \sum_{i=1}^{4} \varepsilon_i(\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V)$, we have

$$\nabla^*\nabla V = -\frac{1}{4}K_1e_1 - \frac{1}{4}K_2e_2 + \frac{1}{2}K_3e_3 + \frac{3K_4}{2}e_4. \tag{5.1}$$

Thus, we have the following

Theorem 5.1. *There does not exist left-invariant harmonic vector fields on the Lorentzian Damek–Ricci space* $\mathbb{S}_4^1$. *Moreover, none of the left-invariant vector fields on the Lorentzian Damek–Ricci space* $\mathbb{S}_4^1$ *defines a harmonic map from* $\mathbb{S}_4^1$ *to* $(T\mathbb{S}_4^1, g^s)$.

From (5.1), we obtain:

$$\begin{aligned}
\nabla^*\nabla V &= -\frac{1}{4}V + \frac{3}{4}K_3e_3 + \frac{7}{4}K_4e_4,\\
\nabla^*\nabla V &= \frac{1}{2}V + K_4e_4 - \frac{3}{4}K_1e_1 - \frac{3}{4}K_2e_2,\\
\nabla^*\nabla V &= \frac{3}{2}V - \frac{7}{4}K_1e_1 - \frac{7}{4}K_2e_2 - K_3e_3.
\end{aligned}$$

So we have the following

Theorem 5.2. *Let* V *be a left-invariant vector field on* $\mathbb{S}_4^1$. *Then* V *is a critical point for the energy functional restricted to vector fields of the same length if and only if* $V = K_1e_1 + K_2e_2$, *or* $V = K_3e_3$, *or* $V = K_4e_4$.

We now determine spatially harmonic vector fields on $\mathbb{S}_4^1$. Let $V = K_1e_1 + K_2e_2 + K_3e_3 + K_4e_4$ be a unit time-like vector field. Then we have

$$\begin{aligned}
\operatorname{div}(V) &= \sum_{i=1}^{4}\varepsilon_i g(\nabla_{e_i}V, e_i) = -2K_4,\\
\nabla_V V &= \left(cK_2K_3 - \frac{1}{2}K_1K_4\right)e_1 - \left(cK_1K_3 + \frac{1}{2}K_2K_4\right)e_2\\
&\quad - K_3K_4e_3 - \left(\frac{1}{2}K_1^2 + \frac{1}{2}K_2^2 + K_3^2\right)e_4,\\
\nabla_V\nabla_V V &= \left(\frac{1}{4}K_1^3 + \frac{1}{4}K_1K_2^2 - \frac{3c}{4}K_2K_3K_4\right)e_1\\
&\quad + \left(\frac{1}{4}K_2^3 + \frac{1}{4}K_1^2K_2 + \frac{3c}{4}K_1K_3K_4\right)e_2 + K_3^3e_3\\
&\quad + \left(K_3^2K_4 + \frac{1}{4}K_1^2K_4 + \frac{1}{4}K_2^2K_4\right)e_4.
\end{aligned}$$

Since

$$(\nabla V)^t\nabla_V V = \sum_{i=1}^{4}\varepsilon_i g(\nabla_V V, \nabla_{e_i}V)e_i,$$

We have

$$(\nabla V)^t\nabla_V V = \left(-\frac{3c}{4}K_2K_3K_4 + \frac{1}{4}K_1K_4^2 - \frac{1}{4}K_1K_2^2 - \frac{1}{4}K_1^3\right)e_1$$

$$+\left(\frac{3c}{4}K_1K_3K_4+\frac{1}{4}K_2K_4^2-\frac{1}{4}K_1^2K_2-\frac{1}{4}K_2^3\right)e_2$$
$$+(-K_3^3+K_3K_4^2)e_3.$$

Therefore, we have

$$\begin{aligned}\widehat{X}_V &= -\nabla^*\nabla V-\nabla_V\nabla_V V-div(V)\nabla_V V+(\nabla V)^t(\nabla_V V)\\ &=\left(-\frac{3}{4}K_1K_4^2-\frac{1}{2}K_1K_2^2-\frac{1}{2}K_1^3+\frac{1}{4}K_1+2cK_2K_3K_4\right)e_1\\ &\quad+\left(-\frac{3}{4}K_2K_4^2-\frac{1}{2}K_1^2K_2-\frac{1}{2}K_2^3+\frac{1}{4}K_2-2cK_1K_3K_4\right)e_2\\ &\quad+\left(-2K_3^3-K_3K_4^2-\frac{1}{2}K_3\right)e_3\\ &\quad-\left(3K_3^2K_4+\frac{5}{4}K_1^2K_4+\frac{5}{4}K_2^2K_4+\frac{3}{2}K_4\right)e_4.\end{aligned}$$

So, V satisfies (2.5) if and only if there exists a real constant λ such that

$$\begin{cases}-\frac{3}{4}K_1K_4^2-\frac{1}{2}K_1K_2^2-\frac{1}{2}K_1^3+\frac{1}{4}K_1+2cK_2K_3K_4=\lambda K_1,\\ -\frac{3}{4}K_2K_4^2-\frac{1}{2}K_1^2K_2-\frac{1}{2}K_2^3+\frac{1}{4}K_2-2cK_1K_3K_4=\lambda K_2,\\ -2K_3^3-K_3K_4^2-\frac{1}{2}K_3=\lambda K_3,\\ -3K_3^2K_4-\frac{5}{4}K_1^2K_4-\frac{5}{4}K_2^2K_4-\frac{3}{2}K_4=\lambda K_4.\end{cases}\tag{5.2}$$

The system of equations (5.2) completely characterizes spatially harmonic unit time-like invariant vector fields (which must satisfy the additional condition $\|V\|^2 = K_1^2+K_2^2+K_3^2-K_4^2=-1$). Now we can prove

Theorem 5.3. *A time-like unit left-invariant vector field V on the Lorentzian Damek–Ricci space $\mathbb{S}_4^1$ is a spatially harmonic vector field if and only if there exist real numbers K_3, K_4 such that $K_4^2=1+K_3^2$ and $V=K_3e_3+K_4e_4$.*

Proof. We first prove the "if" part. Suppose $V=K_3e_3+K_4e_4$, where $K_3^2=1+K_4^2$. Then by (5.2), we have $\lambda=-3K_3^2-\frac{3}{2}$. Thus V is a spatially harmonic vector field.

Now we prove the "only if" part. For this, we need to prove the existence of non-trivial solutions of (5.2). Since $K_4\neq 0$, from the last equation of (5.2), we have $\lambda=-\frac{3}{2}-\frac{5}{4}K_2^2-\frac{5}{4}K_1^2-3K_3^2$. Then, from the first and second equations of (5.2), we have $(K_1^2+K_2^2)(\frac{9}{4}K_3^2+1)=0$. Thus $K_1=K_2=0$. Since $K_4^2=1+K_3^2$, the third equation also holds. Therefore, we have $V=K_3e_3+K_4e_4$ and $K_4^2=1+K_3^2$. $\square$

Next we calculate the energy of a smooth vector field $V : (M, g) \to (TM, g^s)$ on $\mathbb{S}_4^1$. Since $\mathbb{S}_4^1$ is not compact, we suppose that D is a relatively compact domain and calculate the energy of $V|_D$.

Proposition 5.4. *Let V be a smooth left-invariant vector field on $\mathbb{S}_4^1$. Then the energy of $V|_D$ is*

$$E_D(V) = \left(2 + \frac{\|V\|^2}{8} - \frac{3}{8}K_3^2 + \frac{7}{8}K_4^2\right) \mathrm{vol}(D),$$

where $E_D(V)$ denotes the energy of $V|_D$.

Proof. Notice that

$$\begin{aligned} \|\nabla V\|^2 &= \sum_{i=1}^{4} \varepsilon_i g(\nabla_{e_i} V, \nabla_{e_i} V) \\ &= \frac{1}{4}K_1^2 + \frac{1}{4}K_2^2 - \frac{1}{2}K_3^2 + \frac{3}{2}K_4^2. \end{aligned}$$

Considering $\|V\|^2 = K_1^2 + K_2^2 + K_3^2 - K_4^2$ in (2.1), we complete the proof. □

5.2. The $\mathbb{S}_4^3$ case. A left-invariant vector field V on the Lorentzian Damek–Ricci space $\mathbb{S}_4^3$ is uniquely determined by its components with respect to the pseudo-orthonormal basis $\{e_i\}$ for which (3.6) holds. Thus in this case one can write $V = K_1e_1 + K_2e_2 + K_3e_3 + K_4e_4$, where K_1, K_2, K_3, K_4 are real constants. Notice that the constant norm of V is given by $\|V\|^2 = K_1^2 + K_2^2 - K_3^2 + K_4^2$.

We now apply (3.7) and (3.9) to calculate $\nabla_{e_i}\nabla_{e_i}V$ and $\nabla_{\nabla_{e_i}e_i}V$ for $i = 1, 2, 3, 4$. It is easily seen that

$$\begin{aligned} \nabla_{e_1}\nabla_{e_1}V &= -\frac{1}{4}K_1e_1 + \frac{1}{4}K_2e_2 + \frac{1}{4}K_3e_3 + \frac{1}{4}K_4e_4, \\ \nabla_{e_2}\nabla_{e_2}V &= \frac{1}{4}K_1e_1 - \frac{1}{4}K_2e_2 + \frac{1}{4}K_3e_3 - \frac{1}{4}K_4e_4, \\ \nabla_{e_3}\nabla_{e_3}V &= -\frac{1}{4}K_1e_1 - \frac{1}{4}K_2e_2 + K_3e_3 + K_4e_4. \end{aligned}$$

Since $\nabla_{e_4}\nabla_{e_4}V = 0$ and $\nabla_{\nabla_{e_i}e_i}V = 0$ for $i = 1, 2, 3, 4$, taking into account the fact that $\nabla^*\nabla V = \sum_{i=1}^{4} \varepsilon_i(\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V)$, we get

$$\nabla^*\nabla V = \frac{1}{4}K_1e_1 + \frac{1}{4}K_2e_2 - \frac{1}{2}K_3e_3 - \frac{3K_4}{2}e_4. \tag{5.3}$$

This proves the following

Theorem 5.5. *There does not exist left-invariant harmonic vector fields on the Lorentzian Damek–Ricci space $\mathbb{S}^3_4$. Moreover, there does not exist any left-invariant vector field on the Lorentzian Damek–Ricci space $\mathbb{S}^3_4$ which defines a harmonic map from $\mathbb{S}^3_4$ to $(T\mathbb{S}^3_4, g^s)$.*

Combining theorem 5.1 with theorem 5.5, we get the following

Proposition 5.6. *There does not exist any left-invariant harmonic vector fields or invariant vector field which defines harmonic maps on 4-dimensional Lorentzian Damek-Ricci space.*

From (5.3), we obtain

$$\begin{aligned}
\nabla^*\nabla V &= \frac{1}{4}V - \frac{3}{4}K_3e_3 - \frac{7}{4}K_4e_4, \\
\nabla^*\nabla V &= -\frac{1}{2}V - K_4e_4 + \frac{3}{4}K_1e_1 + \frac{3}{4}K_2e_2, \\
\nabla^*\nabla V &= -\frac{3}{2}V + \frac{7}{4}K_1e_1 + \frac{7}{4}K_2e_2 + K_3e_3.
\end{aligned}$$

So we have the following

Theorem 5.7. *Let V be a left-invariant vector field on $\mathbb{S}^3_4$. Then V is a critical point for the energy functional restricted to vector fields of the same length if only if $V = K_1e_1 + K_2e_2$, or $V = K_3e_3$, or $V = K_4e_4$.*

Next we determine spatially harmonic vector fields on $\mathbb{S}^3_4$. Let $V = K_1e_1 + K_2e_2 + K_3e_3 + K_4e_4$ be a unit time-like vector field. Then a direct computation shows that

$$\begin{aligned}
\operatorname{div}(V) &= \sum_{i=1}^{4} \varepsilon_i g(\nabla_{e_i}V, e_i) = -2K_4, \\
\nabla_V V &= \left(-cK_2K_3 - \frac{1}{2}K_1K_4\right)e_1 + \left(cK_1K_3 - \frac{1}{2}K_2K_4\right)e_2 \\
&\quad - K_3K_4e_3 + \left(\frac{1}{2}K_1^2 + \frac{1}{2}K_2^2 - K_3^2\right)e_4, \\
\nabla_V\nabla_V V &= \left(-\frac{1}{4}K_1^3 - \frac{1}{4}K_1K_2^2 + K_1K_3^2 + \frac{3c}{4}K_2K_3K_4\right)e_1 \\
&\quad + \left(-\frac{1}{4}K_2^3 - \frac{1}{4}K_1^2K_2 + K_2K_3^2 - \frac{3c}{4}K_1K_3K_4\right)e_2 + K_3^3e_3 \\
&\quad + \left(K_3^2K_4 - \frac{1}{4}K_1^2K_4 - \frac{1}{4}K_2^2K_4\right)e_4.
\end{aligned}$$

Since

$$(\nabla V)^t \nabla_V V = \sum_{i=1}^{4} \varepsilon_i g(\nabla_V V, \nabla_{e_i} V) e_i,$$

We have

$$\begin{aligned}(\nabla V)^t \nabla_V V &= \left(\frac{3c}{4}K_2K_3K_4 + \frac{1}{4}K_1K_4^2 + \frac{1}{4}K_1K_2^2 + \frac{1}{4}K_1^3\right)e_1 \\ &\quad + \left(-\frac{3c}{4}K_1K_3K_4 + \frac{1}{4}K_2K_4^2 + \frac{1}{4}K_1^2K_2 + \frac{1}{4}K_2^3\right)e_2 \\ &\quad + (-K_3^3 + K_3K_4^2)e_3\end{aligned}$$

Thus

$$\begin{aligned}\widehat{X}_V &= -\nabla^*\nabla V - \nabla_V\nabla_V V - \mathrm{div}(V)\nabla_V V + (\nabla V)^t(\nabla_V V) \\ &= \left(-\frac{3}{4}K_1K_4^2 + \frac{1}{2}K_1K_2^2 + \frac{1}{2}K_1^3 - \frac{1}{4}K_1 - 2cK_2K_3K_4\right)e_1 \\ &\quad + \left(-\frac{3}{4}K_2K_4^2 + \frac{1}{2}K_1^2K_2 + \frac{1}{2}K_2^3 - \frac{1}{4}K_2 + 2cK_1K_3K_4\right)e_2 \\ &\quad + \left(-2K_3^3 - K_3K_4^2 + \frac{1}{2}K_3\right)e_3 \\ &\quad + \left(-3K_3^2K_4 + \frac{5}{4}K_1^2K_4 + \frac{5}{4}K_2^2K_4 + \frac{3}{2}K_4\right)e_4.\end{aligned}$$

Therefore, V satisfies (2.5) if and only if there exists a real constant λ, such that

$$\begin{cases} -\frac{3}{4}K_1K_4^2 + \frac{1}{2}K_1K_2^2 + \frac{1}{2}K_1^3 - \frac{1}{4}K_1 - 2cK_2K_3K_4 = \lambda K_1, \\ -\frac{3}{4}K_2K_4^2 + \frac{1}{2}K_1^2K_2 + \frac{1}{2}K_2^3 - \frac{1}{4}K_2 + 2cK_1K_3K_4 = \lambda K_2, \\ -2K_3^3 - K_3K_4^2 + \frac{1}{2}K_3 = \lambda K_3, \\ -3K_3^2K_4 + \frac{5}{4}K_1^2K_4 + \frac{5}{4}K_2^2K_4 + \frac{3}{2}K_4 = \lambda K_4. \end{cases} \tag{5.4}$$

Solutions of system (5.4) completely characterize spatially harmonic unit time-like invariant vector fields (which must satisfy the additional condition $\|V\|^2 = K_1^2 + K_2^2 - K_3^2 + K_4^2 = -1$). Thus we have the following:

Theorem 5.8. *A time-like unit left-invariant vector field V on the Lorentzian Damek–Ricci space $\mathbb{S}_4^3$ is a spatially harmonic vector field if and only if V has the form $V = K_3e_3 + K_4e_4$ with $K_3^2 = 1 + K_4^2$.*

Proof. We first prove the "if" part. If $V = K_3e_3 + K_4e_4, K_3^2 = 1 + K_4^2$, then by (5.4), we have $\lambda = -3K_4^2 - \frac{3}{2}$. So V is a spatially harmonic vector field.

Now we prove the "only if" part. We only need to find a non-trivial solution of (5.4). Notice that $K_3 \neq 0$. If $K_4 = 0$, then by (5.4), we get the following system of equations:

$$\begin{cases} \frac{1}{2}K_1K_2^2 + \frac{1}{2}K_1^3 - \frac{1}{4}K_1 = \lambda K_1, \\ \frac{1}{2}K_2K_1^2 + \frac{1}{2}K_2^3 - \frac{1}{4}K_2 = \lambda K_2, \\ -2K_3^3 + \frac{1}{2}K_3 = \lambda K_3. \end{cases}$$

If $K_1^2 + K_2^2 \neq 0$, then it follows from the first and the second equations that

$$\lambda = \frac{1}{2}(K_1^2 + K_2^2) - \frac{1}{4}.$$

Moreover, by the third equation, we also have

$$\lambda = -2K_3^2 + \frac{1}{2}.$$

Since $K_1^2 + K_2^2 = K_3^2 - 1$, from the above two equations, we obtain $K_3^2 = \frac{1}{2}$, but $K_3^2 = 1 + K_1^2 + K_2^2 \geq 1$, which is a contradiction. So $K_1 = K_2 = 0$.

If $K_4 \neq 0$, then it follows from the last equation of (5.4) that $\lambda = \frac{3}{2} + \frac{5}{4}K_1^2 + \frac{5}{4}K_2^2 - 3K_3^2$. Then, from the first and second equations of (5.4), we get $(K_1^2 + K_2^2)(\frac{9}{4}K_3^2 - 1) = 0$. Notice also that $K_3^2 \geq 1$. Thus $K_1 = K_2 = 0$. Since $K_3^2 = 1 + K_4^2$, the third equation automatically holds. So $V = K_3e_3 + K_4e_4$. □

Finally, we calculate the energy of a smooth vector field $V : (M, g) \to (TM, g^s)$ on $\mathbb{S}_4^3$. Since $\mathbb{S}_4^3$ is not compact, we suppose that D is a relatively compact domain in $\mathbb{S}_4^3$ and calculate the energy of $V|_D$.

Proposition 5.9. *Let V be a smooth left-invariant vector field on $\mathbb{S}_4^3$. Then the energy of $V|_D$ is*

$$E_D(V) = \left(2 - \frac{\|V\|^2}{8} - \frac{3}{8}K_3^2 + \frac{7}{8}K_4^2\right)\mathrm{vol}D$$

where $E_D(V)$ denotes the energy of $V|_D$.

Proof. Notice that

$$\begin{aligned} \|\nabla V\|^2 &= \sum_{i=1}^{4} \varepsilon_i g(\nabla_{e_i}V, \nabla_{e_i}V) \\ &= -\frac{1}{4}K_1^2 - \frac{1}{4}K_2^2 - \frac{1}{2}K_3^2 + \frac{3}{2}K_4^2. \end{aligned}$$

Considering $\|V\|^2 = K_1^2 + K_2^2 - K_3^2 + K_4^2$ in (2.1), we complete the proof. □

References

[1] M. T. K. Abbassi, G. Calvaruso and D. Perrone, Harmonicity of unit vector fields with respect to Riemannian g-natural metrics, *Differential Geom. Appl.* **27** (2009), 157–169.

[2] P. Baird and L. Danielo, Three-dimensional Ricci solitons which project to surfaces, *J. Reine Angew. Math.* **608** (2007), 65–91.

[3] W. Batat, G. Calvaruso and B. de Leo, Curvature properties of Lorentzian manifolds with large isometry groups, *Math. Phys. Anal. Geom.* **12** (2009), 201–217.

[4] J. Berndt, F. Tricerri and L. Vanhecke, Generalized Heisenberg Groups and Damek–Ricci Harmonic Spaces, Lecture Notes in Mathematics, Vol. **1598**, *Springer-Verlag, Berlin*, 1995.

[5] M. Brozos-Vázquez, G. Calvaruso, E. Garca-Río and S. Gavino-Fernández, Three--dimensional Lorentzian homogeneous Ricci solitons, *Israel J. Math.* **188** (2012), 385–403.

[6] M. Brozos-Vázquez, E. García-Río, P. Gilkey, S. Nikčević and R. Vázquez-Lorenzo, The Geometry of Walker Manifolds, Synthesis Lectures on Mathematics and Statistics, Vol. **5**, *Morgan and Claypool Publishers*, *Williston, VT*, 2009.

[7] P. Bueken and L. Vanhecke, Three- and four-dimensional Einstein-like manifolds and homogeneity, *Geom. Dedicata* **75** (1999), 123–136.

[8] G. Calvaruso, Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds, *Geom. Dedicata* **127** (2007), 99–119.

[9] G. Calvaruso, Harmonicity of vector fields on four-dimensional generalized symmetric spaces, *Cent. Eur. J. Math.* **10** (2012), 411–425.

[10] G. Calvaruso, Harmonicity properties of invariant vector fields on three-dimensional Lorentzian Lie groups, *J. Geom. Phys.* **61** (2011), 498–515.

[11] G. Calvaruso and B. de Leo, Curvature properties of four-dimensional generalized symmetric spaces, *J. Geom.* **90** (2008), 30–46.

[12] G. Calvaruso and A. Fino, Four-dimensional pseudo-Riemannian homogeneous Ricci solitons, *Int. J. Geom. Methods Mod. Phys.* **12** (2015), 1550056, 21 pp.

[13] G. Calvaruso and A. Fino, Ricci solitons and geometry of four-dimensional non-reductive homogeneous spaces, *Canad. J. Math.* **64** (2012), 778–804.

[14] G. Calvaruso and A. Zaeim, A complete classification of Ricci and Yamabe solitons of non-reductive homogeneous 4-spaces, *J. Geom. Phys.* **80** (2014), 15–25.

[15] A. A Cintra, F Mercuri and I. I Onnis, Minimal surfaces in 4-dimensional Lorentzian Damek–Ricci spaces, arXiv preprint, 2015, arXiv: 1501.03427.

[16] L. F. Di Cerbo, Generic properties of homogeneous Ricci solitons, *Adv. Geom.* **14** (2014), 225–237.

[17] O. Gil-Medrano, Relationship between volume and energy of vector fields, *Differential Geom. Appl.* **15** (2001), 137–152.

[18] R. S. Hamilton, The Ricci flow on surfaces, *Contemp. Math.* **71** (1988), 237–261.

[19] S. Hervik, Ricci nilsoliton black holes, *J. Geom. Phys.* **58** (2008), 1253–1264.

[20] T. Ishihara, Harmonic sections of tangent bundles, *J. Math. Tokushima Univ.* **1979**, 23–27.

[21] J. Lauret, Ricci soliton solvmanifolds, *J. Reine Angew. Math.* **650** (2011), 1–21.

[22] O. Nouhaud, Applications harmoniques d'une variété Riemannienne dans son fibré tangent, *C. R. Acad. Sci. Paris* **284** (1977), 815–818.

[23] T. L. Payne, The existence of soliton metrics for nilpotent Lie groups, *Geom. Dedicata* **145** (2010), 71–88.

[24] G. Wiegmink, Total bending of vector fields on Riemannian manifolds, *Math. Ann.* **303** (1995), 325–344.

[25] C. M. Wood, On the energy of a unit vector field, *Geom. Dedicata* **64** (1997), 319–330.

JU TAN
SCHOOL OF MATHEMATICAL SCIENCE
NANKAI UNIVERSITY
TIANJIN 300071
P. R. CHINA

E-mail: tanju2007@mail.nankai.edu.cn

SHAOQIANG DENG
SCHOOL OF MATHEMATICAL SCIENCES AND LPMC
NANKAI UNIVERSITY
TIANJIN 300071
P. R. CHINA

E-mail: dengsq@nankai.edu.cn

(Received June 2, 2015; revised December 11, 2015)