Publ. Math. Debrecen
89/1-2 (2016), 125–135
DOI: 10.5486/PMD.2016.7400

GK-dimension of 2×2 generic Lie matrices

By VESSELIN DRENSKY (Sofia), PLAMEN KOSHLUKOV (Campinas)
and GUSTAVO GRINGS MACHADO (Santa Maria)

Abstract. Recently Machado and Koshlukov have computed the Gelfand–Kirillov dimension of the relatively free algebra $F_m = F_m(\mathrm{var}(sl_2(K)))$ of rank m in the variety of algebras generated by the three-dimensional simple Lie algebra $sl_2(K)$ over an infinite field K of characteristic different from 2. They have shown that $\mathrm{GKdim}(F_m) = 3(m-1)$. The algebra F_m is isomorphic to the Lie algebra generated by m generic 2×2 matrices. Now we give a new proof for $\mathrm{GKdim}(F_m)$ using classical results of Procesi and Razmyslov combined with the observation that the commutator ideal of F_m is a module of the center of the associative algebra generated by m generic traceless 2×2 matrices.

1. Introduction

Let R be a (not necessarily associative) algebra generated by m elements $r_1, \dots, r_m$ over a field K and let V_n be the vector subspace of R spanned by all products $r_{i_1} \dots r_{i_k}$, $k \leq n$. The growth function of R with respect to the given system of generators is

$$g_R(n) = \dim(V_n), \quad n \geq 0.$$

The Gelfand–Kirillov dimension of R is defined as

$$\mathrm{GKdim}(R) = \limsup_{n \to \infty} \log_n(g_R(n)).$$

Mathematics Subject Classification: Primary: 16P90; Secondary: 16R30, 17B01, 17B20.

Key words and phrases: Gelfand–Kirillov dimension, generic matrices, matrix invariants, relatively free Lie algebras.

This project was partially supported by Grant I02/18 "Computational and Combinatorial Methods in Algebra and Applications" of the Bulgarian National Science Fund. P. Koshlukov acknowledges partial support by FAPESP grant No. 2014/08608-0, and by CNPq grant No. 304632/2015-5.

It does not depend on the choice of the generators of R. See the book [9] for a background on GKdim. If the algebra R is graded,

$$R = \bigoplus_{n \geq 0} R^{(n)},$$

where $R^{(n)}$ is the homogeneous component of degree n of R, then the Hilbert series of R is the formal power series

$$H(R,t) = \sum_{n \geq 0} \dim(R^{(n)}) t^n.$$

If R is generated by its homogeneous elements of first degree, then its growth function is

$$g_R(n) = \sum_{l=0}^{n} \dim(R^{(l)}).$$

In the general case, if R is a graded algebra generated by a finite system of (homogeneous) elements of arbitrary degree, its Gelfand–Kirillov dimension can be expressed using again its Hilbert series as

$$\mathrm{GKdim}(R) = \limsup_{n \to \infty} \log_n \left(\sum_{l=0}^{n} \dim(R^{(l)}) \right).$$

When studying varieties of K-algebras $\mathfrak{V}$, all information for the m-generated algebras in $\mathfrak{V}$ is carried by the relatively free algebra $F_m(\mathfrak{V})$ of rank m in $\mathfrak{V}$. When the base field K is of characteristic 0, a lot is known for the Gelfand–Kirillov dimension of relatively free associative algebras, see the book [9], the survey article [4], or the paper [11]. In particular, $\mathrm{GKdim}(F_m(\mathfrak{V}))$ is an integer for all proper varieties of associative algebras. Almost nothing is known for relatively free Lie algebras. Using the bases of free nilpotent-by-abelian Lie algebras given by Shmelkin [17], it is easy to see that

$$\mathrm{GKdim}(F_m(\mathfrak{N}_c\mathfrak{A})) = \mathrm{GKdim}(L_m/(L_m')^{c+1}) = mc,$$

where $m > 1$ and L_m is the free m-generated Lie algebra. Together with free nilpotent Lie algebras where the Gelfand–Kirillov dimension is equal to 0, these are the only free polynilpotent Lie algebras of finite Gelfand–Kirillov dimension, see Petrogradsky [12].

Recently Machado and Koshlukov [11] have computed the Gelfand–Kirillov dimension of the relatively free algebra $F_m = F_m(\mathrm{var}(sl_2(K)))$ of rank

$m > 2$ in the variety of algebras generated by the three-dimensional simple Lie algebra $sl_2(K)$ over an infinite field K of characteristic different from 2. They have shown that $\mathrm{GKdim}(F_m) = 3(m-1)$. Their proof is based on a careful analysis of the explicit expression of the Hilbert series of F_m obtained by DRENSKY [3]. The case $m = 2$ was handled before by BAHTURIN [2] who showed that $\mathrm{GKdim}(F_2) = 3$. The algebra F_m is isomorphic to the Lie algebra generated by m generic traceless 2×2 matrices. The purpose of our paper is to give a new proof for $\mathrm{GKdim}(F_m)$ using classical results of PROCESI [13], [14] on Gelfand–Kirillov dimension of the algebra of generic matrices and RAZMYSLOV [16] on the weak polynomial identities of matrices, combined with the observation that the commutator ideal of F_m is a module over the center of the associative algebra generated by m generic traceless 2×2 matrices. We believe that the present approach is more adequate for generalizations for other finite dimensional simple Lie algebras than the approach in [11].

2. The proof

The following statement and its corollary are folklorely known. We include the proof for self-completeness of the exposition and also because we were not able to find an explicit reference.

Lemma 1. *Let R be a finitely generated graded algebra with Hilbert series of the form*

$$H(R,t) = h(t) \prod_{i=1}^{s} \frac{1}{(1-t^{d_i})},$$

where $h(t) \in \mathbb{C}[t]$ is a polynomial and the d_i's are positive integers. Then the Gelfand–Kirillov dimension of R is equal to the multiplicity of 1 as a pole of $H(R,t)$.

PROOF. It is sufficient to consider the case when R is not finite dimensional and hence its Hilbert series has a nontrivial denominator. Let d be the least common multiple of the degrees d_i. Then

$$\begin{aligned} H(R,t) = \sum_{n\geq 0} a_n t^n &= f(t) + \sum_{p=1}^{k} \sum_{q=0}^{d-1} \frac{\alpha_{pq}}{(1-\omega_q t)^p} \\ &= f(t) + \sum_{n\geq 0} \left(\sum_{p=1}^{k} \binom{n+p-1}{p-1} \sum_{q=0}^{d-1} \alpha_{pq} \omega_q^n \right) t^n, \end{aligned}$$

where $f(t) \in \mathbb{C}[t]$, $\alpha_{pq} \in \mathbb{C}$, $\omega_0 = 1, \omega_1, \ldots, \omega_{d-1}$ are the d-th roots of 1, and at least one of the coefficients α_{kq} is different from zero. Since $\omega_q^d = 1$, the sequences

$$\beta_{pn} = \sum_{q=0}^{d-1} \alpha_{pq}\omega_q^n, \quad p = 1, \ldots, k,$$

are periodic with period d and for n large enough the coefficients a_n of the Hilbert series $H(R,t)$ are bounded by polynomials of degree $k-1$ in n. Hence the sequence

$$\sum_{l=0}^{n} a_l = \sum_{l=0}^{n} \dim(R^{(l)})$$

needed for the definition of the Gelfand–Kirillov dimension of R is bounded by a polynomial of degree k in n and

$$\mathrm{GKdim}(R) \leq k.$$

The asymptotics of the coefficients a_n of

$$H(R,t) = f(t) + \sum_{n\geq 0}\left(\sum_{p=1}^{k}\binom{n+p-1}{p-1}\beta_{pn}\right)t^n,$$

is determined by β_{kn}. Since a_n are positive integers, we derive that the periodic sequence β_{kn}, $n = 0, 1, 2, \ldots$, consists of nonnegative reals and at least one of them is positive. Since $\omega_q^d = 1$, if $\omega_q \neq 1$, then $1 + \omega_q + \omega_q^2 + \cdots + \omega_q^{d-1} = 0$. Hence

$$0 < \sum_{l=0}^{d-1} \beta_{k,dn+l} = \sum_{l=0}^{d-1}\sum_{q=0}^{d-1} \alpha_{kq}\omega_q^{dn+l} = \sum_{q=0}^{d-1} \alpha_{kq} \sum_{l=0}^{d-1} \omega_q^l = d\alpha_{k0}.$$

Therefore $\alpha_{k0} > 0$. We consider the partial sum $p_{dn} = a_0 + a_1 + \cdots + a_{dn}$ of the coefficients of the Hilbert series $H(R,t)$. Its asymptotics is determined by

$$\begin{aligned}\tilde{p}_{dn} &= \sum_{c=0}^{dn}\binom{c+k-1}{k-1}\beta_{kc} \approx \frac{1}{(k-1)!}\sum_{c=0}^{dn} c^{k-1}\beta_{kc} \\ &\approx \frac{1}{(k-1)!}\sum_{e=0}^{n}(ed)^{k-1}\sum_{l=0}^{d-1}\beta_{k,ed+l} = \frac{d\alpha_{k0}}{(k-1)!}\sum_{e=0}^{n}(ed)^{k-1}\end{aligned}$$

and this is a polynomial of degree k in n. Hence

$$\begin{aligned}\mathrm{GKdim}(R) &= \limsup_{n\to\infty} \log_n\left(\sum_{l=0}^{n} a_l\right) \geq \limsup_{n\to\infty} \log_n\left(\sum_{c=0}^{dn} a_c\right)\\ &= \limsup_{n\to\infty} \log_{dn}(p_{dn}) = \limsup_{n\to\infty} \log_{dn}(\tilde{p}_{dn}) = k\end{aligned}$$

which, together with the opposite inequality $\mathrm{GKdim}(R) \leq k$, completes the proof. □

Corollary 2. *Let R be a finitely generated graded algebra and let C be a finitely generated graded subalgebra of the center of R such that R is a finitely generated C-module. Then the Gelfand–Kirillov dimension of R is equal to the multiplicity of 1 as a pole of $H(R,t)$.*

Proof. By the Hilbert–Serre theorem (see e.g., [1]), the Hilbert series of any finitely generated graded module M over a finitely generated graded commutative algebra C is of the form

$$H(M,t) = h(t)\prod_{i=1}^{k}\frac{1}{(1-t^{d_i})}, \quad h(t)\in\mathbb{C}[t], d_i > 0.$$

Hence the proof follows immediately from Lemma 1. □

In the sequel we assume that the base field K is of characteristic 0. Let

$$\Omega_{km} = K[Y_{km}] = K[y_{pq}^{(i)} \mid p,q = 1,\dots,k, i = 1,\dots,m]$$

be the algebra of polynomials in k^2m commuting variables and let

$$y_i = (y_{pq}^{(i)}), \quad i = 1,\dots,m,$$

be m generic $k\times k$ matrices. We consider the following algebras:

R_{km} – the generic matrix algebra. This is the subalgebra generated by $y_1,\dots,y_m$ of the associative $k\times k$ matrix algebra $M_k(\Omega_{km})$ with entries from Ω_{km}.
C_{km} – the pure trace algebra. This is the subalgebra of Ω_{km} generated by the traces of the products, $\mathrm{tr}(y_{i_1}\cdots y_{i_l})$. We embed C_{km} in $M_k(\Omega_{km})$ by $f(Y_{km}) \to f(Y_{km})I_k$, where I_k is the identity matrix.
T_{km} – the mixed trace algebra. This is the subalgebra of $M_k(\Omega_{km})$ generated by R_{km} and C_{km}.

For a background on generic matrices see e.g., [14] or [7]. Below we summarize the results we need.

Proposition 3. *Let $k,m\geq 2$. Then:*

(i) *The mixed trace algebra T_{km} has no zero divisors;*

(ii) *The pure trace algebra C_{km} coincides with the center of T_{km}. It is finitely generated and T_{km} is a finitely generated C_{km}-module;*

(iii) (Kirillov [8], Procesi [13])

$$\mathrm{GKdim}(T_{km})=\mathrm{GKdim}(C_{km})=\mathrm{GKdim}(R_{km})=k^2(m-1)+1.$$

Further, we consider the generic traceless $k\times k$ matrices

$$z_i=(z_{pq}^{(i)})=y_i-\frac{1}{k}\mathrm{tr}(y_i)I_k,\quad i=1,\dots,m,$$

and the subalgebra W_{km} of T_{km} generated by $z_1,\dots,z_m$, the subalgebra $C_{km}^{(0)}$ of C_{km} generated by the traces of the products, $\mathrm{tr}(z_{i_1}\cdots z_{i_l})$, and the subalgebra $T_{km}^{(0)}$ of T_{km} generated by W_{km} and $C_{km}^{(0)}$. Finally, let L_{km} be the Lie subalgebra of W_{km} generated by $z_1,\dots,z_m$.

Proposition 4. *Let $k,m\geq 2$. Then*

(i) (Procesi [15])

$$T_{km}\cong K[\mathrm{tr}(y_1),\dots,\mathrm{tr}(y_m)]\otimes_K T_{km}^{(0)},$$

$$C_{km}\cong K[\mathrm{tr}(y_1),\dots,\mathrm{tr}(y_m)]\otimes_K C_{km}^{(0)};$$

(ii) (Razmyslov [16])

$$W_{km}\cong K\langle x_1,\dots,x_m\rangle/\mathrm{Id}(M_k(K),sl_k(K))$$

where $\mathrm{Id}(M_k(K),sl_k(K))$ is the ideal of all weak polynomial identities in m variables for the pair $(M_k(K),sl_k(K))$, i.e., the polynomials in the free associative algebra $K\langle x_1,\dots,x_m\rangle$ which vanish when evaluated on $sl_k(K)$ considered as a subspace in $M_k(K)$.

(iii) (Razmyslov [16]) *The Lie algebra L_{km} is isomorphic to the relatively free algebra $F_m(\mathrm{var}(sl_k)(K))$ in the variety of Lie algebras generated by $sl_k(K)$.*

Corollary 5. *For $k,m\geq 2$*

$$\mathrm{GKdim}(T_{km}^{(0)})=\mathrm{GKdim}(C_{km}^{(0)})=(k^2-1)(m-1).$$

Proof. The algebras T_{km} and C_{km} satisfy the conditions of Corollary 2. Hence the multiplicity of 1 as a pole of the Hilbert series of T_{km} and C_{km} is equal to their Gelfand–Kirillov dimension $k^2(m-1)+1$ (see Proposition 3 (iii)). Proposition 4 (i) gives that

$$H(T_{km},t) = H(K[\mathrm{tr}(y_1),\dots,\mathrm{tr}(y_m)],t)H(T_{km}^{(0)},t) = \frac{1}{(1-t)^m}H(T_{km}^{(0)},t),$$

$$H(C_{km},t) = \frac{1}{(1-t)^m}H(C_{km}^{(0)},t).$$

Hence the multiplicity of 1 as a pole of $H(T_{km}^{(0)},t)$ and $H(C_{km}^{(0)},t)$ is equal to $(k^2(m-1)+1)-m = (k^2-1)(m-1)$. Both algebras $T_{km}^{(0)}$ and $C_{km}^{(0)}$ are finitely generated and graded. Hence the proof follows from Corollary 2. $\square$

Now we shall summarize the information for 2×2 generic matrices.

Proposition 6. *Let $k=2$ and $m \geq 2$. Then:*

(i) (Sibirskii [18]) *The trace polynomials*

$$\mathrm{tr}(y_i),\ i=1,\dots,m, \quad \mathrm{tr}(y_iy_j),\ 1 \leq i \leq j \leq m,$$
$$\mathrm{tr}(y_{i_1}y_{i_2}y_{i_3}), \quad 1 \leq i_1 < i_2 < i_3 \leq m,$$

form a minimal system of generators of C_{2m}.

(ii) (Procesi [15]) *The algebras $T_{2m}^{(0)}$ and W_{2m} coincide. The algebra $C_{2m}^{(0)}$ is generated by*

$$\mathrm{tr}(z_iz_j),\ 1 \leq i \leq j \leq m, \quad \mathrm{tr}(z_{i_1}z_{i_2}z_{i_3}),\ 1 \leq i_1 < i_2 < i_3 \leq m,$$

which belong to W_{2m}.

(iii) (Drensky [5]) *The algebra $C_{2m}^{(0)}$ is generated by*

$$z_i^2,\ i=1,\dots,m, \quad z_iz_j+z_jz_i,\ 1 \leq i \leq j \leq m,$$
$$s_3(z_{i_1},z_{i_2},z_{i_3}), \quad 1 \leq i_1 < i_2 < i_3 \leq m,$$

where

$$s_3(x_1,x_2,x_3) = \sum_{\sigma \in S_3} \mathrm{sign}(\sigma)x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$$

is the standard polynomial of degree 3.

Proof. We shall present the proof of (ii) and (iii) as a consequence of (i). Clearly $C_{2m}^{(0)}$ is generated by $\mathrm{tr}(z_iz_j)$, $1 \leq i \leq j \leq m$, and $\mathrm{tr}(z_{i_1}z_{i_2}z_{i_3})$, $1 \leq i_1 < i_2 < i_3 \leq m$. Now the proof of (ii) and (iii) follows immediately from the equalities in $T_{2m}^{(0)}$

$$\mathrm{tr}(z_1^2) = 2z_1^2, \quad \mathrm{tr}(z_1z_2) = z_1z_2+z_2z_1, \quad \mathrm{tr}(z_1z_2z_3) = \frac{1}{3}s_3(z_1,z_2,z_3)$$

which may be checked by direct verification. $\square$

Lemma 7. *The commutator ideal L'_{2m} is a $C_{2m}^{(0)}$-module.*

Proof. The following equalities which can be verified directly hold in W_{2m}:

$$\begin{aligned}
[z_1,z_2]z_3^2 &= \frac{1}{4}([z_1,z_2,z_3,z_3]-[[z_1,z_3],[z_2,z_3]]),\\
[z_1,z_2](z_3z_4+z_4z_3) &= \frac{1}{4}([z_1,z_2,z_3,z_4]+[z_1,z_2,z_4,z_3]\\
&\quad -[[z_1,z_3],[z_2,z_4]]-[[z_1,z_4],[z_2,z_3]]),\\
z_4s_3(z_1,z_2,z_3) &= \frac{3}{8}\sum_{\sigma\in S_3}\operatorname{sign}(\sigma)[z_4,z_{\sigma(1)},z_{\sigma(2)},z_{\sigma(3)}].
\end{aligned}$$

The elements of the commutator ideal are linear combinations of (left normed) commutators $u_i=[z_{i_1},z_{i_2},\ldots,z_{i_n}]$. If v is a generator of $C_{2m}^{(0)}$, then

$$u_iv=[z_{i_1},z_{i_2},\ldots,z_{i_n}]v=[[z_{i_1},z_{i_2}]v,\ldots,z_{i_n}]$$

and the above equalities guarantee that u_iv is a linear combination of commutators, i.e., belongs to L'_{2m} again. Hence $L'_{2m}C_{2m}^{(0)}\subset L'_{2m}$. □

Remark 8. It is known that W_{2m} is a $C_{2m}^{(0)}$-module generated by 1, z_i, $i=1,\ldots,m$, and $[z_i,z_j]$, $1\le i<j\le m$. Using the equality

$$[z_1,z_2,z_3]=2(z_1(z_2z_3+z_3z_2)-z_2(z_1z_3+z_3z_1)),$$

as in the proof of Lemma 7 we can show that L'_{2m} is a $C_{2m}^{(0)}$-module generated by all commutators $[z_i,z_j]$ and $[z_{i_1},z_{i_2},z_{i_3}]$. For $m=2$, the commutator ideal L'_{22} is a free $C_{22}^{(0)}$-module generated by $[z_1,z_2]$, $[z_1,z_2,z_1]$, $[z_1,z_2,z_2]$, see [6].

The proof of the following theorem established in [11] is the main result of our paper.

Theorem 9. *Let K be a field of characteristic 0 and let L_{2m} be the Lie algebra generated by m generic traceless 2×2 matrices, $m\ge 2$. Then*

$$\operatorname{GKdim}(L_{2m})=\operatorname{GKdim}(F_m(\operatorname{var}(sl_2(K))))=3(m-1).$$

Proof. Let

$$H(C_{2m}^{(0)},t)=\sum_{n\ge 0}c_nt^n,\quad H(L_{2m},t)=\sum_{n\ge 1}l_nt^n,\quad H(W_{2m},t)=\sum_{n\ge 1}w_nt^n$$

be the Hilbert series of $C_{2m}^{(0)}$, L_{2m}, and W_{2m}, respectively. Since the algebra L_{2m} is finitely generated, its Gelfand–Kirillov dimension is

$$\mathrm{GKdim}(L_{2m}) = \limsup_{n\to\infty} \log_n \left(\sum_{k=1}^{n} l_k \right).$$

The algebra W_{2m} has no zero divisors and hence $[z_1, z_2]C_{2m}^{(0)} \subset L'_{2m} \subset L_{2m}$ is a free $C_{2m}^{(0)}$-module. Therefore

$$\sum_{k=0}^{n-2} c_k \le \sum_{k=1}^{n} l_k \le \sum_{k=0}^{n} w_k,$$

which implies that

$$3(m-1) = \mathrm{GKdim}(C_{2m}^{(0)}) \le \mathrm{GKdim}(L_{2m}) \le \mathrm{GKdim}(W_{2m}) = 3(m-1). \quad \square$$

Remark 10. As in [11], the formula for the Gelfand–Kirillov dimension of $F_m(\mathrm{var}(sl_2(K)))$ obtained in characteristic 0 holds also for any infinite field K of characteristic different from 2.

Remark 11. In characteristic 2, the algebra $sl_2(K)$ is nilpotent of class 2 and hence $F_m(\mathrm{var}(sl_2(K)))$ is isomorphic to the free nilpotent of class 2 Lie algebra $F_m(\mathfrak{N}_2)$ which is finite dimensional. Therefore $\mathrm{GKdim}(F_m(\mathrm{var}(sl_2(K)))) = 0$. When K is an infinite field of characteristic 2, a much more interesting object is the relatively free algebra $F_m(\mathrm{var}(M_2(K)^{(-)}))$ of the variety generated by the 2×2 matrix algebra $M_2(K)$ considered as a Lie algebra. Vaughan-Lee [19] showed that the algebra $M_2(K)^{(-)}$ does not have a finite basis of its polynomial identities. (It is easy to see that the four-dimensional Lie algebra constructed in [19] is isomorphic to $M_2(K)^{(-)}$.) The algebra $M_2(K)^{(-)}$ satisfies the center-by-metabelian polynomial identity

$$[[[x_1, x_2], [x_3, x_4]], x_5] = 0.$$

It is well known that the free center-by-metabelian Lie algebra $F_m([\mathfrak{A}^2, \mathfrak{E}])$ over any field K is spanned by

$$[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n}], \quad [[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n}], [x_{i_{n+1}}, x_{i_{n+2}}]],$$

where $i_1 > i_2 \le i_3 \le \cdots \le i_n$ and the commutators are left normed, e.g., $[x_1, x_2, x_3] = [[x_1, x_2], x_3]$. (A basis of $F_m([\mathfrak{A}^2, \mathfrak{E}])$ is given by Kuzmin [10].) Since the commutators $[x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_n}]$ form a basis of the free metabelian Lie algebra $F_m(\mathfrak{A}^2)$ and are linearly independent in $F_m(\mathrm{var}(M_2(K)^{(-)}))$, we obtain immediately that

$$\mathrm{GKdim}(F_m(\mathrm{var}(M_2(K)^{(-)}))) = m, \quad m > 1.$$

In characteristic 2 there is another three-dimensional simple Lie algebra which is an analogue of the Lie algebra of the three-dimensional real vector space with the vector multiplication. It is interesting to see whether this algebra has a finite basis of its polynomial identities (probably not) and, when the field is infinite, to compute the Gelfand–Kirillov dimension of the corresponding relatively free algebras.

Acknowledgements. The authors are grateful to the anonymous referees for the useful suggestions for improving of the exposition.

References

[1] M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra, *Addison–Wesley*, *Reading, Mass.*, 1969.

[2] Yu. A. Bahturin, Two-variable identities in the Lie algebra $sl(2, k)$, *Trudy Sem. Petrovsk.* **5** (1979), 205–208 (in *Russian*); Translation: Contemp. Soviet Math. Petrovskiĭ Seminar **5**, Topics in Modern Math. *Plenum*, *New York – London*, (1985), 259–263.

[3] V. Drensky, Codimensions of T-ideals and Hilbert series of relatively free algebras, *J. Algebra* **91** (1984), 1–17.

[4] V. Drensky, Gelfand–Kirillov dimension of PI-algebras, In: Methods in Ring Theory, Proc. of the Trento Conf., Lect. Notes in Pure and Appl. Math. Vol. **198**, *Dekker*, *New York*, 1998, 97–113.

[5] V. Drensky, Defining relations for the algebra of invariants of 2×2 matrices, *Algebr. Represent. Theory* **6** (2003), 193–214.

[6] V. Drensky and Ş. Findik, Inner automorphisms of Lie algebras related with generic 2×2 matrices, *Algebra and Discrete Math.* **14** (2012), 49–70.

[7] V. Drensky and E. Formanek, Polynomial Identity Rings, Advanced Courses in Mathematics, *CRM Barcelona, Birkhäuser*, *Basel – Boston*, 2004.

[8] A. A. Kirillov, Certain division algebras over a field of rational functions, *Funkts. Anal. Prilozh.* **1** (1967), 101–102 (in *Russian*); Translation: *Funct. Anal. Appl.* **1** (1967), 87–88.

[9] G. R. Krause and T. H. Lenagan, Growth of Algebras and Gelfand-Kirillov Dimension, *Pitman Publ.*, *London*, (1967); Revised Edition: Graduate Studies in Mathematics, Vol. **22**, *AMS*, *Providence, RI*, 2000.

[10] Yu. V. Kuzmin, Free center-by-metabelian groups, Lie algebras and $\mathcal{D}$-groups, *Izv. Akad. Nauk SSSR Ser. Mat.* **41** (1977), 3–33, 231 (in *Russian*); Translation: *Math. USSR Izvestija* **11** (1977), 1–30.

[11] G. G. Machado and P. Koshlukov, GK dimension of the relatively free algebra for sl_2, *Monatsh. Math.* **175** (2014), 543–553.

[12] V. M. Petrogradsky, Intermediate growth in Lie algebras and their enveloping algebras, *J. Algebra* **179** (1996), 459–482.

[13] C. Procesi, Non-commutative affine rings, *Atti Accad. Naz. Lincei, Ser. 8* **8** (1967), 237–255.

[14] C. Procesi, The invariant theory of $n \times n$ matrices, *Adv. Math.* **19** (1976), 306–381.

[15] C. Procesi, Computing with 2×2 matrices, *J. Algebra* **87** (1984), 342–359.

[16] Yu. P. Razmyslov, The existence of a finite basis of the identities of a matrix algebra of second order over a field of characteristic 0, *Algebra i Logika* **12** (1973), 83–113 (in *Russian*); Translation: *Algebra and Logic* **12** (1973), 43–63.

[17] A. L. Shmelkin, Wreath products of Lie algebras and their application in the theory of groups, *Trudy Moskov. Mat. Obshch.* **29** (1973), 247–260 (in *Russian*); Translation: *Trans. Moscow Math.* **29** (1973), 239–252.

[18] K. S. Sibirskii, Algebraic invariants for a set of matrices, *Sib. Mat. Zh.* **9** (1968), 152–164 (in *Russian*); Translation: *Siber. Math. J.* **9** (1968), 115–124.

[19] M. R. Vaughan-Lee, Varieties of Lie algebras, *Quart. J. Math. Oxford Ser. 2* **21** (1970), 297–308.

VESSELIN DRENSKY
INSTITUTE OF MATHEMATICS AND INFORMATICS
BULGARIAN ACADEMY OF SCIENCES
ACAD. G. BONCHEV STR., BLOCK 8
1113 SOFIA
BULGARIA

E-mail: drensky@math.bas.bg

PLAMEN KOSHLUKOV
DEPARTMENT OF MATHEMATICS
IMECC, UNICAMP
SÉRGIO BUARQUE DE HOLANDA 651
CAMPINAS, SP 13083-859
BRAZIL

E-mail: plamen@ime.unicamp.br

GUSTAVO GRINGS MACHADO
DEPARTMENT OF MATHEMATICS
CCNE, UFSM
FAIXA DE CAMOBI, CAMPUS UFSM
SANTA MARIA, RS 97105-900
BRAZIL

E-mail: grings@smail.ufsm.br

(Received June 12, 2015; revised January 6, 2016)