*Publ. Math. Debrecen*
**46 / 1-2** (1995), 33–39

# Generalized power means for matrix functions

By B. MOND (Bundoora) and J. E. PEČARIĆ (Zagreb)

## §1. Introduction

Generalized power means are defined by [1]

$$(1)\qquad \begin{aligned} M_{n,a}(x;w)_p &= \left\{ \frac{\sum_{i=1}^{n} w_i x_i^{a+p}}{\sum_{i=1}^{n} w_i x_i^p} \right\}^{1/a}, \quad a \neq 0 \\ &= \exp \left\{ \frac{\sum_{i=1}^{n} w_i x_i^p \log x_i}{\sum_{i=1}^{n} w_i x_i^p} \right\}, \quad a = 0 \end{aligned}$$

where $a, p \in \mathbb{R}$, $x, w \in \mathbb{R}_+^n$, $n \in \mathbb{N}$. Concerning inequalities for these means see [2–5]. Further generalizations of these means are given in ([6,7]).

Let $\phi : I \to \mathbb{R}_+$ be a strictly positive function, $F : I \to \mathbb{R}$ be a strictly monotone function, $x \in I^n$, $w > 0 (\in \mathbb{R}^n)$. Then define

$$(2)\qquad F_n(x,\phi) = F^{-1}\left( \frac{\sum_{i=1}^{n} w_i \phi(x_i) F(x_i)}{\sum_{i=1}^{n} w_i \phi(x_i)} \right).$$

*Remark.* Note that in [8,9] means are defined with $\phi_i(x_i)$ instead of $w_i \phi(x_i)$.

Concerning inequalities for these means see, for example, [8], [9] or the monograph [10, pp. 261–269].

In this paper, we give analogous means for matrix functions.

## 2. Preliminaries and definitions

Let $A \in C^{n\times n}$ be a normal matrix, i.e., $A^*A = AA^*$. Here $A^*$ means $\bar{A}^t$, the transpose conjugate of $A$. There exists [11] a unitary matrix $U$ such that

$$A = U^*[\lambda_1, \lambda_2, \ldots, \lambda_n]U \tag{3}$$

where $[\lambda_1, \lambda_2, \ldots, \lambda_n]$ is the diagonal matrix $(\lambda_j \delta_{ij})$, and where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$, each appearing as often as its multiplicity. $A$ is Hermitian if and only if $\lambda_i$, $i \in I_n = \{1, 2, \ldots, n\}$ are real. If $A$ is Hermitian and all $\lambda_i$ are strictly positive, then $A$ is said to be positive definite.

Assume now that $f(\lambda_i) \in C$, $i \in I_n$ is well-defined. Then $f(A)$ may be defined by (see e.g. [11, p.71] or [12, p.90])

$$f(A) = U^*[f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)]U. \tag{4}$$

As before, if $f(\lambda_i)$, $i \in I_n$ are real, then $f(A)$ is Hermitian. If, also, $f(\lambda_i) > 0$, $i \in I_n$, then $f(A)$ is positive definite.

We note that for the inner product

$$(f(A)x, x) = \sum_{i=1}^{n} |y_i|^2 f(\lambda_i) \tag{5}$$

where $y \in C^n$, $y = Ux$, and so $\sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} |x_i|^2$. Thus if $x$ is a unit vector, then so is $y$.

If $A$ is positive definite, so that $\lambda_i > 0$, $i \in I_n$, and $f(t) = t^r$, where $t > 0$ and $r \in \mathbb{R}$, we have $f(A) = A^r$. This representation is used in [13] and [14] to obtain matrix inequalities involving powers of $A$.

It is obvious that the above representations can also be used to obtain matrix versions of means (1) and (2).

*Definition 1.* Let $A$ be an $n \times n$ positive definite Hermitian matrix, $0 \neq x \in C^n$; $a, p \in \mathbb{R}$. Then the generalized power means of $A$ is given by

$$\begin{aligned} M_a^p(A; x) &= \left[\frac{(A^{a+p}x, x)}{(A^p x, x)}\right]^{1/a}, \quad a \neq 0 \\ &= \exp\left\{\frac{((A^p \log A)x, x)}{(A^p x, x)}\right\}, \quad a = 0. \end{aligned} \tag{6}$$

*Definition 2.* Let $\phi : I \to \mathbb{R}_+ (I \subset \mathbb{R})$ be a positive function, $F : I \to \mathbb{R}$ a strictly monotone function, and let $A$ be a Hermitian matrix with eigenvalues in $I$. The generalized quasi-arithmetic mean $F(A;x)$ is defined by

$$\tilde{F}(A;x;\phi) = F^{-1}\left(\frac{((\phi \cdot F)(A)x,x)}{(\phi(A)x,x)}\right) \tag{7}$$

where $x \in C^n$, $x \neq 0$.

## 3. Inequalities for generalized power means

**Theorem 1.** *Let $a,b,p,q \in \mathbb{R}$ satisfy*

$$|\,|a| - |b|\,| + a + 2p \leq b + 2q. \tag{8}$$

*Then for every positive definite Hermitian matrix $A$ and every $x \in C^n$, $x \neq 0$,*

$$M_a^p(A;x) \leq M_b^q(A;x). \tag{9}$$

Proof. This is a simple consequence of (5) and of the following ([2]):

**Lemma 1.** *Let $a,b,p,q \in \mathbb{R}$. Then*

$$M_{n,a}(x;w)_p \leq M_{n,b}(x;w)_q$$

*holds for every $x \in \mathbb{R}^n (x \neq 0)$, if and only if (8) holds.*

**Theorem 2.** *Let $a$, $b_1,\ldots,b_k$, $p,q_1,\ldots,q_k \in \mathbb{R}$, $k \geq 2$. Further, let*

$$Q_0 = a^- - p,\ Q_i = b_i^+ + q_i\ i = 1,\ldots,k$$
$$Q_0^* = a^+ + p,\ Q_i^* = b_i^- - q_i,\ i = 1,\ldots,k,$$

*where $a^+ = (|a|+a)/2$ and $a^- = (|a|-a)/2$ and for $i = 0,1,\ldots,k$, let*

$$H_i = \begin{cases} \left(\sum\limits_{\substack{j=0\\ j\neq i}}^{k} Q_j^{-1}\right)^{-1}, & \text{when } \prod\limits_{\substack{j=0\\ j\neq i}}^{k} Q_j \neq 0 \\ 0, & \text{when } \prod\limits_{\substack{j=0\\ j\neq i}}^{k} Q_j = 0. \end{cases}$$

*Let $A$ be a normal matrix with eigenvalues in $I (I \subset C)$; $f_j : I \to \mathbb{R}_+$ for $j = 1,\ldots,k$. If*

$$Q_i \geq 0 \text{ and } H_i \geq Q_i^* \quad (i = 0,\ldots,k), \tag{10}$$

*then*

$$M_a^p((f_1 \dots f_k)(A); x) \leq M_{b_1}^{q_1}(f_1(A); x) \dots M_{b_k}^{q_k}(f_k(A); x). \tag{11}$$

Proof. This is again a simple consequence of the following ([3]):

**Lemma 2.** *Let $a, b_1, \dots, b_k,\ p, q_1, \dots, q_k \in \mathbb{R}$, $k \geq 2$ and let $Q_i, Q_i^*$, $H_i$ be defined as in Theorem 2. Then the inequality*

$$M_{n,a}(x_1 \dots x_k; w)_p \leq M_{n,b_1}(x_1; w)_{q_1} \dots M_{n,b_k}(x_k; w)_{q_k}$$

*holds for all $x_1, \dots, x_n \in \mathbb{R}_+^n,\ n \in \mathbb{N}$, iff* (10) *holds.*

*Remark.* If $x, y \in \mathbb{R}^n$ we use notation $xy \equiv (x_1 y_1, \dots, x_n y_n)$.

**Theorem 3.** *Let $a, b_1, \dots, b_k, p, q_1, \dots, q_k \in \mathbb{R}$, $k \geq 2$. Let $A$ be a normal matrix with eigenvalues in $I$ ($I \subseteq C$), and let $f_i : I \to \mathbb{R}_+$ for $i = 1, \dots, k$, $x \in C^n$, $x \neq 0$. If*

$$\max\{p + a^+, 1\} \leq q_i + b_i^+ \tag{12}$$

*and*

$$\max\{p - a^-, 0\} \leq \min\{q_i - b_i^-, 1\} \tag{13}$$

*hold for every $i = 1, \dots, k$, then*

$$M_a^p((f_1 + \dots + f_k)(A); x) \leq M_{b_1}^{q_1}(f_1(A); x) + \dots + M_{b_k}^{q_k}(f_k(A); x). \tag{14}$$

*The reverse inequality holds in* (14) *if*

$$\min\{p + a^+, 1\} \geq \max\{q_i + b_i^+, 0\} \tag{15}$$

*and*

$$\min\{p - a^-, 0\} \geq q_i - b_i^- \tag{16}$$

*hold for $i = 1, \dots k$.*

Proof. This is a consequence of the following ([4]):

**Lemma 3.** *Let $a, b_1, \dots, b_k, p, q_1, \dots, q_k \in \mathbb{R}, k \geq 2$. Then the inequality*

$$M_{n,a}(x_1 + \dots + x_k; w)_p \leq M_{n,b_1}(x_1; w)_{q_1} + \dots + M_{n,b_k}(x_k; w) q_k$$

*holds for every $n \in \mathbb{N}$, $x_1, \dots, x_k \in \mathbb{R}_+^n$, iff* (12) *and* (13) *hold for every $i = 1, \dots, k$. The reverse inequality holds if* (15) *and* (16) *hold for $i = 1, \dots, k$.*

**Theorem 4.** *Let $A$ be a positive definite Hermitian matrix with eigenvalues $\lambda_i$ such that*

$$0 < m \le \lambda_i \le M \quad (i = 1, \dots, n).$$

*Then*

$$M_b^q(A; x) \le C(m, M) M_a^p(A; x) \tag{16}$$

*where $a, b, p, q$ are fixed numbers such that* (8) *holds, where $C(m, M)$ is defined by*

$$C(m, M) = \Gamma_{b,q}(t_0, \gamma)/\Gamma_{a,p}(t_0, \gamma) \tag{18}$$

*$\gamma = M/m$ and $t_0$ is the unique positive root of the equation*

$$\lambda_{a,p}(\gamma)(\gamma^q + t)(\gamma^{b+q} + t) = \lambda_{b,q}(\gamma)(\gamma^p + t)(\gamma^{a+p} + t)$$

*where, for $t > 0$,*

$$\lambda_{a,p}(t) = \begin{cases} t^p \dfrac{t^{a-1}}{a}, & a \ne 0 \\ t^p \log t, & a = 0, \end{cases}$$

*and*

$$\Gamma_{a,p}(t, \gamma) = \begin{cases} ((\gamma^{a+p} + t)/(\gamma^p + t))^{1/a}, & a \ne 0 \\ \exp((\gamma^p \log \gamma)/(\gamma^p + t)), & a = 0. \end{cases}$$

PROOF. This is a simple consequence of the following ([5]):

**Lemma 4.** *Let $0 < m < M < \infty$, let*

$$C(m, M) = \sup_{x \in [m,M]^n} M_{n,b}(x; w)_q / M_{n,a}(x; w)_p$$

*where $a$, $b$, $p$, $q$ are fixed numbers such that* (8) *holds. Further, let $\lambda_{a,p}$, $\Gamma_{a,p}$, $t_0$, $\gamma$ be defined as in Theorem 4. Then $C(m, M)$ is given by* (18).

*Remark.* Note that results in [3–5] are obtained in the non-weighted case, i.e., for $w = (1, \dots, 1)$. However, the previous lemmas can easily be obtained from the non-weighted case.

## 4. Inequalities for quasi-arithmetic means

**Theorem 5.** *Let $K$, $L$, $M$ be three differentiable strictly monotone functions from the closed interval $I$ to $\mathbb{R}$; and let $\phi, \psi, \chi$ be three functions* *from $I$ to $\mathbb{R}_+$, $f : I^2 \to I$ such that for all $u, v, s, t \in I$, the following inequality holds.*

$$\left(\frac{M \circ f(u,v) - M \circ f(t,s)}{M' \circ f(t,s)}\right) \frac{\chi \circ f(u,v)}{\chi \circ f(t,s)} \leq \tag{19}$$
$$\leq \left(\frac{K(u) - K(t)}{K'(t)}\right) \frac{\phi(u)}{\phi(t)} f_1'(t,s) + \left(\frac{L(v) - L(s)}{L'(s)}\right) \frac{\psi(v)}{\psi(s)} f_2'(t,s).$$

*Let $A$ be a normal matrix with eigenvalues in $J$ and $g, h : J \to I$ are given functions. Then for $x \in C^n$, $(x \neq 0)$*

$$f(\tilde{K}(g(A); x, \phi), \tilde{L}(h(A); x, \psi)) \geq \tilde{M}(f(g(A), h(A)); x; \chi). \tag{20}$$

(Note that $f(g(A), h(A))$ is the matrix $U^*[f(g(\lambda_1), h(\lambda_1)), f(g(\lambda_2), h(\lambda_2)), \ldots, f(g(\lambda_n), h(\lambda_n))]U$).

Proof. This is a simple consequence of the following lemma [10, p.262].

**Lemma 5.** *Let functions $K, L, M, \phi, \psi, \chi$ be defined as in Theorem 5 and let $a, b, \in I^n$ then*

$$f(K_n(a;\phi),\ L_n(b;\psi)) \geq M_n(f(a,b);\chi)$$

*holds if and only if, for all $u, v, s, t \in I$, (19) holds. Here $f(a,b)$ is the vector whose $j^{\text{th}}$ component is $f(a_j, b_j)$.*

*Remark.* Note that in [8–10] a more general result than our Lemma 5 was proved. Of course, Lemma 5 follows from this result in the cases when $\phi_i = w_i\phi$, $\psi_i = w_i\psi$, and $\chi_i = w_i\chi$ $(i = 1, \ldots, n)$.

**Theorem 6.** *With the notations of Theorem 5 with $\lambda_i \in I$ $(i = 1, \cdots, n)$ ($\lambda_i$ are the eigenvalues of $A$),*

$$\tilde{M}(A; x; \chi) \leq \tilde{K}(A; x; \phi) \tag{21}$$

*if, for all $u, t \in I$,*

$$\left(\frac{M(u) - M(t)}{M'(t)}\right) \frac{\chi(u)}{\chi(t)} \leq \left(\frac{K(u) - K(t)}{K'(t)}\right) \frac{\phi(u)}{\phi(t)}.$$

Proof. Immediate from Theorem 5 taking $f(x,y) = x$, $g(x) = x$.

A very important particular case of Theorem 5 is when $f(x,y) = x + y$. Then, we get

$$\tilde{K}(g(A); x; \phi) + \tilde{L}(h(A); x; \psi) \geq \tilde{M}(g(A) + h(A); x; \chi)$$

holds if, for all $u, v, s, t \in I$,

$$\frac{M(u+v)-M(t+s)}{M'(t+s)}\frac{\chi(u+v)}{\chi(t+s)} \leq \frac{K(u)-K(t)}{K'(t)}\frac{\phi(u)}{\phi(t)} + \frac{L(v)-L(s)}{L'(s)}\frac{\chi(v)}{\chi(s)}.$$

## References


[1] J. Aczél and Z. Daróczy, Über verallgemeinerte guasilineare Mittelwerte, die mit Gewichts functionen gebildet sind, *Publ. Math. Debrecen* **10** (1963), 171–190.

[2] Z. Daróczy and L. Losonczi, Über den Vergleich von Mittelwerten, *Ibid.* **17** (1970), 289–297.

[3] Z. Páles, On Hölder-type inequalities, *J. Math. Anal. Appl.* **95** (1983), 457–466.

[4] Z. Páles, A generalization of the Minkowski inequality, *Ibid.* **90** (1982), 456–462.

[5] Z. Páles, On complementary inequalities, *Publ. math. Debrecen* **30** (1983), 75–88.

[6] M. Bajraktarević, Sur une généralisation des moyennes quasilineare, *Publ. Inst. Math. Beograd* 3 **(17)** (1963), 69–76.

[7] M. Bajraktarević, Über die Vergleichbarkeit der mit Gerwichtsfunktionen gebildeten Mittelwerte, *Stud. Sci. Math. Hungar.* **4** (1969), 3–8.

[8] L. Losonczi, Über eine neue klasse von Mittelwerte, *Acta Sci. Math. (Szeged)* **32** (1971), 71–81.

[9] P. S. Bullen, On a Theorem of L. Losonczi, *Univ. Beograd Publ. Elektrotehn Fak. Ser. Mat. Fiz.* **412-460** (1973), 105–108.

[10] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, Means and their inequalities, *D. Reidel Publ. Comp., Dordrecht, Boston, Lancaster, Tokyo*, 1988.

[11] M. Marcus and H. Minc, A survey of Matrix theory and inequalities, *Allyn and Bacon*, *Boston*, 1964.

[12] R. Bellman, Introduction to matrix Analysis, *McGraw-Hill*, *New York*, 1960.

[13] B. Mond, A matrix inequality including that of Kantorovich, *J. Math. Anal. Appl.* **13** (1966), 49–52.

[14] J. E. Pečarić and B. Mond, A matrix inequality including that of Kantorovich–Hermite II, *Ibid.* **168** (1992), 381–384.



B. MOND
DEPARTMENT OF MATHEMATICS
LA TROBE UNIVERSITY
BUNDOORA, VICTORIA 3083, AUSTRALIA

J. E. PEČARIĆ
FACULTY OF TEXTILE TECHNOLOGY
UNIVERSITY OF ZAGREB
ZAGREB, CROATIA


*(Received December 20, 1993)*