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Geodesics in a Finsler surface with one-parameter group of motions

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Abstract. The surfaces with one-parameter groups of motions are classified to define the Finsler surfaces of revolution. We generalize Clairaut's relation between a geodesic and parallel circles in a Finsler surface of revolution and its consequences, stating the global behavior of geodesics in a Finsler 2-torus of revolution with non-symmetric distance. As for the local behavior of geodesics in a Finsler manifold, we recall the reversibility of geodesics, using the symmetric part of the Finsler metric.

1. Introduction

Let (M, F) be a *Finsler n-manifold* which is by definition a smooth n-manifold equipped with fundamental function $F : TM \to \mathbb{R}$ such that F is smooth on $TM \setminus \{0\}$, $F(x, t\dot{x}) = tF(x, \dot{x})$ for all $t > 0$ and $\dot{x} \in T_xM$, and F is strictly convex on all tangent spaces T_xM. Here TM denotes the tangent bundle of M. We define, as usual, the length $L_F(c)$ of a piecewise smooth curve c in M with respect to F and an intrinsic distance d on M induced by F. The distance d is not symmetric, in general.

We call a curve $\sigma : [a, b] \to M$ a (*forward*) *geodesic* if it satisfies the Euler–Lagrange equation $EL(F, \sigma) = 0$. Namely, a geodesic is an extremal of the variation problem of lengths of curves c. A geodesic is locally minimizing in (M, d). Moreover, if (M, d) is complete, then, for any two points $p, q \in M$, there

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exists a minimizing geodesic σ from p to q, i.e., $d(p,q) = L_F(\sigma)$. We say that a geodesic $\gamma : (-\infty, \infty) \to M$ is a *straight line* if $L_F(\gamma|_{[s,t]}) = d(\gamma(s), \gamma(t))$ for all $s, t \in (-\infty, \infty)$ with $s < t$.

Busemann and Pedersen [7] have determined how the straight lines behave in the universal covering planes of 2-tori with one-parameter groups of motions. Their method is for G-spaces defined by Busemann [6] but can be applied to our case of non-symmetric distance. Their method was influenced by and is influential to related topics.

The geodesics satisfy Clairaut's relation (cf. [25]) in a Riemannian surface of revolution, from which we can evaluate the behavior of geodesics on it. However, we need to study more for the complete description of geodesics, such as conjugate points, rays and straight lines, etc.. The geodesics on tori of revolution embedded in the Euclid space $\mathbb{E}^3$ is studied by Bliss [5] and Kimball [17]. Gravesen–Markvorsen–Sinclair–Tanaka [10] has studied the cut locus in a torus of revolution.

Morse [20] and Hedlund [12] studied the geodesics on arbitrary Riemannian tori whose lifts into the universal covering space are straight lines. Their methods are unified by Bangert [2] with those of Mather [19] and Aubry–Le Daeron [1] to study a monotone twist map of the annulus and the discrete Frenkel–Kontrova model. In this way, the method of finding straight lines by displacement functions can be applied in more general situations. Indeed, in [2], we can see the complete classification of straight lines in the universal covering plane of an arbitrary 2-torus, as an application. The classification is described in terms of rotation numbers.

A complete Finsler manifold $(\mathbb{R}^n, F)$ is without conjugate points if and only if all geodesics are straight in $(\mathbb{R}^n, F)$. Any Riemannian metric on an n-torus without conjugate points is flat. This theorem is proved by Hopf [11] for $n = 2$ and Burago–Ivanov [4] for $n \geq 3$. However, this theorem is not true for Finsler manifolds. Zinoviev [26] gives examples of symmetric Finsler metrics on n-tori without conjugate points by showing some condition that generalized metrics on n-tori of revolution has no conjugate points.

A surface with one-parameter group of motions φ_t is one of Lagrangian systems which are invariant under the action of Lie groups. In studying those systems, a geodesic is said to be a *relative equilibrium* if it coincides with an integral curve of a fundamental vector field of the action of φ_t. The problem of finding relative equilibrium points for the Euler–Lagrange field of an invariant Lagrangian has been studied by many researchers (cf. [9], [18], [22]).

Though the study of geodesics in Finsler tori of revolution has been developed, it still remains to determine the behavior of geodesics whose lifts into its universal covering plane are not straight lines. The main purpose of this paper is to generalize Clairaut's theorem and determine the global behavior of all geodesics in a Finsler surface of revolution. We devise certain methods for showing the global behavior of those geodesics, although it is obvious from Clairout's relation in the Riemannian case.

In §2 we classify the Finsler surfaces with one-parameter groups of motions, observing the number of fixed points. From this we can define a Finsler surface of revolution. Roughly speaking, a surface of revolution is a surface with one-parameter group of motions such that its period is finite and the number of its fixed points is zero, one or two. However, in case its one-parameter group of motions is not like a rotation, we do not consider it to be a surface of revolution. In this manner we can regard the one-parameter group of motions as a rotation and its orbits as parallel circles.

In §3 we define a local Riemannian metric g in a neighborhood U_γ around a curve γ in M such that it is a geodesic in (U_γ, g) if and only if so in (U_γ, F). Such a Riemannian metric g has been used to study the implication of the Chern connection (cf. [24]). When (M, F) is a surface of revolution, we can introduce a local Riemannian surface (U_γ, g) of revolution and prove some properties of geodesics, using Clairaut's relation for geodesics in Riemannian surfaces of revolution.

In §4 we generalize Clairaut's relation for geodesics in a Riemannian surface of revolution. In fact, we find not only Clairaut's constant, but also the global behavior of geodesics in a strip between two parallel circles.

In §5 we state the main theorem which shows the global behavior of all geodesics in a Finsler torus of revolution. The behavior of the straight lines are mentioned in the theorem for the complete description, although they are direct consequences from [7] and [2].

In §6 we discuss the reversibility of geodesics in (M, F) by using the symmetric part A and the skew-symmetric part B of F. The Euler–Lagrange equations of geodesics for F and A are compared. In order to evaluate the reversibility of geodesics in our examples of §7, we especially treat the case where B arises from a 1-form on M. Namely, we study some conditions implying that $EL(B, \gamma) = 0$ for a curve γ in M. Crampin [8] and Nagano [21] have studied some conditions for the reversibility of geodesics in (M, F) by comparing the equation of geodesics for F with those for its reversed metric $\bar{F}(x, \dot{x}) := F(x, -\dot{x})$.

In §7 we discuss Example 7.1 in which all types of geodesics stated in the main theorem appear. Their fundamental functions are Randers metrics on a plane $\mathbb{R}^2$. One of our examples shows a fact which does not take place in the Riemannian and reversible Finslerian cases, i.e., there exists a Finsler 2-torus of revolution having no pole.

In §8 we show that any strip S bounded by two parallel circles in a Finsler surface of revolution can be embedded in a Finsler torus of revolution M. Thanks to existing straight lines in the universal covering surface $\widetilde{M}$, we can get some information on the behavior of geodesics in S via its lift into $\widetilde{M}$. Indeed, we use the embedding strip to generalize Clairaut's theorem (see Theorem 4.5).

In §9 we discuss the relations between some distances induced by F and clarify the difference between symmetric and non-symmetric distances. The symmetrization of the distance d is used in [23] to find a minimal geodesic loop. The symmetrization of the Finsler metric F is used to evaluate the reversibility of geodesics in (M, F) (see §6).

2. Surfaces with one-parameter groups of motions

Let (M, F) be a complete oriented Finsler surface with one-parameter group of motions φ_t on (M, F). Namely, φ_t satisfies $F(\varphi_t(x), \varphi_{t*}(\dot{x})) = F(x, \dot{x})$ for all $x \in M$, $\dot{x} \in T_xM$ and $t \in (-\infty, \infty)$. Here a *motion* on (M, F) is by definition an isometry which preserves the orientation of M. The distance d is invariant under φ_t. Let H be the vector field on M generating φ_t, i.e., $H(x)$ is the tangent vector of the curve $e(t) = \varphi_t(x)$ at $t = 0$. Let $\mathrm{Sing}(H) := \{x \in M \mid H(x) = 0\}$ and let $\#\,\mathrm{Sing}(H)$ denote the number of points in $\mathrm{Sing}(H)$. The set of all fixed points of φ_t is $\mathrm{Sing}(H)$.

Lemma 2.1. *Let (M, F) be a complete oriented Finsler surface and φ_t a one-parameter group of motions on (M, F). If φ_t is not the identity map for all $t \in (-\infty, \infty) \setminus \{0\}$, then the interior $\mathrm{Sing}(H)^0$ of $\mathrm{Sing}(H)$ is empty and the number of elements of $\mathrm{Sing}(H)$ is at most two.*

Proof. We first claim that if the interior $\mathrm{Sing}(H)^0$ of $\mathrm{Sing}(H)$ is not empty, then φ_t is the identity map for all $t \in (-\infty, \infty)$, because any geodesic segment is determined by two points which are sufficiently close on it. Therefore, from the assumption, we have $\mathrm{Sing}(H)^0 = \emptyset$.

Let a point $p \in \mathrm{Sing}(H)$ exist and let N_p be the normal neighborhood around p. We next prove that $\mathrm{Sing}(H) \cap N_p \setminus \{p\} = \emptyset$. Suppose for indirect proof that there exists a point $q \in \mathrm{Sing}(H) \cap N_p \setminus \{p\}$. Then we can have a unique minimizing geodesic segment $T(p, q)$ from p to q. From this, $\varphi_t(x) = x$

for all $x \in T(p,q)$ and $t \in (-\infty, \infty)$. Take a point $q_1 \in N_p$ such that a minimizing geodesic T_1 from q_1 to $T(p,q)$ is contained in N_p and $p_1 := T_1 \cap T(p,q) \in T(p,q) \setminus \{p,q\}$. We show that both cases $H(q_1) = 0$ and $H(q_1) \neq 0$ do not happen.

Assume that $H(q_1) = 0$. We then have $T(q_1,p_1) \subset \mathrm{Sing}(H)$ and furthermore, $\cup_{x \in T(q_1,p_1)} T(x,p_2) \subset \mathrm{Sing}(H)$ for a point $p_2 \in T(p,q)$ sufficiently close to p_1, since q_1, p_1, p_2 and $x \in T(q_1,p_1)$ are fixed points of all φ_t. This implies that $\mathrm{Sing}(H)^0 \neq \emptyset$, a contradiction.

Assume that $H(q_1) \neq 0$. Choose a parameter t_0 sufficiently close to 0 such that $T(\varphi_{t_0}(q_1), p_1)$ passes through the side of $T(q_1,p_1)$ which is different from the side of $T(q_1,p_1)$ containing $T(p_1,q)$. This is possible, because p_1 is a fixed point of φ_t and, hence, $d(\varphi_{t_0}(q_1), p_1) = d(q_1,p_1)$. Then we can choose a point $p_2 \in T(p_1,q)$ close to p_1 such that $T(\varphi_{t_0}(q_1), p_2)$ intersects $T(q_1,p_1)$ at $T(\varphi_{t_0}(q_1), p_2) \cap T(q_1,p_1) =: z$.

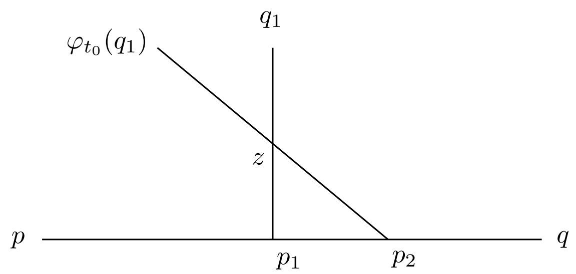

Figure 1. p, q, p_1, p_2 and z

Since p_1 and p_2 are fixed points of all φ_t, we have

$$\begin{aligned} d(q_1,p_1) + d(q_1,p_2) &= d(\varphi_{t_0}(q_1), p_1) + d(q_1,p_2) \\ &< d(\varphi_{t_0}(q_1), z) + d(z,p_1) + d(q_1,z) + d(z,p_2) \\ &= d(\varphi_{t_0}(q_1), z) + d(z,p_2) + d(q_1,z) + d(z,p_1) \\ &= d(\varphi_{t_0}(q_1), p_2) + d(q_1,p_1) \\ &= d(q_1,p_1) + d(q_1,p_2), \end{aligned}$$

a contradiction, proving $\mathrm{Sing}(H) \cap N_p \setminus \{p\} = \emptyset$.

As was seen in the above, the set $\mathrm{Sing}(H)$ is discrete in M. Moreover, if $p \in \mathrm{Sing}(H)$, then $\mathrm{Sing}(H) \cap N_p = \{p\}$. Assume that there exists another fixed point q of φ_t. We then have $\varphi_t(T(p,q) \setminus \{q\}) \subset N_p$. This implies that $M \setminus \{q\} = N_p$, meaning that any points other than q are not fixed by φ_t. Therefore, $\mathrm{Sing}(H)$ consists of at most two points. □

Lemma 2.2. *Let (M, F) be a complete oriented Finsler surface with one-parameter group of motions φ_t on (M, F). Then the following are true.*

(1) *If $\# \mathrm{Sing}(H) \geq 3$, then $H = 0$ identically in M.*

(2) *If $\# \mathrm{Sing}(H) = 2$, then M is homeomorphic to a sphere.*

(3) *If $\# \mathrm{Sing}(H) = 1$, then M is homeomorphic to a plane.*

(4) *If $\# \mathrm{Sing}(H) = 0$, then M is homeomorphic to a plane, a cylinder or a torus.*

Proof. If p is a fixed point of φ_t, then the trajectories of $\varphi_t(q)$, $t \in (-\infty, \infty)$, are the circles with center p and radii $d(p, q)$ (or $d(q, p)$). Hence, (1), (2) and (3) follow from the proof of Lemma 2.1. (4) follows from the Poincaré–Hopf index theorem for the vector field H. □

Let $\mathrm{Orb}(x)$ denote the orbit of $x \in M$ by φ_t: $\mathrm{Orb}(x) := \{\varphi_t(x) \,|\, t \in (-\infty, \infty)\}$. Let $x \notin \mathrm{Sing}(H)$. We define the *period* $\tau(x)$ of $x \in M$ for φ_t by $\tau(x) := \min\{s > 0 \,|\, \varphi_s(x) = x\}$ if $\{s > 0 \,|\, \varphi_s(x) = x\} \neq \emptyset$, and, otherwise, ∞.

Lemma 2.3. *Let (M, F) be a complete oriented Finsler surface with one-parameter group of motions φ_t on (M, F). The period $\tau(x)$ is constant for $x \in M \smallsetminus \mathrm{Sing}(H)$.*

Proof. Let $x, y \in M \smallsetminus \mathrm{Sing}(H)$, $x \neq y$. If $y \in \mathrm{Orb}(x)$, then $\tau(y) = \tau(x)$. In fact, if $y = \varphi_s(x)$, then

$$y = \varphi_s(x) = \varphi_s(\varphi_{\tau(x)}(x)) = \varphi_{\tau(x)}(\varphi_s(x)) = \varphi_{\tau(x)}(y).$$

Assume that $y \notin \mathrm{Orb}(x)$. We then have $\mathrm{Orb}(x) \cap \mathrm{Orb}(y) = \emptyset$. Let $x_1 \in \mathrm{Orb}(x)$ be a foot of y on $\mathrm{Orb}(x)$, i.e., $x_1 \in \mathrm{Orb}(x)$ and $d(y, x_1) = d(y, \mathrm{Orb}(x))$. Recall that x_1 is the unique foot of any point $y_1 \in T(y, x_1) \smallsetminus \{y\}$ on $\mathrm{Orb}(x)$. We claim that there exists only one foot of y on $\mathrm{Orb}(x)$. If there exists another foot x_2 of y on $\mathrm{Orb}(x)$, then there exists a parameter $s \neq 0$ such that $|s|$ is sufficiently small, $\varphi_s(x_2) \neq x_1$ and $T(y, x_1) \cap \varphi_s(T(y, x_2)) \neq \emptyset$. Then the intersection point has two feet on $\mathrm{Orb}(x)$, x_1 and $\varphi_s(x_2)$, a contradiction. By the same argument above, there is the unique minimizing geodesic segment $T(y, x_1)$. From this fact, for any point $y_1 = \varphi_s(y) \in \mathrm{Orb}(y)$, if x_2 is the foot of y_1 on $\mathrm{Orb}(x)$, we then have $T(y_1, x_2) = \varphi_s(T(y, x_1))$ and $\varphi_s(x_1) = x_2$, because φ_s is a motion on M. In particular, since $\varphi_{\tau(x)}(x_1) = x_1$, we have $\varphi_{\tau(x)}(y) = y$. This shows $\tau(x) = \tau(y)$. □

Set $\tau(\varphi_t) := \tau(p)$ for a point $p \notin \mathrm{Sing}(H)$. From Lemma 2.3, $\tau(\varphi_t)$ is independent of the choice of the point $p \in M \setminus \mathrm{Sing}(H)$. We say that $\mathrm{Orb}(x)$ is a *parallel circle* through x if $\tau(\varphi_t) < \infty$. If $p \in \mathrm{Sing}(H)$, then a parallel circle through q is the (forward) circle $S^+(p, d(p,q)) := \{x \in M \mid d(p,x) = d(p,q)\}$ with center p and radius $d(p,q)$ and, at the same time, the backward circle $S^-(p, d(q,p)) := \{x \in M \mid d(x,p) = d(q,p)\}$ with center p and radius $d(q,p)$. Note that when a point $p \in M$ is a fixed point of φ_t, $S^+(p, d(p,q)) = S^-(p, d(p,q))$ even if $d(p,q) \neq d(q,p)$.

We say that (M, F) is a *Finsler surface of revolution* if it is in the cases of (2), (3) of Lemma 2.2 and the following cases appearing in (4):

(4-1) M is topologically a torus $S^1 \times S^1$ and $S^1 \times \{y\} = \{\varphi_t(x,y) \mid t \in (-\infty, \infty)\}$ for all $(x,y) \in S^1 \times S^1$.

(4-2) M is topologically a cylinder $S^1 \times \mathbb{R}$ and $S^1 \times \{y\} = \{\varphi_t(x,y) \mid t \in (-\infty, \infty)\}$ for all $(x,y) \in S^1 \times \mathbb{R}$.

When M is a Finsler surface of revolution with one-parameter group of motions φ_t, we call φ_t a *rotation* of M.

A Finsler surface of revolution is topologically a sphere, a torus, a cylinder or a plane. Any strip bounded by two parallel circles in those surfaces is isometrically embedded into a certain torus of revolution (see §8). Hence, it is important to determine the behavior of geodesics in a torus of revolution.

3. Local Riemannian metrics and geodesics

In order to study the behavior of geodesics in (M, F), we define a Riemannian metric g in some neighborhood U around the geodesics under consideration, in such a way that they become geodesics in (U, g). Applying the geometry of geodesics for a Riemannian manifold in (U, g), we obtain certain informations on the original geodesics in (M, F) (cf. [3], [14], [15], [16], [23]).

We begin with a general discussion on Finsler n-manifolds. Let $g(x, \dot{x})$, $\dot{x} \in T_xM$, denote a Riemannian metric in $T_xM \setminus \{0\}$ defined by

$$g_{ij}(x, \dot{x}) = \frac{1}{2} \frac{\partial^2 F^2}{\partial \dot{x}^i \partial \dot{x}^j}(x, \dot{x}).$$

The following lemma is a well-known elementary fact and its proof is omitted here.

Lemma 3.1 (cf. [3])**.** *Let (M, F) be a Finsler manifold. Let $y, z, w \in T_xM$ with $y \neq 0$. The following are true.*

(1) $g(x,y)(z,w) = \dfrac{\partial^2 f}{\partial t \partial s}(0,0)$ *where* $f(s,t) = F(x, y + sz + tw)^2/2$.

(2) $g(x,y)(y,y) = F(x,y)^2$.

(3) $g_{ij}(x,ty) = g_{ij}(x,y)$ *for* $t > 0$.

Let X be a vector field in an open set U such that $X(x) \neq 0$ for all $x \in U$ and $a_x(t)$ the integral curve of X with $a_x(0) = x$. Assume that there exists a hypersurface S with coordinate system $y : V \subset \mathbb{R}^{n-1} \to S$ in U such that the map $(y,t) \to a_y(t) \in U$ from $\{(y,t) \,|\, y \in V, a_y(t) \in U\}$ is a coordinate system of U. We define a Riemannian metric g on U associated with X by $g_{ij}(x) = g_{ij}(x, X(x))$ for $x = (y,t) \in U$. Some relations of the Levi–Civita connection of (U,g) and the Chern connection of (U,F) are described in [23], [24]. In this article, we only use it in the form of the following lemma.

Lemma 3.2. *Let* (M,F) *be a Finsler manifold. In the notation above, let* $x_0 \in S$*. Then* $a_{x_0}(t)$ *is a constant speed geodesic in* (U,g) *if and only if so is it in* (U,F)*.*

Proof. Let $e_n = (0, \cdots, 0, 1) \in \mathbb{R}^n$. Since $(x, X(x)) = (x, e_n)$ by the coordinate system given as above, we have

$$\frac{\partial g_{ij}}{\partial x^k}(x) = \frac{\partial g_{ij}}{\partial x^k}(x, X(x)) + \sum_{h=1}^{n} \frac{\partial g_{ij}}{\partial \dot{x}^h}(x, X(x)) \frac{\partial X^h}{\partial x^k}(x) \ = \frac{\partial g_{ij}}{\partial x^k}(x, X(x)),$$

and, hence, ${}^g\Gamma_i{}^k{}_j(x) = {}^F\Gamma_i{}^k{}_j(x, X(x))$ for $x \in U$ where ${}^g\Gamma_i{}^k{}_j$ and ${}^F\Gamma_i{}^k{}_j$ are the Christoffel symbols with respect to g and F, respectively. Therefore, $a_x(t)$ is a constant speed geodesic in (U,g) if and only if so is it in (M,F) (see p. 125 in [3]). □

From now on, we assume that M is a surface of revolution with rotation φ_t. Let $\gamma : (0,a) \to M$ be the unit speed geodesic in (M,F) such that $\dot{\gamma}(s)$ is not parallel to $H(\gamma(s))$ for all $s \in (0,a)$. Set

$$U_\gamma := \{\varphi_t(\gamma(s)) \,|\, t \in (-\infty, \infty), s \in (0,a)\}.$$

Let $\widetilde{U}_\gamma$ denote the universal covering surface of U_γ. As to be shown in Lemma 3.3, it is homeomorphic to a strip $(-\infty, \infty) \times (0,a)$ in a plane. Let $\tilde{\gamma}$ and $\tilde{\varphi}_t$ be lifts of γ and φ_t into $\widetilde{U}_\gamma$, respectively.

Lemma 3.3. *Let* (M,F) *be a Finsler surface of revolution with rotation* φ_t*. Let* $\gamma : (0,a) \to M$ *be a unit speed geodesic in* (M,F) *such that* $H(\gamma(s)) \neq 0$ *and* $\dot{\gamma}(s)$ *is not parallel to* $H(\gamma(s))$ *for all* $s \in (0,a)$*. The following are true.*

(1) *U_γ is homeomorphic to a cylinder $S^1 \times (0,a)$.*

(2) *If $\psi : (-\infty,\infty) \times (0,a) \to \widetilde{U}_\gamma$ is a map given by $\psi(t,s) = \tilde{\varphi}_t(\tilde{\gamma}(s))$, then it makes a coordinate system for $\widetilde{U}_\gamma$. Under the identification through ψ with this coordinate system, if $x = (t,s)$ and $y = (u,v) \in T_x\widetilde{U}_\gamma$, then $\tilde{\varphi}_r(t,s) = (t+r,s)$ and $\tilde{\varphi}_{r*}(u,v) = (u,v) \in T_{\tilde{\varphi}_r(x)}\widetilde{U}_\gamma$.*

(3) *Let $\gamma_t(s) = \varphi_t(\gamma(s))$ for all $t \in (-\infty,\infty)$ and $s \in (0,a)$. We then have a foliation of geodesics γ_t for U_γ, i.e., for any point $x \in U_\gamma$, there exists the unique number t_0, $-\tau(\varphi_t)/2 \le t_0 < \tau(\varphi_t)/2$, such that γ_{t_0} passes through x.*

Proof. From the definition of a surface of revolution, U_γ is homeomorphic to a cylinder $S^1 \times (0,a)$, proving (1).

Notice that the set of orbits $\mathrm{Orb}(\gamma(s))$, $s \in (0,a)$, makes a foliation of U_γ. Since $\dot{\gamma}(s)$ and $H(\gamma(s))$ are not parallel, the different two points in $\gamma((0,a))$ do not belong to the same orbit. This shows that ψ is a coordinate system for $\widetilde{U}_\gamma$. The other part of (2) follows from the construction of ψ.

(3) follows from Lemma 2.3. □

Using the coordinate system obtained in Lemma 3.3, we make a Riemannian metric $g_{(t,s)} = g(\varphi_t(\gamma(s)), \varphi_{t*}(\dot{\gamma}(s)))$ on U_γ of γ. Let $\widetilde{F}$ and $\tilde{g}$ denote the lifts of F and g into $\widetilde{U}_\gamma$, respectively.

Lemma 3.4. *Let (M,F) be a Finsler surface of revolution with rotation φ_t. Let $\gamma : (0,a) \to M$ be a unit speed geodesic in (M,F) such that $\dot{\gamma}(s)$ is not parallel to $H(\gamma(s))$ for all $s \in (0,a)$. The following are true.*

(1) *φ_t is also a one-parameter group of motions on (U_γ, g).*

(2) *$\gamma : (0,a) \to U_\gamma$ is also a unit speed geodesic in (U_γ, g).*

(3) *If $\dot{\gamma}(0) = cH(\gamma(0))$ (resp., $\dot{\gamma}(a) = cH(\gamma(a))$) for some number $c > 0$, then $\ddot{\gamma}(0)$ (resp., $\ddot{\gamma}(a)$) points to the interior of U_γ.*

Proof. From (2) of Lemma 3.3, in the suitable coordinate system, we have $\widetilde{F}(t,s,u,v) = \widetilde{F}(t+r,s,u,v)$ for $x = (t,s)$ and $y = (u,v) \in T_xM$ and $r \in (-\infty,\infty)$. Therefore, we have $\tilde{g}_{ij}(t,s,u,v) = \tilde{g}_{ij}(t+r,s,u,v)$, meaning that $g(x,y)(z,w) = g(\varphi_r(x), \varphi_{r*}(y))(\varphi_{r*}(z), \varphi_{r*}(w))$ for $x \in U_\gamma$, $y \in T_xM$ and $r \in (-\infty,\infty)$. Hence, φ_t is a one-parameter group of motions in (U_γ, g). This proves (1).

(2) follows from Lemma 3.2, i.e., the fact that the differential equations of a constant speed geodesic $\gamma(s) = (0,s)$ in (U_γ, F) and in (U_γ, g) equal along γ, since $\tilde{g}_{(t,s)} = \tilde{g}(t,s,0,1)$ for $(t,s) \in (-\infty,\infty) \times (0,a)$.

(3) follows from (1) and (2). Namely, in a Riemannian surface U_γ of revolution, we see that $\ddot\gamma(0) = \lim_{s\to+0}\ddot\gamma(s)$ (resp., $\ddot\gamma(a) = \lim_{s\to a-0}\ddot\gamma(s)$) points to the interior of U_γ. □

4. Clairaut's theorem

Let (M, F) be a Finsler surface of revolution with rotation φ_t. Let $h(x) := F(x, H(x))$ and $h^-(x) = F(x, -H(x))$ for $x \in M$. Obviously, h and h^- are constant on each orbit $\mathrm{Orb}(x)$.

The function h and h^- defined above are called the *locked Lagrangian* in the literature on relative equilibria (cf. [18]). Although the statement of the following proposition is known (cf. [9], [18]), we give the proof as an application of Clairaut's theorem for Riemannian case.

Proposition 4.1. *Let (M, F) be a Finsler surface of revolution with rotation φ_t. Assume that there exists a point $p \in M$ such that $h(p) \neq 0$ and $dh_p = 0$. Then, $e(t) = \varphi_t(p)$, $t \in (-\infty.\infty)$, is a geodesic in (M, F).*

Proof. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a unit speed geodesic such that $\gamma(0) = p$ and $\dot\gamma(s)$ is not parallel to $H(\gamma(s))$ for all $s \in (-\varepsilon, \varepsilon)$. As before, let U_γ denote the φ_t-invariant neighborhood around $\mathrm{Orb}(p)$. We introduce a coordinate system $(t, s) = \varphi_t(\gamma(s))$ on U_γ. This time, we define a Riemannian metric g on U_γ by $g_x = g(x, H(x))$. Then, φ_t is a one-parameter group of motions and $h(x)$ is the length of $H(x)$ in (U_γ, g). In other words, it is considered to be a subset of a Riemannian surface of revolution and $\mathrm{Orb}(x)$ is a parallel circle. Therefore, the orbit $\mathrm{Orb}(p)$ is a geodesic in (U_γ, g) because of $h(p) \neq 0$ and $dh_p = 0$ (cf. p. 13 in [25]). Lemma 3.2 proves this proposition. □

We say that a geodesic $\gamma : [0, \infty) \to M$ is a *ray* if $L(\gamma|_{[s,t]}) = d(\gamma(s), \gamma(t))$ for all $s, t \in [0, \infty)$ with $s < t$. We say that a ray $\alpha : [0, \infty) \to M$ is a *co-ray* to a ray γ if there exist a sequence of points p_n and a sequence of numbers t_n such that $p_n \to \alpha(0)$, $t_n \to \infty$ and $T(p_n, \gamma(t_n)) \to \alpha$ as $n \to \infty$. A straight line $\alpha : (-\infty, \infty) \to M$ is called an *asymptote* to a ray γ if $\alpha|_{[s,\infty)}$ is a co-ray from $\alpha(s)$ to γ for any $s \in (-\infty, \infty)$.

A motion $\psi : M \to M$ is said to be *axial* if there exist a straight line γ with unit speed and a constant $a > 0$ such that $\psi(\gamma(t)) = \gamma(t + a)$ for all $t \in (-\infty, \infty)$. Such a straight line γ is called an *axis* of ψ. We define the *displacement function* $d_\psi : M \to \mathbb{R}$ of a motion ψ by $d_\psi(x) = d(x, \psi(x))$, $x \in M$.

The following Lemmas 4.2 and 4.3 have been proved in [7] for a G-space defined by BUSEMANN [6]. We emphasize that the Lemmas are valid for complete Finsler planes with non-symmetric distances.

Let $D[\psi, \tilde{\varphi}_\tau]$ be the group of motions generated by ψ and $\tilde{\varphi}_\tau$ on M.

Lemma 4.2 ([7, (2.6) Theorem]). *Let (M, F) be a complete Finsler plane and ψ a motion of (M, F). If there exists a point $p \in M$ such that $d_\psi(p) = \inf\{d_\psi(x) \,|\, x \in M\} > 0$, then ψ is an axial motion with axis through p and $\psi(p)$. All axes of ψ are asymptotes to each other. In particular, if d_ψ is constant on M, then the axes of ψ simply cover M. If M is the universal covering plane of a torus $T^2 = M/D[\psi, \tilde{\varphi}_\tau]$ of revolution, then M is simply covered with the axes of $\psi^n \circ \tilde{\varphi}_t$ for every integer $n \neq 0$ and a number t.*

Lemma 4.3. [7, (2.7) Corollary] *Let (M, F) be a complete Finsler plane and φ_t a one-parameter group of motions on M. If a number t_0 and $p \in M$ exist such that $0 < d(p, \varphi_{t_0}(p)) = \inf\{d_{\varphi_{t_0}}(x) \,|\, x \in M\}$, then the curve $e(t) = \varphi_t(p)$, $t \in (-\infty, \infty)$, is a straight line.*

Lemma 4.4. *Let (M, F) be a complete Finsler plane and φ_t a one-parameter group of motions on (M, F). If there exists a point $p \in M$ such that $h(p) = \inf\{h(x) \,|\, x \in M\} > 0$ (resp., $h^-(p) = \inf\{h^-(x) \,|\, x \in M\}$), then $e(t) = \varphi_t(p)$ (resp., $e(t) = \varphi_{-t}(p)$) is a straight line. The straight lines through the minimum points of h (and h^-) are asymptotes to each other (see Figure 2).*

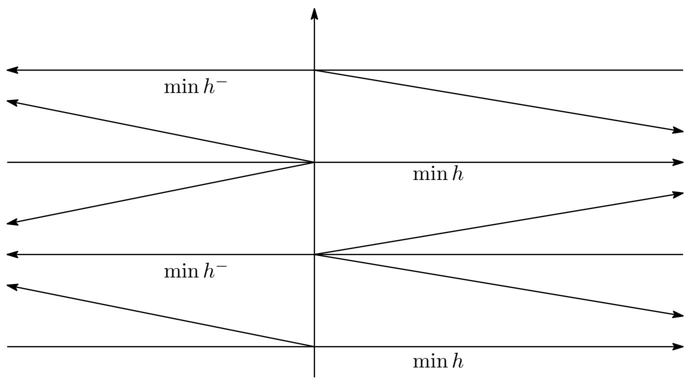

Figure 2. Axes of φ_t, φ_{-t}, co-rays and asymptotes

PROOF. We first assume that there exist a point $q \in M$ and a number t_0 such that $0 < d(q, \varphi_{t_0}(q)) = \inf\{d_{\varphi_{t_0}}(x) \,|\, x \in M\}$. We prove $h(q) = \inf\{h(x) \,|\, x \in M\}$.

We then have $h(q) = h(p)$. In fact, from Lemma 4.3, $e_q(t) = \varphi_t(q)$, $t \in (-\infty, \infty)$, is a straight line in M. Hence, we have, for $s > 0$,

$$\begin{aligned} sh(q) &= \int_0^s F(\varphi_t(q), H(\varphi_t(q))\, dt \; = d(q, \varphi_s(q)) \\ &\le d(q,p) + d(p, \varphi_s(p)) + d(\varphi_s(p), \varphi_s(q)) \\ &\le d(q,p) + \int_0^s F(\varphi_t(p), H(\varphi_t(p))\, dt + d(p,q) \\ &= d(q,p) + sh(p) + d(p,q). \end{aligned}$$

Divide both sides by s and then $s \to \infty$, and we have $h(q) \le h(p)$, meaning that $h(q) = \inf\{h(x) \,|\, x \in M\}$.

Using the point q as in the above argument, we can prove that $e(t) = \varphi_t(p)$ is a straight line as follows. Since $e_q(t)$ is a straight line, we have $0 < d(q, \varphi_s(q)) = \inf\{d_{\varphi_s}(x) \,|\, x \in M\}$ for all $s > 0$ (see [7]). From Proposition 4.1, we know that $e(t)$ is a geodesic. Hence, there exists a sufficiently small $s_0 > 0$ such that $d(p, \varphi_{s_0}(p)) = s_0 h(p)$. From

$$s_0 h(p) = s_0 h(q) = \inf\{d_{\varphi_{s_0}}(x) \,|\, x \in M\},$$

we have $d_{\varphi_{s_0}}(p) = \inf\{d_{\varphi_{s_0}}(x) \,|\, x \in M\}$. It follows from Lemma 4.3 that $e(t)$ is a straight line.

We have to prove that there exist a point $q \in M$ and a number t_0 such that $0 < d(q, \varphi_{t_0}(q)) = \inf\{d_{\varphi_{t_0}}(x) \,|\, x \in M\}$, which we assumed in the above arguments, and then complete the proof. Let X be any strip bounded by two parallel circles $\mathrm{Orb}(x)$ and $\mathrm{Orb}(y)$ such that $p \in X$. We make, from Lemma 8.1 in Appendix 1, a Finsler plane $(\widetilde{M}, \widetilde{F}) \supset (X, F)$ with one-parameter group $\tilde{\varphi}_t$ of motions and a motion ψ on $\widetilde{M}$ with $\tilde{\varphi}_t \circ \psi = \psi \circ \tilde{\varphi}_t$ and $\psi^i(\widetilde{X}) \cap \psi^j(\widetilde{X}) = \emptyset$ for all $i \ne j \in \mathbb{Z}$. We may assume that $h(p) = \inf\{h(x) \,|\, x \in \widetilde{M}\}$. All motions $d_{\tilde{\varphi}_s}$, $s > 0$, attain their minima in $(\widetilde{M}, \widetilde{F})$, because $T^2 = \widetilde{M}/D[\tilde{\varphi}_1, \psi]$ is compact where $D[\tilde{\varphi}_1, \psi]$ is a group of motions generated by $\tilde{\varphi}_1$ and ψ, meaning the existence of q and t_0. From the above argument, we know that $e(t) = \varphi_t(p)$ is a straight line in $\widetilde{M}$. Since X is an arbitrary strip containing p, the straightness of $e(t)$ is valid in M. □

We prove Clairaut's theorem for Finsler surfaces of revolution. To write the statement, we set

$$C(\gamma(s)) = \frac{1}{2} \frac{\partial^2 F(\gamma(s), \dot{\gamma}(s) + u\dot{\gamma}(s) + vH(\gamma(s)))^2}{\partial u \partial v}\bigg|_{u=0, v=0}.$$

Theorem 4.5. *Let (M, F) be a Finsler surface of revolution with rotation φ_t. Let $\gamma : (0, a) \to M$ be a unit speed geodesic in (M, F) such that $\dot\gamma(s)$ is linearly independent of $H(\gamma(s))$ for all $s \in (0, a)$. Then $C(\gamma(s))$ is constant for $s \in (0, a)$. Moreover, if $\dot\gamma(0) = cH(\gamma(0))$ and $\dot\gamma(a) = c'H(\gamma(a))$ for some numbers $c > 0$ and $c' > 0$, then $h(\gamma(0)) = h(\gamma(a))$ and $h(\gamma(s)) > h(\gamma(0))$ for all $s \in (0, a)$. The same is true if h and H are replaced by h^- and $-H$. (See Figure 3.)*

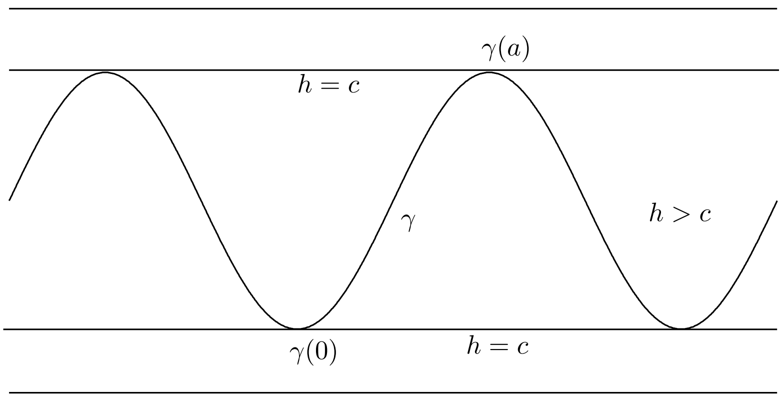

Figure 3. Geodesics between parallel circles

Proof. From Lemma 3.4 and Clairaut's relation for a geodesic in a Riemannian surface of revolution, it follows that $g_{\gamma(s)}(\dot\gamma(s), H(\gamma(s)))$ is constant for s in (U_γ, g). Hence, (1) of Lemma 3.1 shows the theorem.

We prove the second part. Suppose $\dot\gamma(0) = cH(\gamma(0))$ and $\dot\gamma(a) = c'H(\gamma(a))$ for some numbers $c > 0$ and $c' > 0$, respectively. Then the first part of the proof shows $h(\gamma(0)) = h(\gamma(a))$.

We suppose for indirect proof that there exists a number $s_0 \in (0, a)$ such that $h(\gamma(s_0)) \le h(\gamma(0))$. Let X be the strip bounded by parallel circles through $\gamma(0)$ and $\gamma(a)$. We assume without loss of generality that $h(\gamma(s_0))$ is the minimum of h in X, i.e., $h(\gamma(s_0)) = \inf\{h(x) \mid x \in X\} \le h(\gamma(0))$. We make, as was mentioned in Appendix 1, the universal covering surface $\widetilde{X}$ of X and a Finsler plane $(\widetilde{M}, \widetilde{F}) \supset (\widetilde{X}, \widetilde{F})$ with one-parameter group of motions $\tilde\varphi_t$ and a motion ψ on $\widetilde{M}$ with $\tilde\varphi_t \circ \psi = \psi \circ \tilde\varphi_t$ and $\psi^i(\widetilde{X}) \cap \psi^j(\widetilde{X}) = \emptyset$ for all $i \ne j \in \mathbb{Z}$. Let $\alpha(s)$ be the axis of ψ through $\gamma(s_0)$ such that $\alpha(0) = \gamma(s_0)$. Note that the axes of ψ simply cover $\widetilde{M}$.

We introduce a coordinate system $(t,s) \in \mathbb{R}^2$ on $\widetilde{M}$ such that $(t,s) = \varphi_t(\alpha(s))$. Then, a t-coordinate function $f_\gamma(s) = t(\gamma(s))$, $s \in (0,a)$, is monotone increasing for s, because the axes of ψ simply cover $\widetilde{M}$ and $\dot\gamma(0) = cH(\gamma(0))$ for some number $c > 0$. From Lemmas 4.4 and 4.2, we see that $\mathrm{Orb}(\gamma(s_0))$, $\mathrm{Orb}(\psi(\gamma(s_0)))$ and $\mathrm{Orb}(\psi^{-1}(\gamma(s_0)))$ are asymptotes to one another. Hence, from the last part of Lemma 4.2, all geodesics $\beta(s)$ through $\gamma(s_0) = \beta(0)$ such that $f_\beta(s)$ is monotone increasing for s are straight lines in $\widetilde{M}$. In particular, it is impossible for γ to be tangent to $\mathrm{Orb}(\gamma(a))$ because (3) of Lemma 3.4, a contradiction. Thus we conclude that $h(\gamma(s)) > h(\gamma(0))$ for all $s \in (0,a)$.

The same is true if h and H are replaced by h^- and $-H$. □

5. Geodesics in a Finsler torus of revolution

The following theorem is a direct consequence of Lemmas 4.2, 4.3, 4.4, Proposition 4.1 and Theorem 4.5. It extends the results for a torus of G-space in [7] to a Finsler torus with non-symmetric distance other than (3). Since we do not assume the symmetric property of F, there is a certain behavior of geodesics different from those in [7].

We say that a curve $\sigma : [a,b] \to M$ is a *backward geodesic* if its reversed curve $\sigma^{-1} : [a,b] \to M$ defined by $\sigma^{-1}(t) := \sigma(a+b-t)$, $t \in [a,b]$, is a geodesic. A geodesic is said to be *reversible* if it is also a backward geodesic.

Theorem 5.1 (Main theorem)**.** *Let (M,F) be a complete Finsler plane with one-parameter group of motions φ_t such that it is the universal covering plane of a torus $T^2 = M/D[\varphi_1,\psi]$ of revolution. Namely, φ_t is the lift of the rotation of T^2. The following are true.*

(1) *If $p \in M$ is a point such that $dh_p = 0$ (resp., $dh^-{}_p = 0$), then $\gamma(t) = \varphi_t(p)$ (resp., $\gamma(t) = \varphi_{-t}(p)$) is a geodesic in (M,F). If $h(p) = \inf\{h(x) \mid x \in M\}$ (resp., $h^-(p) = \inf\{h^-(x) \mid x \in M\}$), then $\gamma(t) = \varphi_t(p)$ (resp., $\gamma(t) = \varphi_{-t}(p)$) is a straight line. In particular, if $p \in M$ is a point such that $h(p) = \inf\{h(x) \mid x \in M\}$ and $h^-(p) = \inf\{h^-(x) \mid x \in M\}$, then $\gamma(t) = \varphi_t(p)$ is a reversible straight line.*

(2) *Let $p \in M$ be a point such that $h(p) \neq \inf\{h(x) \mid x \in M\}$. Let $X \ni p$ be a domain bounded by two straight lines $\gamma_i(t) = \varphi_t(q_i)$, $i = 1,2$, where q_i are points such that $h(q_i) = \inf\{h(x) \mid x \in M\}$ and there is no minimum point of h in the interior of X. Then there exist exactly two asymptotes α_i*

through $p = \alpha_i(0)$ to the straight lines γ_i which are contained entirely in X. The same result is true if h and φ_t are replaced by h^- and φ_{-t}.

(3) *Using the same notation as in (2), we define the sector $Y \subset X$ bounded by $\alpha_1([0,\infty))$ and $\alpha_2([0,\infty))$. If a unit vector $v \in T_pM$ is tangent to the interior of Y, then $\gamma_v(t)$, $t \geq 0$, lies in Y where γ_v is a unit speed geodesic with $\dot{\gamma}_v(0) = v$. Moreover, there exist two parallel orbits $e_i(t) = \varphi_t(r_i)$, $i =$ 1, 2, such that $\gamma_v(t)$, $t \in (-\infty,\infty)$, is alternately tangent to them and lies in the strip between e_1 and e_2. Here $h(r_i)$ are Clairaut's constant $C(\gamma_v(t))$ along γ_v. The same result is true if h and φ_t are replaced by h^- and φ_{-t}.*

(4) *The displacement function $d_{\psi\circ\varphi_t}$ is constant on M for every $t \in (-\infty,\infty)$. The axes of each $\psi \circ \varphi_t$ cover M simply. Its axis is a straight line passing through p which is obtained as the extension of a minimizing geodesic segment from p to $\psi \circ \varphi_t(p)$. Similarly, we have the simple cover of M by the axes of $\psi^{-1} \circ \varphi_{-t}$. The axes of $\psi \circ \varphi_t$ and $\psi^{-1} \circ \varphi_{-t}$ through a point p pass through the points $(\psi \circ \varphi_t)^k(p)$ for all integer k but, in general, they do not have the same images.*

(5) *If there exists a point $p \in M$ such that $h(p) = \inf\{h(x) \mid x \in M\}$ and $h^-(p) = \inf\{h^-(x) \mid x \in M\}$, then all geodesics through p are straight lines.*

We define a map $\exp_p : T_pM \to M$ by $\exp_p(v) = \gamma_v(1)$ and a map $\exp^-{}_p : T_pM \to M$ by $\exp^-{}_p(v) = \gamma_{-v}(-1)$ at $p \in M$ where γ_v is a constant speed geodesic such that $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$. We say that $p \in M$ is a (*forward*) *pole* (resp., *backward pole*) if the differential map of $\exp_p$ (resp., $\exp^-{}_p$) is nonsingular on T_pM.

Corollary 5.2. *Let (T^2, F) be a Finsler torus of revolution with rotation φ_t. Then the following are equivalent.*

(1) *There exists a point $p \in T^2$ such that $h(p) = \inf\{h(x) \mid x \in M\}$ and $h^-(p) = \inf\{h^-(x) \mid x \in M\}$.*

(2) *There exists a pole in T^2.*

(3) *There exists a backward pole in T^2.*

Theorem 5.1 directly shows this corollary.

Remark 5.3. By the method used in [13], we can prove that, without existence of one-parameter group of motions, if there exists a pole p in a Finsler torus (T^2, F), then any homotopy class of closed curves in T^2 has a minimal closed geodesic passing through the point p in it. This property implies that a pole is also a backward pole in T^2.

Corollary 5.4. *Let (M, F) be a complete Finsler plane with one-parameter group of motions φ_t and let h and h^- assume their minima at $p \in M$. Then, the point p is a pole in (M, F).*

Proof. From Lemma 2.2, if $\mathrm{Sing}(H) \neq \emptyset$, then p is the unique point where h attains its minimum 0. Therefore, p is a forward and backward pole. The orbits of φ_t are the parallel circles centered at p.

We first treat the case that there exists a motion ψ of (M, F) such that $M/D[\varphi_1, \psi]$ is topologically a torus. For every $\delta = \varphi_t \circ \psi$ with $t \in \mathbb{R}$, there exists the unique axis of δ through p. Further, $\mathrm{Orb}(p)$ is the axes of both φ_1 and ${\varphi_1}^{-1}$ through p, since $h(p) = \min\{h(x) \mid x \in M\}$ and $h^-(p) = \min\{h^-(x) \mid x \in M\}$. This implies that all geodesics through p are straight lines in M. Therefore, the sub-rays from p of those straight lines are rays from p, meaning that p is a forward and backward pole in M.

We next treat the case where there is no such a motion ψ of M. Let K be any compact set containing p in M and let U be a φ_t-invariant subset of M containing K. We construct a Finsler torus T^2 of revolution such that

(1) $U/[\varphi_1] \subset T^2$, isometrically,

(2) φ_t is extended to a motion of T^2,

(3) $h(p) = \min\{h(x) \mid x \in T^2\}$ and $h^-(p) = \min\{h^-(x) \mid x \in T^2\}$.

From the above argument, the point p is a pole in the universal covering plane of T^2. Since K is an arbitrary compact set, the point p is a pole in M. □

6. Reversibility of geodesics

Let (M, F) be a Finsler manifold. A (*forward*) *geodesic* $\sigma : [a, b] \to M$ satisfies the Euler–Lagrange equation:

$$EL_i(F, \sigma) := F_{x^i}(\sigma, \dot{\sigma}) - \frac{d}{dt} F_{\dot{x}^i}(\sigma, \dot{\sigma}) = 0, \quad i = 1, 2, \cdots, n.$$

We briefly write this equation $EL(F, \sigma) = 0$. Since F is positively homogeneous in TM, all curves $\alpha(s) = \sigma(t(s))$ with $t'(s) > 0$ are also geodesics if σ is a geodesic. However, if $t'(s) < 0$, then $\alpha(s)$ may not be a geodesic. All geodesics are reversible if F is absolutely homogeneous. If F is the sum of an absolutely homogeneous fundamental function and a closed 1-form, then all geodesics in (M, F) are reversible but the distance d is not symmetric.

We decompose F into the symmetric part A and the skew-symmetric part B by setting

$$F(x,\dot{x}) = A(x,\dot{x}) + B(x,\dot{x}), \qquad \dot{x} \in T_xM \setminus \{0\}$$

where

$$A(x,\dot{x}) := \frac{F(x,\dot{x}) + F(x,-\dot{x})}{2}, \quad B(x,\dot{x}) := \frac{F(x,\dot{x}) - F(x,-\dot{x})}{2}.$$

Note that all geodesics in (M,A) are reversible.

Lemma 6.1. *Let (M,F) be a Finsler manifold. The following are true for $\dot{x} \in T_xM \setminus \{0\}$.*

(1) *$F(x,\dot{x}) = A(x,\dot{x}) + B(x,\dot{x})$.*

(2) *$A(x,t\dot{x}) = |t|A(x,\dot{x})$ and $A_{\dot{x}^i}(x,t\dot{x}) = sgn(t)A_{\dot{x}^i}(x,\dot{x})$ for all $t \in \mathbb{R}$. Here $sgn(t) = 1$ if $t > 0$ and $sgn(t) = -1$ if $t < 0$.*

(3) *$B(x,t\dot{x}) = tB(x,\dot{x})$ for all $t \in \mathbb{R}$. Hence, $B_{x^i}(x,t\dot{x}) = tB_{x^i}(x,\dot{x})$ and $B_{\dot{x}^i}(x,t\dot{x}) = B_{\dot{x}^i}(x,\dot{x})$ for all $t \in \mathbb{R} \setminus \{0\}$.*

(4) *If $c(t)$ is a curve in M, $t = t(s)$ is a change of parameter, i.e., $t'(s) \neq 0$ and $\bar{c}(s) = c(t(s))$, then*

$$EL(F,\bar{c}(s)) = |t'(s)|EL(A,c(t)) + t'(s)EL(B,c(t)).$$

Proof. (1)–(3) are well known elementary facts and their proofs are omitted here.

We prove (4). From (2) and (3), we have

$$F_{x^i}(\bar{c}(s),\dot{\bar{c}}(s)) = |t'|A_{x^i}(c(t),\dot{c}(t)) + t'B_{x^i}(c(t),\dot{c}(t)),$$
$$F_{\dot{x}^i}(\bar{c}(s),\dot{\bar{c}}(s)) = sgn(t')A_{\dot{x}^i}(c(t),\dot{c}(t)) + B_{\dot{x}^i}(c(t),\dot{c}(t)),$$

and

$$\frac{d}{ds}F_{\dot{x}^i} = \frac{dt}{ds}\left(sgn(t')\frac{d}{dt}A_{\dot{x}^i} + \frac{d}{dt}B_{\dot{x}^i}\right) = |t'|\frac{d}{dt}A_{\dot{x}^i} + t'\frac{d}{dt}B_{\dot{x}^i}.$$

Therefore, we have the equation in (4). □

It is well known that B is a closed 1-form on M if and only if the Euler–Lagrange equations of F and A equal. In order to see the behavior of all geodesics, we state how to evaluate the reversibility of each geodesic.

Lemma 6.2. *Let $\gamma(t)$ be a geodesic in (M,F) and $t = t(s)$ a reversed change of parameter for $s \in \mathbb{R}$, i.e., $t'(s) < 0$. Set $\alpha(s) = \gamma(t(s))$. Then α is a geodesic in (M,F) if and only if $EL(B,\gamma) = 0$. In other words, γ is reversible in (M,F) if and only if it is a geodesic in (M,A).*

Proof. Since $\gamma(t)$ is a geodesic in (M, F), we have

$$EL(F, \gamma(t)) = EL(A, \gamma(t)) + EL(B, \gamma(t)) = 0.$$

Therefore, because of $t' < 0$ and (4) of Lemma 6.1, we have

$$EL(F, \alpha(s)) = 2t' EL(B, \gamma(t)) = 2|t'| EL(A, \gamma(t)).$$

From this equation, $\alpha(s)$ is a geodesic if and only if $EL(B, \gamma(t)) = 0$ if and only if $EL(A, \gamma(t)) = 0$. Therefore, $\gamma(t)$ is reversible in (M, F) if and only if $\gamma(t)$ is a geodesic in (M, A). □

We restate a theorem in [8] from our point of view.

Lemma 6.3. *Assume that B is a 1-form in M. Then a geodesic γ is reversible in (M, F) if and only if $i_{\dot\gamma(t)} dB = 0$ where i_v is the interior product for $v \in TM$.*

Proof. When B is a 1-form, we have $EL(B, \gamma) = -i_{\dot\gamma} dB$. Therefore, Lemma 6.2 shows that γ is reversible if and only if $i_{\dot\gamma(t)} dB = 0$. □

7. Examples

We can see the behavior of geodesics mentioned in Theorem 5.1.

Example 7.1. Let $M = \mathbb{R}^2 = \{(x, y) \,|\, x, y \in \mathbb{R}\}$. Let $0 < \varepsilon < b < a$. We define Randers metrics F_1 and F_2 on M by

$$F_1(x, y, \dot x, \dot y) := \sqrt{\dot x^2 + \dot y^2} - \varepsilon(\dot x \cos 2\pi y + \dot y \sin 2\pi y) \tag{1}$$

$$F_2(x, y, \dot x, \dot y) := \sqrt{(a + b\cos 2\pi y)^2 \dot x^2 + b^2 \dot y^2} - \varepsilon(\dot x \cos 2\pi y + \dot y \sin 2\pi y) \tag{2}$$

for a sufficiently small $\varepsilon > 0$.

Obviously, $\varphi_t(x, y) := (x + t, y)$ and $\psi^n(x, y) := (x, y + n)$ are motions in M for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$.

In both examples, we have, for all tangent vectors $(\dot x, \dot y) \in T_{(x,y)}M$,

$$B(x, y) := B(x, y, \dot x, \dot y) = -\varepsilon(\cos 2\pi y\, dx + \sin 2\pi y\, dy).$$

Hence, we have $i_v dB = 2\pi\varepsilon \sin 2\pi y(\dot y\, dx - \dot x\, dy)$ for all $v = (\dot x, \dot y) \in T_{(x,y)}M$. It follows from Lemma 6.3 and Proposition 4.1 that a curve $\gamma(t) = (x(t), y(t))$ is a reversible geodesic if and only if $y(t) = n/2$ for all $t \in (-\infty, \infty)$ and an integer

$n \in \mathbb{Z}$. Let A_i be the symmetric part of F_i for $1 = 1, 2$. A curve $\gamma(t) = (x(t), y(t))$ is a geodesic in both (M, F_i) and (M, A_i) if and only if $y(t) = n/2$ for all $t \in (-\infty, \infty)$ and an integer $n \in \mathbb{Z}$.

For integers $n \neq 0$ and a number t, we have $(\psi^n \circ \varphi_t)^m(x, y) = (x+mt, y+mn)$ for all $(x, y) \in M$ and integers m. There exists a unique straight line through those points $(x+mt, y+mn)$ in the order of increasing m such that it is not reversible, because the displacement function $d_{\psi^n \circ \varphi_t}$ is constant on M and $i_{\dot\gamma} dB \neq 0$ along an axis γ of $\psi^n \circ \varphi_t$. Hence, there exists another straight line through the same points in the reversed order.

This is interpreted as existence of the shortest closed geodesics. Let $T^2 = M/D[\psi, \varphi_{t_0}]$ for a number t_0. Each homotopy class of close curves contains a shortest closed curve in T^2 such that those lifts to its universal covering plane are straight lines which are axes of certain motions. Let $[c]$ be a homotopy class of closed curves containing c. The above argument shows that the shortest closed curves in $[c]$ and $[c^{-1}]$ are different with only one exception $(\pi \circ \gamma)(t) = \pi((x \pm t, 1/2))$ in the example (7.2) where $\pi : M \to T^2$ is a natural projection.

We find geodesics invariant under φ_t. Let $h(x, y) := F(x, y, H(x, y))$ and $h^-(x, y) = F(x, y, -H(x, y))$ for $(x, y) \in M$. Since $H(x, y) = (1, 0) \in T_{(x,y)}M$, we have, in the example (7.1),

$$h(x, y) = 1 - \varepsilon \cos 2\pi y, \quad h^-(x, y) = 1 + \varepsilon \cos 2\pi y,$$

and, in the example (7.2),

$$h(x, y) = a + (b - \varepsilon) \cos 2\pi y, \quad h^-(x, y) = a + (b + \varepsilon) \cos 2\pi y.$$

From Proposition 4.1 and Lemma 4.4, for the example (7.1) we conclude the following.

(1) If n is even, $\gamma_+(t) := (x + t, n/2)$ is a straight line, and $\gamma_-(t) := (x - t, n/2)$ is a geodesic but not a straight line.

(2) If n is odd, $\gamma_-(t) := (x - t, n/2)$ is a straight line, and $\gamma_+(t) := (x + t, n/2)$ is a geodesic but not a straight line.

For the example (7.2), we conclude the following.

(1) If n is even, $\gamma(t) := (x \pm t, n/2)$ is a geodesic but not a straight line.

(2) If n is odd, $\gamma(t) := (x \pm t, n/2)$ is a reversible straight line.

Therefore, there exists a pole in (M, F_2) but not in (M, F_1).

We show the behavior of geodesics $\gamma(t)$ such that $\gamma(0) \neq (0, n/2)$ and $\dot\gamma(0)$ is parallel to H. From Theorem 4.5 or (3) of Theorem 5.1, there exist two parallel circles $e_i(t) = \varphi_t(r_i)$, $i = 1, 2$, such that $\gamma(t)$ is alternately tangent to them and lies in the strip between them. For the example (7.1) we conclude the following.

(1) If n is even and $v = H(0, r)$ for $r \in (n/2, (n+1)/2)$, then the geodesic $\gamma_v(t)$, $t \in (-\infty, \infty)$, lies in the strip between $\mathrm{Orb}((0, r))$ and $\mathrm{Orb}((0, n/2 + 1 - r))$.

(2) If n is odd and $v = H(0, r)$ for $r \in (n/2, (n+1)/2)$, then the geodesic $\gamma_{-v}(t)$, $t \in (-\infty, \infty)$, lies in the strip between $\mathrm{Orb}((0, r))$ and $\mathrm{Orb}((0, n/2 + 1 - r))$.

For the example (7.2), we conclude that if n is odd and $v = H(0, r)$ for $r \in (n/2, (n+1)/2)$, then the geodesic $\gamma_{\pm v}(t)$, $t \in (-\infty, \infty)$, lies in the strip between $\mathrm{Orb}((0, r))$ and $\mathrm{Orb}((0, n/2 + 1 - r))$.

8. Appendix 1: A strip bounded by parallel circles

A Finsler surface of revolution has a one-parameter group of motions φ_t with period $\tau(\varphi_t) < \infty$. If a geodesic $\gamma : [0, 1] \to M$ in a Finsler surface of revolution does not pass through the fixed points of φ_t, we then find a strip X bounded by two parallel circles containing γ. No matter what the topological structure of M is, X is homeomorphic to a cylinder $S^1 \times T$ where T is a closed interval and its universal covering surface $\widetilde{X}$ is homeomorphic to $\mathbb{R} \times T$. Let $\pi : (\widetilde{X}, \widetilde{F}) \to (X, F)$ be the natural projection. Then there exists a one-parameter group of motions $\tilde{\varphi}_t$ in $(\widetilde{X}, \widetilde{F})$ such that $\varphi_t = \pi \circ \tilde{\varphi}_t$.

Lemma 8.1. *Under the notation above, there exists a complete Finsler plane $(\widetilde{M}, \widetilde{F})$ with one-parameter group of motions $\tilde{\varphi}_t$ such that $(\widetilde{X}, \widetilde{F})$ is a subspace embedded in $(\widetilde{M}, \widetilde{F})$ and there exists a motion ψ on $(\widetilde{M}, \widetilde{F})$ with $\tilde{\varphi}_t \circ \psi = \psi \circ \tilde{\varphi}_t$ and $\psi^i(\widetilde{X}) \cap \psi^j(\widetilde{X}) = \emptyset$ for all $i \neq j \in \mathbb{Z}$.*

Proof. For a sufficiently small $\varepsilon > 0$ we have a Finsler surface $(\widetilde{X}_\varepsilon, \widetilde{F}_\varepsilon) \subset (\widetilde{X}, \widetilde{F})$ with one-parameter group of motions $\tilde{\varphi}_t$ where X_ε is the ε-neighborhood of X. We assume that $\widetilde{X}_\varepsilon$ is homeomorphic to $\mathbb{R} \times [0, b]$ and $\tilde{\varphi}_t(x, y) = (x+t, y)$ for all $(x, y) \in \mathbb{R} \times [0, b]$. We define a fundamental function $\widetilde{F}_0$ on a plane $\widetilde{M} = \mathbb{R}^2$ by $\widetilde{F}_0(x, y, \dot{x}, \dot{y}) = \widetilde{F}_\varepsilon(x, y_0, \dot{x}, \dot{y})$ where $(\dot{x}, \dot{y}) \in T_{(x,y)}\widetilde{M}$ and y_0 satisfies $y = nb + y_0$, $0 \leq y_0 < b$, for some integer n. Then $\widetilde{F}_0$ may not be continuous at $y = nb$ for all integers $n \in \mathbb{Z}$. We have to make it smooth. Let $k(t)$, $t \in \mathbb{R}$, be a smooth function such that $0 \leq k(t) \leq 1$ for all $t \in \mathbb{R}$ and

$$k(t) = \begin{cases} 1 & \text{if } t \in [nb + \varepsilon, (n+1)b - \varepsilon], \\ 0 & \text{if } t \in [nb, nb + \varepsilon/2] \cup [(n+1)b - \varepsilon/2, (n+1)b]. \end{cases}$$

Let $g(x,y)$ denote a Riemannian metric on $\widetilde{M}$ such that $g(x,y) = g(x+t, y+nb)$ for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. We define a fundamental function $\widetilde{F} : T\widetilde{M} \to \mathbb{R}$ by

$$\widetilde{F}(x,y,\dot{x},\dot{y}) = k(y)\widetilde{F}_0(x,y,\dot{x},\dot{y}) + (1-k(y))\|(\dot{x},\dot{y})\|_g$$

where $\|\cdot\|_g$ denotes the norm with respect to g. Then the fundamental function $\widetilde{F}$ has a motion $\psi(x,y) = (x, y+b)$ satisfying the condition. $\square$

Thanks to existing straight lines in $\widetilde{M}$, we can get some information on the behavior of γ via $\tilde{\gamma}$ where $\tilde{\gamma}$ is a lift of γ into $\widetilde{M}$ constructed in Lemma 8.1.

From this reason, it is very important to study geodesics in a Finsler plane (M,F) with one-parameter group of motions φ_t and a motion ψ on (M,F) such that $\varphi_t \circ \psi = \psi \circ \varphi_t$ for all $t \in \mathbb{R}$ and the quotient surface $T^2 = M/D[\varphi_{t_0}, \psi]$ is a 2-torus of revolution.

9. Appendix 2: Distances induced by F

Let (M,F) be a complete Finsler manifold. Let $\Omega(p,q)$ denote the set of all piecewise smooth curves from p to q. We recall the distance d induced by F, $d(p,q) := \inf\{L_F(c) \in \mathbb{R} \,|\, c \in \Omega(p,q)\}$, where $L_F(c)$ is the length of a curve $c : [0,1] \to M$ given by

$$L_F(c) := \int_0^1 F(c(t), \dot{c}(t))\, dt.$$

We define a symmetric distance m on M by

$$m(p,q) := d(p,q) + d(q,p), \qquad p,q \in M,$$

and a symmetric fundamental function G by

$$G(x,\dot{x}) := F(x,\dot{x}) + F(x,-\dot{x}), \qquad \dot{x} \in T_xM.$$

Note that $G = 2A$, where A is the symmetric part of F.

Let $T(p,q)$ denote a minimizing geodesic segment from p to q in (M,d). We then have $L_F(T(p,q)) = d(p,q)$ and $L_F(T(q,p)) = d(q,p)$. Obviously, $T(p,q) \cup T(q,p)$ is the shortest round path from p through q in (M,d). Its length is $m(p,q)$. In general, (M,m) is not an intrinsic metric space. As usual, we define the length of a curve in (M,m) and, then, an intrinsic distance m_L in M, i.e., $m_L(p,q) := \inf_{c \in \Omega(p,q)} L_m(c)$. Thus we have an intrinsic metric space (M, m_L).

When we need to measure the length with respect to G, it is denoted by $L_G(c)$ for a piecewise smooth curve $c : [0,1] \to M$. Set as usual $d_G(p,q) := \inf\{L_G(c) \in \mathbb{R} \,|\, c \in \Omega(p,q)\}$. We discuss the relations between F, G, d, d_G, m and m_L.

Let $p, q \in M$. For curves $c_1, c_2 : [0,1] \to M$ such that $c_1(0) = p$, $c_1(1) = q$, $c_2(0) = q$ and $c_2(1) = p$, the curve $c_1 \cup c_2 : [0,1] \to M$ is defined by

$$c_1 \cup c_2(t) := \begin{cases} c_1(2t) & \text{if } 0 \le t \le 1/2, \\ c_2(2-2t) & \text{if } 1/2 < t \le 1. \end{cases}$$

We define three families of piecewise smooth curves for $p, q \in M$.

(1) $\Omega(p,q) := \{c : [0,1] \to M \,|\, c(0) = p, c(1) = q\}$,

(2) $\Gamma(p,q) := \{c : [0;1] \to M \,|\, c(0) = c(1) = p, c(1/2) = q\}$,

(3) $\Gamma_0(p,q) := \{c \cup c^{-1} : [0,1] \to M \,|\, c \in \Omega(p,q)\}$.

Lemma 9.1. *Let (M,F) be a complete Finsler manifold and $p, q \in M$ such that $p \neq q$. Let $T_G(p,q)$ be a minimizing geodesic segment from p to q in (M,G). The following are true.*

(1) *$T(p,q)$ is the shortest curve in $\Omega(p,q)$ with length $d(p,q)$.*

(2) *$T(p,q) \cup T(q,p)$ is the shortest curve in $\Gamma(p,q)$ with length $m(p,q)$.*

(3) *$T_G(q,p) = T_G(p,q)^{-1}$. Moreover, $T_G(p,q) \cup T_G(q,p)$ is the shortest curve in $\Gamma_0(p,q)$ with length $d_G(p,q) \ge m(p,q)$.*

Proof. (1) and (2) are well known. We prove (3). Let $c_0 \in \Gamma_0(p,q)$. Then there exists a curve $c : [0,1] \to M$ in $\Omega(p,q)$ such that $c_0 = c \cup c^{-1}$. From the definition of lengths, $L_F(c_0) = L_G(c)$. Hence we have $d_G(p,q) = \inf\{L_G(c) \,|\, c \in \Omega(p,q)\} = \inf\{L_F(c_0) \,|\, c_0 \in \Gamma_0(p,q)\} \ge m(p,q)$. □

A geodesic in (M,G) is an extremal of the variation problem of lengths of curves $c \cup c^{-1}$ for $c \in \Omega(p,q)$ in (M,F). A minimizing geodesic $T_G(p,q)$ connecting p and q is the shortest in $\Omega(p,q)$ under the condition that we go along a curve c and return along the reversed curve c^{-1}.

Lemma 9.2. *Let (M, m_L) denote the length space made from m. Then m_L equals the distance induced by G, i.e., $m_L = d_G$.*

Proof. Let $\Delta : 0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of $[0,1]$. We then have $L_G(c) = L_F(c) + L_F(c^{-1})$ because of the definition of length with respect to F and G. From the definition of L_m, we have $L_m(c) = \sup_\Delta \sum_{i=1}^n m(c(t_{i-1}), c(t_i)) = \sup_\Delta \sum_{i=1}^n (d(c(t_{i-1}), c(t_i)) + d(c(t_i), c(t_{i-1}))) = L_F(c) + L_F(c^{-1}) = L_G(c)$. This implies that m_L is the distance induced by G. □

We say that a metric space (X, d) is *Menger convex* if, for any points $p, q \in X$ with $p \neq q$, there exists a point $r \in X$ such that $r \neq p$, $r \neq q$ and $d(p, r) + d(r, q) = d(p, q)$ (see [6]).

Lemma 9.3. *Let (M, F) be a complete Finsler manifold. Then, (M, m) is Menger convex if and only if $m = d_G$.*

Proof. If (M, m) is Menger convex, we then have $m = m_L$ because there exists a minimizing geodesic connecting any two points in (M, m). From Lemma 9.2, we have $m = d_G$.

Assume that $m = d_G$. Then, (M, m) is Menger convex, because d_G is an intrinsic distance. □

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