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On (m,n)-Jordan centralizers of semiprime rings

By IRENA KOSI-ULBL (Maribor) and JOSO VUKMAN (Maribor)

Abstract. In this paper we prove the following result. Let $m \geq 1, n \geq 1$ be fixed integers and let R be an $mn(m+n)$-torsion free semiprime ring. Suppose there exists an additive mapping $T : R \to R$ satisfying the relation $(m+n)T(x^2) = mT(x)x + nxT(x)$ for all $x \in R$ ((m,n)-Jordan centralizer). In this case T is a two-sided centralizer.

Throughout, R will represent an associative ring with center $Z(R)$. As usual, the commutator $xy - yx$ will be denoted by $[x,y]$. We shall use the commutator identity $[xy,z] = [x,z]y + x[y,z]$. Given an integer $n \geq 2$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime if for $a,b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. We denote by $\mathrm{char}(R)$ the characteristic of a prime ring R. We denote by Q_r and C Martindale right ring of quotients and extended centroid of a semiprime ring R, respectively. For explanation of Q_r and C, we refer the reader to [2]. An additive mapping $T : R \to R$ is called a left centralizer in case $T(xy) = T(x)y$ holds for all pairs $x,y \in R$ and is called a left Jordan centralizer in case $T(x^2) = T(x)x$ holds for all $x \in R$. In case R has the identity element, $T : R \to R$ is a left centralizer iff T is of the form $T(x) = ax$ for all $x \in R$, where $a \in R$ is a fixed element. For a semiprime ring R all left centralizers are of the form $T(x) = qx$ for all $x \in R$, where q is a fixed element from Q_r (see [2, Chapter 2]). The definition of right centralizer and right Jordan centralizer should be self-explanatory. We call $T : R \to R$ a two-sided centralizer in case T is both a left and a right centralizer. In case $T : R \to R$ is a two-sided centralizer, where R is a semiprime ring with extended centroid C, then there exists element

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$\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ (see [2, Theorem 2.3.2]). Zalar [21] has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer (Zalar theorem). For results concerning centralizers (also called multipliers) on rings and algebras, we refer to [1], [3], [4], [6], [9]–[13], [16]–[19], [21] where further references can be found.

Let R be an arbitrary ring and let $m \geq 0$, $n \geq 0$ be some fixed integers with $m+n \neq 0$. An additive mapping $T : R \to R$ is called an (m,n)-Jordan centralizer in case

$$(m+n)T(x^2) = mT(x)x + nxT(x) \tag{1}$$

holds for all $x \in R$. The concept of (m,n)-Jordan centralizer, which has been introduced by Vukman in [20], covers the concept of left Jordan centralizer as well as the concept of right Jordan centralizer. Namely, putting in the relation above $m = 1$, $n = 0$ one obtains left Jordan centralizer, in case $m = 0$, $n = 1$ the relation (1) reduces to right Jordan centralizer. Moreover, in case $m = n = 1$, we obtain the relation

$$2T(x^2) = T(x)x + xT(x), \quad x \in R. \tag{2}$$

Vukman [16] has proved that in case an additive mapping $T : R \to R$, where R is a 2-torsion free semiprime ring, satisfies the relation (2) for any $x \in R$, then T is a two-sided centralizer. For results concerning (m,n)-Jordan centralizers, we refer to [8], [14], [15], [20]. In [20], one can find the following conjecture.

Conjecture 1 ([20, Conjecture 2]). *Let $m \geq 1, n \geq 1$ be some fixed integers, let R be a semiprime ring with suitable torsion restrictions, and let $T : R \to R$ be an (m,n)-Jordan centralizer. In this case T is a two-sided centralizer.*

Peršin and Vukman [15] have proved the following result.

Theorem 1 ([15, Theorem 2]). *Let $m \geq 1$, $n \geq 1$ be some fixed integers, let R be a prime ring with $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) > (m+n)^2$ and let $T : R \to R$ be an additive mapping satisfying the relation*

$$2(m+n)^2T(x^3) = m(2m+n)T(x)x^2 + 2mnxT(x)x + n(2n+m)x^2T(x) \tag{3}$$

for all $x \in R$. In this case T is a two-sided centralizer.

It is easy to see that any (m,n)-Jordan centralizer on arbitrary ring satisfies the relation (3), which means that Theorem 1 proves Conjecture 1 in a special case we have a prime ring.

The result below proves Conjecture 1 in general.

Theorem 2. *Let $m \geq 1$, $n \geq 1$ be some fixed integers, let R be an $mn(m+n)$-torsion free semiprime ring, and let $T : R \to R$ be an (m,n)-Jordan centralizer. In this case T is a two-sided centralizer.*

The methods used in the proof of Theorem 2 differ from those used in Theorem 1. As the main tool in the proof of Theorem 1 the theory of functional identities (Brešar–Beidar–Chebotar theory) is used. We refer the reader to [5] for introductory account of functional identities and to [7] for full treatment of this theory. The proof of Theorem 2 is, as we shall see, rather long, but it is elementary in the sense that one needs no specific knowledge concerning semiprime rings in order to follow the proof. For the proof of Theorem 2, we need Theorem 3, Lemma 1 and Zalar theorem.

Theorem 3 ([20, Proposition 3])**.** *Let $m \geq 0$, $n \geq 0$ be some integers with $m+n \neq 0$, let R be a ring and let $T : R \to R$ be an (m,n)-Jordan centralizer. In this case we have*

$$\begin{aligned} 2(m+n)^2T(xyx) &= mnT(x)xy + m(2m+n)T(x)yx - mnT(y)x^2 \\ &+2mnxT(y)x - mnx^2T(y) + n(2n+m)xyT(x) + mnyxT(x), \end{aligned} \tag{4}$$

for all pairs $x, y \in R$.

Lemma 1 ([17, Lemma 1])**.** *Let R be a semiprime ring. Suppose that the relation*

$$axb + bxc = 0$$

holds for all $x \in R$ and some $a, b, c \in R$. In this case

$$(a+c)xb = 0$$

is satisfied for all $x \in R$.

Proof of Theorem 2. Let us point out that from the assumption of the theorem that R is $mn(m+n)$-torsion free, it follows, that R is 2-torsion free.

The linearization of the relation (1) gives

$$(m+n)T(xy+yx) = mT(x)y + mT(y)x + nxT(y) + nyT(x), \quad x, y \in R. \tag{5}$$

Putting $xy + yx$ for y in (4) and using (5), we obtain

$$\begin{aligned} 2(m+n)^3T(x^2yx + xyx^2) &= mn(m+n)T(x)x^2y + 2m(m+n)^2T(x)xyx \\ &+m(2m^2+2mn+n^2)T(x)yx^2 - m^2nT(y)x^3 + mn(2m-n)xT(y)x^2 \end{aligned}$$

$$-mn^2yT(x)x^2+2m^2nxT(x)yx+mn(2n-m)x^2T(y)x+2mn^2xyT(x)x$$
$$-m^2nx^2T(x)y-mn^2x^3T(y)+n(2n^2+2mn+m^2)x^2yT(x)$$
$$+2n(m+n)^2xyxT(x)+mn(m+n)yx^2T(x),\quad x,y\in R. \tag{6}$$

On the other hand, putting xyx for y in (5) and applying (4), one obtains after some calculation

$$2(m+n)^3T(x^2yx+xyx^2)=m(2m^2+5mn+2n^2)T(x)xyx+m^2(2m+n)T(x)yx^2$$
$$-m^2nT(y)x^3+mn(2m-n)xT(y)x^2+mn(2n-m)x^2T(y)x+mn(2n+m)xyT(x)x$$
$$+m^2nyxT(x)x+mn^2xT(x)xy+mn(2m+n)xT(x)yx-mn^2x^3T(y)$$
$$+n^2(2n+m)x^2yT(x)+n(2m^2+5mn+2n^2)xyxT(x),\quad x,y\in R. \tag{7}$$

Comparing the relations (6) and (7), we obtain, since R is an mn-torsion free ring,

$$(m+n)T(x)x^2y-mT(x)xyx+(m+n)T(x)yx^2-nyT(x)x^2$$
$$-nxT(x)yx-mxyT(x)x-mx^2T(x)y+(m+n)x^2yT(x)-nxyxT(x)$$
$$+(m+n)yx^2T(x)-myxT(x)x-nxT(x)xy=0,\quad x,y\in R. \tag{8}$$

Putting first yx for y in the above relation, then multiplying the relation (8) by x from the right side, and subtracting the relations so obtained one from another, we obtain

$$ny[T(x),x]x^2+mxy[T(x),x]x-(m+n)x^2y[T(x),x]+nxyx[T(x),x]$$
$$-(m+n)yx^2[T(x),x]+myx[T(x),x]x=0,\quad x,y\in R. \tag{9}$$

Putting $T(x)y$ for y in the above relation, then multiplying the relation (9) by $T(x)$ from the left side, we arrive after subtraction

$$m[T(x),x]y[T(x),x]x-(m+n)[T(x),x^2]y[T(x),x]$$
$$+n[T(x),x]yx[T(x),x]=0,\quad x,y\in R.$$

The above relation can be written in the form

$$-(m+n)[T(x),x^2]y[T(x),x]$$
$$+[T(x),x]y(m[T(x),x]x+nx[T(x),x])=0,\quad x,y\in R.$$

According to Lemma 1, we obtain from the above relation

$$(-(m+n)[T(x),x^2]+m[T(x),x]x+nx[T(x),x])y[T(x),x]=0,\quad x,y\in R,$$

which reduces to

$$(n[T(x),x]x+mx[T(x),x])y[T(x),x]=0, \qquad x,y\in R. \tag{10}$$

Right multiplication of the above relation by nx, then putting ymx for y in the relation (10), and combining the relations so obtained, we obtain

$$(n[T(x),x]x+mx[T(x),x])y(n[T(x),x]x+mx[T(x),x])=0, \qquad x,y\in R,$$

whence it follows

$$n[T(x),x]x+mx[T(x),x]=0, \quad x\in R, \tag{11}$$

by semiprimeness of R. The linearization of the above relation gives

$$\begin{aligned}&n[T(x),x]y+n[T(x),y]x+n[T(y),x]x+mx[T(x),y]\\&+mx[T(y),x]+my[T(x),x]+n[T(y),y]x+n[T(y),x]y+n[T(x),y]y\\&+my[T(y),x]+my[T(x),y]+mx[T(y),y]=0, \quad x,y\in R.\end{aligned} \tag{12}$$

Putting $-x$ for x in the above relation and comparing the relation so obtained with the relation (12), we arrive to

$$\begin{aligned}&n[T(x),x]y+n[T(x),y]x+n[T(y),x]x\\&+mx[T(x),y]+\ mx[T(y),x]+my[T(x),x]=0, \quad x,y\in R.\end{aligned} \tag{13}$$

In the procedure above we used the fact that R is 2-torsion free. Putting $(m+n)(xy+yx)$ for y in the above relation and using first the relation (5) and then the relation (11) we obtain after some calculation

$$\begin{aligned}&(3mn+2n^2)[T(x),x]yx+(3mn+2m^2)xy[T(x),x]+m^2x[T(x),x]y+n^2y[T(x),x]x\\&+n(m+n)[T(x),y]x^2+m(m+n)x^2[T(x),y]+(m+n)^2x[T(x),y]x\\&+mnT(x)[y,x]x+mnx[y,x]T(x)+n^2[y,x]T(x)x+m^2xT(x)[y,x]\\&+mn[T(y),x]x^2+mnx^2[T(y),x]+(m^2+n^2)x[T(y),x]x=0, \quad x,y\in R.\end{aligned}$$

Rearranging the above relation, one obtains

$$\begin{aligned}&(3mn+2n^2)[T(x),x]yx+(3mn+2m^2)xy[T(x),x]+m^2x[T(x),x]y+n^2y[T(x),x]x\\&+n(m+n)[T(x),y]x^2+m(m+n)x^2[T(x),y]+(m+n)^2x[T(x),y]x\\&+mnT(x)[y,x]x+mnx[y,x]T(x)+n^2[y,x]T(x)x+m^2xT(x)[y,x]\end{aligned}$$

$$+m(n[T(y),x]x+mx[T(y),x])x+nx(n[T(y),x]x+mx[T(y),x])=0, \quad x,y\in R. \quad (14)$$

According to the relation (13) one can replace in the above relation

$$n[T(y),x]x+mx[T(y),x]$$

by

$$-n[T(x),x]y-n[T(x),y]x-mx[T(x),y]-my[T(x),x],$$

which gives

$$\begin{gathered}(2mn+2n^2)[T(x),x]yx+(2mn+2m^2)xy[T(x),x]+(m^2-n^2)x[T(x),x]y\\ -(m^2-n^2)y[T(x),x]x+n^2[T(x),y]x^2+m^2x^2[T(x),y]+2mnx[T(x),y]x\\ +mnT(x)[y,x]x+mnx[y,x]T(x)+n^2[y,x]T(x)x+m^2xT(x)[y,x]=0, \quad x,y\in R. \quad (15)\end{gathered}$$

Putting first yx for y in the above relation, then multiplying the relation (15) by x from the right side, we arrive after subtraction to

$$\begin{gathered}2m(m+n)xy[[T(x),x],x]-(m^2-n^2)y[[T(x),x],x]x-n^2y[T(x),x]x^2-m^2x^2y[T(x),x]\\ -2mnxy[T(x),x]x+mnx[y,x][T(x),x]+n^2[y,x][T(x),x]x=0, \quad x,y\in R,\end{gathered}$$

which can be written in the form

$$\begin{gathered}2m(m+n)xy[[T(x),x],x]-(m^2-n^2)y[[T(x),x],x]x-n^2y[T(x),x]x^2\\ -m^2x^2y[T(x),x]-2mnxy[T(x),x]x+mnxyx[T(x),x]\\ -mnx^2y[T(x),x]+n^2yx[T(x),x]x-n^2xy[T(x),x]x=0, \quad x,y\in R. \quad (16)\end{gathered}$$

Substitution $T(x)y$ for y, then left multiplication of the relation (16) by $T(x)$, then subtraction, gives

$$\begin{gathered}2m(m+n)[T(x),x]y[[T(x),x],x]-m^2[T(x),x^2]y[T(x),x]\\ -2mn[T(x),x]y[T(x),x]x+mn[T(x),x]yx[T(x),x]-mn[T(x),x^2]y[T(x),x]\\ -n^2[T(x),x]y[T(x),x]x=0, \quad x,y\in R.\end{gathered}$$

We rewrite the above relation in the form

$$\begin{gathered}[T(x),x]y(2m(m+n)[[T(x),x],x]-2mn[T(x),x]x+mnx[T(x),x]-n^2[T(x),x]x)\\ -(m^2+mn)[T(x),x^2]y[T(x),x]=0, \quad x,y\in R.\end{gathered}$$

According to the relation (11), one can replace in the above relation $-n^2[T(x),x]x$ with $mnx[T(x),x]$. Thus we have

$$[T(x),x]y(2m^2[[T(x),x],x]) - (m^2+mn)[T(x),x^2]y[T(x),x] = 0, \quad x,y \in R,$$

whence it follows, since R is m-torsion free,

$$[T(x),x]y(2m[[T(x),x],x]) - (m+n)[T(x),x^2]y[T(x),x] = 0, \quad x,y \in R.$$

Using Lemma 1, we obtain from the above relation

$$(2m[[T(x),x],x] - (m+n)[T(x),x^2])y[T(x),x] = 0, \ x,y \in R. \tag{17}$$

Using the relation (11), we obtain

$$\begin{aligned}(m+n)[T(x),x^2] &= m[T(x),x]x + (mx[T(x),x] + n[T(x),x]x) + nx[T(x),x]\\ &= m[T(x),x]x + nx[T(x),x], \quad x \in R,\end{aligned}$$

so we can replace in the relation (17) $(m+n)[T(x),x^2]$ with $m[T(x),x]x + nx[T(x),x]$. Thus we have

$$\begin{aligned}&2m[[T(x),x],x] - m[T(x),x]x - nx[T(x),x]\\ &\quad = m[[T(x),x],x] + m[T(x),x]x - mx[T(x),x] - m[T(x),x]x - nx[T(x),x]\\ &\quad = m[[T(x),x],x] + n[T(x),x]x - nx[T(x),x] = (m+n)[[T(x),x],x], \quad x \in R.\end{aligned}$$

Now, the relation (17) reduces to

$$(m+n)[[T(x),x],x]y[T(x),x] = 0, \quad x,y \in R,$$

whence it follows

$$[[T(x),x],x]y[T(x),x] = 0, \quad x,y \in R.$$

From the above relation one obtains

$$[[T(x),x],x]y[[T(x),x],x] = 0, \quad x,y \in R,$$

whence it follows

$$[[T(x),x],x] = 0, \quad x \in R.$$

The relation above makes it possible to replace in the relation (11) $x[T(x),x]$ with $[T(x),x]x$, which gives $(m+n)[T(x),x]x=0$, $x\in R$, and, since R is $(m+n)$-torsion free,

$$[T(x),x]x=0,\quad x\in R. \tag{18}$$

Of course we have also

$$x[T(x),x]=0,\quad x\in R. \tag{19}$$

From the relation (18) we obtain

$$[T(x),x]y+[T(x),y]x+[T(y),x]x=0,\quad x,y\in R.$$

Multiplying the above relation by $[T(x),x]$ and applying the relation (19), we obtain

$$[T(x),x]y[T(x),x]=0,\quad x,y\in R,$$

whence it follows

$$[T(x),x]=0,\quad x\in R.$$

This relation makes it possible to replace in (1) $xT(x)$ with $T(x)x$, which gives $(m+n)T(x^2)=(m+n)T(x)x$, whence it follows

$$T(x^2)=T(x)x,\quad x\in R.$$

Similarly, we have also

$$T(x^2)=xT(x),\quad x\in R.$$

We have therefore proved that T is a left and a right Jordan centralizer. By Zalar theorem, T is a two-sided centralizer. The proof of the theorem is complete. $\square$

In case $m=n=1$, Theorem 2 reduces to Theorem 1 in [16].

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JOSO VUKMAN
DEPARTMENT OF MATHEMATICS
AND COMPUTER SCIENCE
FACULTY OF NATURAL SCIENCES
AND MATHEMATICS
UNIVERSITY OF MARIBOR
KOROŠKA 160
2000 MARIBOR
SLOVENIA

E-mail: joso.vukman@guest.um.si

IRENA KOSI-ULBL
FACULTY OF MECHANICAL ENGINEERING
UNIVERSITY OF MARIBOR
SMETANOVA 17
2000 MARIBOR
SLOVENIA

E-mail: irena.kosi@um.si

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