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Hölder equivalence of homogeneous Moran sets

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Abstract. For two homogeneous Moran sets $E = \mathcal{C}([0,1],\{n_k\},\{c_k\})$ and $E' = \mathcal{C}([0,1],\{n'_k\},\{c'_k\})$ with Hausdorff dimensions s and s' with $s' < s$ such that $\{n_k\}$ and $\{n'_k\}$ are bounded and the spacings are uniform in some sense, we prove that there exists a homeomorphism $f : E \to E'$ such that f is $\left(\frac{s'}{s} - \epsilon\right)$-Hölder continuous but not $\left(\frac{s'}{s} + \epsilon\right)$-Hölder continuous for any $\epsilon > 0$.

1. Introduction

The class of homogeneous Moran sets are defined and studied by Dejun Feng, Zhiying Wen and Jun Wu [4]. Let $n_k \geq 2$ be integers and c_k be positive numbers satisfying that $0 < c_k n_k < 1$ $(k = 1, 2, \dots)$. Let $D_k = \prod_{i=1}^{k}\{1, 2, \dots n_i\}$ and $D = \cup_{k=0}^{\infty} D_k$, where an element in D_k is denoted by a finite sequence $\sigma_1\sigma_2\dots\sigma_k$ of $\sigma_i \in \{1, 2, \dots, n_i\}$ $(i = 1, 2, \dots, k)$ and D_0 consists of the empty sequence $\emptyset$. Let $\mathbb{J}_\emptyset = [0,1]$ and define closed intervals $\mathbb{J}_\sigma \subset [0,1]$ for $\sigma \in D$ inductively. Let $\sigma = \sigma' i \in D_k$ with $\sigma' \in D_{k-1}$ and $i \in \{1, 2, \dots, n_k\}$. Let $\mathbb{J}_{\sigma'} = [a, b]$ with $b - a = c_1 \dots c_{k-1}$. Then, $\mathbb{J}_{\sigma'1}, \mathbb{J}_{\sigma'2}, \dots, \mathbb{J}_{\sigma' n_k}$ are disjoint closed intervals of length $c_1 \dots c_{k-1} c_k$ contained in $\mathbb{J}_{\sigma'}$ arranged from left to right in this order. To determine these sets, we introduce another quantity, a sequence of positive numbers $(d_k^1, \dots, d_k^{n_k - 1})$ called *spacing* satisfying that

$$d_k^1 + \dots + d_k^{n_k-1} + n_k c_k \leq 1.$$

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Then define

$$\mathbb{J}_{\sigma' i} = [a + ((i-1)c_k + d_k^1 + \cdots + d_k^{i-1})\delta,\ a + (ic_k + d_k^1 + \cdots + d_k^{i-1})\delta]$$
$$(i = 1, 2, \ldots, n_k),$$

where $\delta = c_1 \ldots c_{k-1}$. Finally, we define a fractal set

$$E = \bigcap_{k=0}^{\infty} \bigcup_{\sigma \in D_k} \mathbb{J}_\sigma \tag{1.1}$$

which we call a *homogeneous Moran set* denoted as $\mathcal{C}([0,1], \{n_k\}, \{c_k\}, \{d_k^i\})$.

In this paper, we always assume that

$$(*) \quad \sup_{k=1,2,\ldots} n_k < \infty, \quad \Delta := \inf_{k=1,2,\ldots;\ i=1,\ldots,n^k-1} d_k^i > 0,$$

and

$$s = \lim_{k\to\infty} \frac{\log N_k}{-\log \delta_k} \text{ exists and } 0 < s < 1,$$

where $\delta_k = c_1 \ldots c_k$ and $N_k = n_1 \ldots n_k$ $(k = 1, 2, \ldots)$.

It is known [4] that for any $E \in \mathcal{C}([0,1], \{n_k\}, \{c_k\}, \{d_k^i\})$ satisfying $(*)$, the Hausdorff dimension $\dim_H E$ is equal to s as above. For the general notions of fractal geometry, refer to [1], [5]. For the multifractal properties of Moran sets, refer to [3], [9]. For the notions of Hölder equivalence or Lipschitz equivalence, refer to [2], [7], [8]. We prove that

Theorem 1. *For homogeneous Moran sets $E = \mathcal{C}([0,1], \{n_k\}, \{c_k\}, \{d_k^i\})$ and $E' = \mathcal{C}([0,1], \{n'_k\}, \{c'_k\}, \{d_k^{i'}\})$ satisfying the condition $(*)$ with $s = \dim_H E$ and $s' = \dim_H E'$ such that $s \le s'$. Then, there exists a homeomorphism $f : E \to E'$ such that*

$$C_1(y-x)^{\frac{s}{s'}+\epsilon} < f(y) - f(x) < C_2(y-x)^{\frac{s}{s'}-\epsilon} \tag{1.2}$$

holds for any $x < y$ in E, where $\epsilon > 0$ is arbitrary and C_1, C_2 are positive constants.

Corollary 1 (Qin Wang and Li-feng Xi [6])**.** *In the above theorem, if $s = s'$, then E and E' are quasi-Lipschitz equivalent.*

2. Basic lemmas

Lemma 1. *Let k and l be positive integers. Let $U_1, U_2, \dots, U_k$ be a disjoint family of sets having the same number l of elements. Let $k < n \le kl$. Then, there exists a disjoint family $V_1, V_2, \dots, V_n$ of nonempty sets such that*

(1) *$V_1 \cup V_2 \cup \cdots \cup V_n = U_1 \cup U_2 \cup \cdots \cup U_k$, and*

(2) *for any $j = 1, 2, \dots, n$, there exists $i = 1, 2, \dots, k$ such that $V_j \subset U_i$.*

(3) *$\#V_i \le 3\#V_j$ holds for any $i, j = 1, 2, \dots, n$, where $\#V$ denotes the number of elements in a set V.*

Proof. Let $d = \lfloor n/k \rfloor$ and $n = kd + r$ with $0 \le r < k$. Then, $n = (k-r)d + r(d+1)$. We partition each of $U_1, U_2, \dots, U_{k-r}$ into d number of subsets and each of $U_{k-r+1}, U_{k-r+2}, \dots, U_k$ into $d+1$ number of subsets. These subsets will become $V_1, V_2, \dots, V_n$. Let $h = \lfloor l/d \rfloor$. Since $l = dh + s$ with $0 \le s < d$, we have $l = (d-s)h + s(h+1)$. Partition each of $U_1, U_2, \dots, U_{k-r}$ into $d - s$ number of subsets with h elements and s number of subsets with $h+1$ elements. Let $m = \lfloor l/(d+1) \rfloor$. Since $l = (d+1)m + t$ with $0 \le t < d+1$, we have $l = (d+1-t)m + t(m+1)$. Partition each of $U_{k-r+1}, U_{k-r+2}, \dots, U_k$ into $d+1-t$ number of subsets with m elements and t number of subsets with $m+1$ elements. The collection of these sets becomes $V_1, V_2, \dots, V_n$. Then, we have (1)(2).

Let us prove (3). We have

$$\max_i \#V_i = \begin{cases} h+1 & \text{if } l \text{ is not a multiple of } d, \\ h & \text{if } l \text{ is a multiple of } d, \end{cases}$$

$$\min_i \#V_i = \begin{cases} m & \text{if } n \text{ is not a multiple of } k, \\ h & \text{if } n \text{ is a multiple of } k. \end{cases}$$

Case 1. If l is not a multiple of d, then $h = \lfloor l/d \rfloor \le (l-1)/d$. Since $l = (d+1)m + t$, $0 \le t \le d$ and $d \ge 1$, we have

$$\begin{aligned} max_i \#V_i &= h + 1 = \lfloor l/d \rfloor + 1 \le (l-1)/d + 1 = ((d+1)m + t - 1)/d + 1 \\ &\le ((d+1)m + d - 1)/d + 1 = m + 1 + (m-1)/d + 1 \\ &\le m + 1 + (m-1) + 1 = 2m + 1 \le 3m \le 3\ min_i \#V_i. \end{aligned}$$

Case 2. If l is a multiple of d, then

$$\begin{aligned} max_i \#V_i &= h = \lfloor l/d \rfloor = l/d = ((d+1)m + t)/d \le ((d+1)m + d)/d \\ &= m + 1 + m/d \le m + 1 + m = 2m + 1 \le 3m \le 3\ min_i \#V_i. \end{aligned}$$ □

Let $D_k = \prod_{i=1}^{k}\{1,2,\dots n_i\}$ and $D'_k = \prod_{i=1}^{k}\{1,2,\dots n'_i\}$ $(k = 0,1,2,\dots)$. For $\sigma \in D_k$ or $\sigma' \in D'_k$, let $\mathbb{J}_\sigma$ or $\mathbb{J}_{\sigma'}$ be the intervals defined in (1.1) with respect to $E = \mathcal{C}([0,1],\{n_k\},\{c_k\},\{d_k^i\})$ or $E' = \mathcal{C}([0,1],\{n'_k\},\{c'_k\},\{d_k^{i'}\})$, respectively. We call it a *basic* interval of E or E' of level k. We denote by δ'_k, N'_k or Δ' the quantities δ_k, N_k or Δ for E'.

Notation. Denote $C_0 := \sup_i n_i$ and $C'_0 = \sup_i n'_i$.

Definition 1. Let $k = 1,2,\dots$ and $e = 0,1,\dots,k$. For basic intervals I and J of E of level k contained in a same basic interval of E of level $k-e$, the minimum interval containing I and J is called an (k,e)-*admissible* interval of E. Specially, a basic interval of E of level k is a $(k,0)$-admissible interval of E. The number of basic intervals of level k contained in an interval H is called the k-*size* of H and denoted by $\#_k H$. Let $\mathcal{I} = \{I_1, I_2, \dots, I_t\}$ be a set of (k,e)-admissible intervals of E. We call it an (k,e)-*admissible partition* of F (in E) if

(1) $I_i \cap I_j = \emptyset$ for any $i \neq j$, and

(2) $(\cup_{i=1}^{t} I_i) \cap E = F$.

Specially, the $(k,0)$-admissible partition of E, that is, the set of all basic intervals of level k is denoted by $\mathcal{E}_k$ and called the k-*basic partition* of E. Just by an *admissible partition*, we mean an (k,e)-admissible partition for some e and k. We define the same things for E' and the k-basic partition of E' is denoted by $\mathcal{E}'_k$

Definition 2. An (l,g)-admissible partition $\mathcal{I}' = \{I_1, I_2, \dots, I_t\}$ of E' is said to be of $\mathcal{E}_k$-*type* if $t = \#\mathcal{I}' = N_k$. In this case, there exists a unique order-preserving bijection φ from $\mathcal{E}_k$ to $\mathcal{I}'$, that is, if $x < y$ holds for any $x \in I$ and $y \in J$ with $I, J \in \mathcal{E}_k$, then $x' < y'$ holds for any $x' \in \varphi(I)$ and $y' \in \varphi(J)$. We call φ the *isomorphism* from $\mathcal{E}_k$ to $\mathcal{I}'$. Let $\mathcal{I}'$ and $\mathcal{J}'$ be admissible partitions of E' of $\mathcal{E}_k$-*type* and $\mathcal{E}_m$-*type* with $k < m$, respectively. They are said to be *consistent* if for any $I \in \mathcal{E}_k$ and $J \in \mathcal{E}_m$ with $I \subset J$, $\varphi(I) \subset \psi(J)$ holds, where φ and ψ are the isomorphisms from $\mathcal{E}_k$ to $\mathcal{I}'$, and from $\mathcal{E}_m$ to $\mathcal{J}'$, respectively.

Lemma 2. (1) *For any $k = 1,2,\dots$, let $N'_{l-1} < N_k \le N'_l$. Then, there exists an $(l,1)$-admissible partition $\mathcal{I}'_l$ of E' of $\mathcal{E}_k$-type such that $\#_l I \le 3\#_l J$ holds for any $I, J \in \mathcal{I}'_l$.*
(2) *Assume that $\mathcal{I}'_l$ is an (l,e)-admissible partition of E' of $\mathcal{E}_k$-type such that $\#_l I \le C\#_l J$ for any $I, J \in \mathcal{I}'_l$ with $C > 1$. Let*

$$g = \lfloor (\log C + \log C_0)/\log 2 + 1 \rfloor.$$

Then for any integer $h > k$, there exists an (s, g)-admissible partition $\mathcal{I}'_s$ of E' of $\mathcal{E}_m$-type with $h \le m < h + \dfrac{\log C'_0}{\log 2} g$ and some s such that $\mathcal{I}'_l$ and $\mathcal{I}'_s$ are consistent. Moreover, $\#_s I \le 9C \#_s J$ holds for any $I, J \in \mathcal{I}'_s$.

PROOF. (1) We consider a basic interval I of E' of level $l-1$ to be the set of basic intervals J of E' of level l such that $J \subset I$. There are N'_{l-1} number of sets as this which are denoted by $U_1, U_2, \dots, U_{N'_{l-1}}$. All of them have n'_l number of elements. Since $N'_{l-1} < N_k \le n'_l N'_{l-1} = N'_l$, applying Lemma 1, we have a disjoint family $V_1, V_2, \dots, V_{N_k}$ of nonempty sets such that (1)(2)(3) of Lemma 1 hold with $k = N'_{l-1}$, $n = N_k$ and $l = n'_l$. Moreover, we may assume that each of $V_1, V_2, \dots, V_{N_k}$ consists of neighboring basic intervals of level l, so that the admissible intervals generated by them are disjoint. Hence, they define a $(l, 1)$-admissible partition $\mathcal{I}'_l$ of E' of $\mathcal{E}_k$-type satisfying that $\#_l I \le 3\#_l J$ for any $I, J \in \mathcal{I}'_l$.

(2) Denote $N_{k,h} = n_{k+1} \cdots n_h$ for $h = k+1, k+2, \cdots$ and $N'_{l,s} = n'_{l+1} \cdots n'_s$ for $s = l+1, l+2, \cdots$. If $h \le k$ or $s \le l$, we define $N_{k,h} = N'_{l,s} = 1$. Let $p = \min_{I \in \mathcal{I}'_l} \#_l I$ and $q = \max_{I \in \mathcal{I}'_l} \#_l I$.

Take any $h > k$ and take an integer s such that $pN'_{l,s-g} < N_{k,h} \le pN'_{l,s}$. Since

$$\frac{pN'_{l,s}}{qN'_{l,s-g}} \ge \frac{n'_{s-g+1} \cdots n'_s}{C} \ge \frac{2^g}{C} \ge \frac{2^{(\log C + \log C_0)/\log 2}}{C} = C_0,$$

there exists m such that

$$qN'_{l,s-g} < N_{k,m} \le pN'_{l,s}. \tag{2.1}$$

If $qN'_{l,s-g} < N_{k,h}$, then we can take $m = h$. Otherwise, since $N_{k,h} \le qN'_{l,s-g} < N_{k,m}$, we must have $h < m$. Moreover, since $pN'_{l,s-g} < N_{k,h}$ and $N_{k,m} \le pN'_{l,s}$, we have

$$\frac{N_{k,m}}{N_{k,h}} < \frac{pN'_{l,s}}{pN'_{l,s-g}} = n'_{s-g+1} \cdots n'_s \le C_0'^g.$$

Therefore,

$$2^{m-h} \le n_{h+1} \cdots n_m = \frac{N_{k,m}}{N_{k,h}} < C_0'^g,$$

and hence, $m < h + \dfrac{\log C'_0}{\log 2}\, g$. Thus, there exists m satisfying (2.1) together with

$$h \le m < h + \frac{\log C'_0}{\log 2}\, g.$$

Construct an (s,g)-admissible partition $\mathcal{I}'_s$ of E' of $\mathcal{E}_m$-type such that $\mathcal{I}'_l$ and $\mathcal{I}'_s$ are consistent and $\#_s I \le 9C\#_s J$ holds for any $I, J \in \mathcal{I}'_s$.

Take any $K \in \mathcal{I}'_l$. Then, we have

$$\#_l K N'_{l,s-g} \le q N'_{l,s-g} < N_{k,m} \le p N'_{l,s} \le \#_l K N'_{l,s-g}.$$

Hence by the same argument as in the proof of (1) applying Lemma 1, there exists a (s,g)-admissible partition $\mathcal{K}_K$ of K in E' with $\#\mathcal{K}_K = N_{k,m}$. Moreover, $\#_s I \le 3\#_s J$ holds for any $I, J \in \mathcal{K}_K$. Let $\mathcal{I}'_s = \cup_{K\in\mathcal{I}'_l}\mathcal{K}_K$. Then, it is clear that $\mathcal{I}'_s$ is a (s,g)-admissible partition of E' of $\mathcal{E}_m$-type which is consistent with $\mathcal{I}'_l$. Take any $I, J \in \mathcal{I}'_s$. Let $I \in \mathcal{K}_K$ and $J \in \mathcal{K}_L$. Then, since

$$\#_s I \le \frac{1}{\#\mathcal{K}_K}\sum_{I'\in\mathcal{K}_K} 3\#_s I' = \frac{3}{N_{k,m}}\sum_{I'\in\mathcal{K}_K}\#_s I' = \frac{3}{N_{k,m}}\, N'_{l,s}\#_l K$$

and

$$\#_s J \ge \frac{1}{\#\mathcal{K}_L}\sum_{I'\in\mathcal{K}_L} (1/3)\#_s I' = \frac{1/3}{N_{k,m}}\sum_{I'\in\mathcal{K}_L}\#_s I' = \frac{1/3}{N_{k,m}}\, N'_{l,s}\#_l L,$$

we have

$$\#_s J \le 9\frac{\#_l K}{\#_l L}\#_s J \le 9C\#_s J,$$

which completes the proof. □

Corollary 2. *There exist sequences of positive integers* $\{k_i\}$, $\{g_i\}$ *and* $\{s_i\}$ *increasing to* ∞ *such that*

(i) $\lim_{i\to\infty} k_i/i = \infty$ *and* $\lim_{i\to\infty} k_{i+1}/k_i = 1$,

(ii) $(1/2)^{k_{i+1}-k_i} < \Delta$ $(i = 1, 2, \dots)$,

(iii) $\sup g_i/i < \infty$, *and*

(iv) *there exists a consistent family of* (s_i, g_i)*-admissible partitions* $\mathcal{I}'_i$ $(i=1,2,\dots)$ *of* E' *of* $\mathcal{E}_{k_i}$*-type.*

Proof. Take j such that $2^{-j} < \Delta$. We construct $k_1 < k_2 < \cdots$ inductively starting by an arbitrary k_1. For $k = k_1$, there exists $(l,1)$-admissible partitions $\mathcal{I}'_l$ of E' of $\mathcal{E}_k$-type by (1) of Lemma 2. Let $(s_1, g_1) = (l, 1)$ and $\mathcal{I}'_1 = \mathcal{I}'_l$. Assume that k_i, (s_i, g_i) and $\mathcal{I}'_i$ are determined. For $k = k_i$, $h = k + j$ and $\mathcal{I}'_i$, apply Lemma 2 and get m and (s,g)-admissible partitions $\mathcal{I}'_s$ of E' of $\mathcal{E}_k$-type. Define $k_{i+1} = m$, $(s_{i+1}, g_{i+1}) = (s, g)$ and $\mathcal{I}'_{i+1} = \mathcal{I}'_s$. Then, we have $(i)(ii)$. We also have (iii) since

$$g_i \le \frac{i\log 9 + \log C_0}{\log 2} + 1 \ (i = 1, 2, \dots).$$

□

3. Proof of the main theorem

Take a sequence $k_1 < k_2 < \cdots$ and a consistent family of (s_i, g_i)-admissible partitions $\mathcal{I}'_i$ $(i = 1, 2, \dots)$ as in Corollary 2. Note that since $\lim_{i\to\infty} g_i/k_i = 0$ and $N_{s_i-g_i} \le N_{k_i} \le N_{s_i}$, we have

$$0 < \liminf_{i\to\infty} N'_{s_i-g_i}/N_{k_i} \le \limsup_{i\to\infty} N'_{s_i}/N_{k_i} < \infty.$$

In particular, we have

(iv) $\lim_{i\to\infty} s_i/i = \infty$ and $\lim_{i\to\infty} s_{i+1}/s_i = 1$.

We may also assume that

(v) $(1/2)^{s_{i+1}-s_i} < \Delta'$ $(i = 1, 2, \dots)$.

For $x \in E$, let $I^i(x)$ be $I \in \mathcal{E}_{k_i}$ such that $x \in I$. Let φ_i be the isomorphism from $\mathcal{E}_{k_i}$ to $\mathcal{I}'_i$. Since $\lim_{i\to\infty} N'_{s_i-g_i} = \infty$, $y \in E'$ such that $y \in \varphi_i(I^i(x))$ for any $i = 1, 2, \dots$ is determined, which is denoted by $f(x)$.

We prove that f satisfies the required conditions. By the construction, it is clear that f is strictly increasing. Take any $x \in E$. If $x + 0 \in E$, then x is not the right end point of $I^i(x)$ for any $i = 1, 2, \dots$. Hence, $f(x)$ and $f(x + 0)$ stay in a same (s_i, g_i)-admissible interval of E' as $s_i - g_i \to \infty$. Hence, $f(x+0) = f(x)+0$. Thus, f is right continuous. The same argument holds for $x - 0$. Thus, f is continuous.

Now, we prove the inequality (1.2). Let $x, y \in E$ satisfy that $x < y$ and $y - x$ is sufficiently small. Take $i = 1, 2, \dots$ such that $\delta_{k_{i+2}} < y - x \le \delta_{k_{i+1}}$. Then, since

$$y - x \le \delta_{k_{i+1}} \le (1/2)^{k_{i+1}-k_i+1}\delta_{k_i-1} < \Delta\delta_{k_i-1} \le d^j_{k_i}\delta_{k_i-1}$$

for any $j \in \{1, 2, \dots, n_{k_i-1} - 1\}$. Hence, there exists a basic interval I of E of level k_i such that $\{x, y\} \subset I$. Therefore, $f(x)$ and $f(y)$ belong to a same (s_i, g_i)-admissible interval of E', and hence, in a same basic interval of E' of level $s_i - g_i$. Denote $l = s_i - g_i$ and $h = s_i$. Since $f(x)$ and $f(y)$ belong to a same basic interval of E' of level l, we have $f(y) - f(x) \le \delta'_l$. For any $\epsilon > 0$, there exists a sufficiently small $\lambda > 0$ such that

$$(1 - \lambda)^2(s' + \lambda)^{-1}(s - \lambda) < \frac{s}{s'} - \epsilon.$$

Take i_0 such that

$$|s - \frac{\log N_{k_{i+2}}}{-\log \delta_{k_{i+2}}}| < \lambda,\ |s' - \frac{\log N'_l}{-\log \delta'_l}| < \lambda,$$

$$\frac{(k_{i+2}-k_i)\log C_0}{k_{i+2}\log 2} < \lambda \quad \text{and} \quad \frac{(h-l)\log C_0'}{h\log 2} < \lambda$$

for any $i \geq i_0$. Assume that $i \geq i_0$. Since $N_l' \leq N_{k_i} \leq N_h'$, we have

$$\begin{aligned}
-\log(f(y)-f(x)) &\geq -\log \delta_l' > (s'+\lambda)^{-1}\log N_l' \\
&= (s'+\lambda)^{-1}\log N_h' \left(1 - \frac{\log(n_{l+1}'\cdots n_h')}{\log N_h'}\right) \\
&\geq (s'+\lambda)^{-1}\log N_h' \left(1 - \frac{(h-l)\log C_0'}{h\log 2}\right) \\
&> (s'+\lambda)^{-1}\log N_h' \,(1-\lambda) \geq (1-\lambda)(s'+\lambda)^{-1}\log N_{k_i} \\
&= (1-\lambda)(s'+\lambda)^{-1}\log N_{k_{i+2}} \left(1 - \frac{\log(n_{k_i+1}\cdots n_{k_{i+2}})}{\log N_{k_{i+2}}}\right) \\
&\geq (1-\lambda)(s'+\lambda)^{-1}\log N_{k_{i+2}} \left(1 - \frac{(k_{i+2}-k_i)\log C_0}{k_{i+2}\log 2}\right) \\
&> (1-\lambda)^2(s'+\lambda)^{-1}\log N_{k_{i+2}} \\
&> (1-\lambda)^2(s'+\lambda)^{-1}(s-\lambda)(-\log \delta_{k_{i+2}}) \\
&> (1-\lambda)^2(s'+\lambda)^{-1}(s-\lambda)(-\log(y-x)) \\
&> (\frac{s}{s'}-\epsilon)(-\log(y-x))
\end{aligned}$$

Hence, for any $\epsilon > 0$, $f(y)-f(x) < (y-x)^{\frac{s}{s'}-\epsilon}$ holds for any $x < y$ in E such that $y-x$ is sufficiently small. Thus, for any $\epsilon > 0$, there exists C_1 such that

$$f(y)-f(x) < C_1(y-x)^{\frac{s}{s'}-\epsilon}$$

for any $x < y$ in E.

Let us prove the inequality of the opposite direction. Let $x, y \in E$ satisfy that $x < y$ and $y-x$ is sufficiently small. Take $i = 1, 2, \ldots$ such that $\delta_{k_{i-1}} < y-x \leq \delta_{k_{i-2}}$. Then, there exists a basic interval I of level k_{i-1} such that $x \in I$ but $y \notin I$. Therefore, $f(x)$ and $f(y)$ belong to distinct (s_{i-1}, g_{i-1})-admissible intervals in $\mathcal{I}_{i-1}'$, and hence, belong to distinct basic interval of E' of level s_{i-1}. Therefore, by (v),

$$f(y)-f(x) \geq d_{s_{i-1}}^{j'}\delta_{s_{i-1}-1}' \geq \Delta\delta_{s_{i-1}-1}' > \delta_{s_i}'.$$

Denote $l = s_i - g_i$ and $h = s_i$. For any $\epsilon > 0$, there exists a sufficiently small $\lambda > 0$ such that

$$(1+\lambda)^2(s'-\lambda)^{-1}(s+\lambda) < \frac{s}{s'} + \epsilon.$$

Take i_0 such that

$$|s - \frac{\log N_{k_{i-2}}}{-\log \delta_{k_{i-2}}}| < \lambda, \ |s' - \frac{\log N'_h}{-\log \delta'_h}| < \lambda,$$

$$\frac{(k_i - k_{i-2}) \log C_0}{k_{i-2} \log 2} < \lambda \ \text{ and } \ \frac{(h-l) \log C'_0}{l \log 2} < \lambda$$

for any $i \geq i_0$. Assume that $i \geq i_0$. Since $N'_l \leq N_{k_i} \leq N'_h$, we have

$$\begin{aligned}
-\log(f(y) - f(x)) &< -\log \delta'_h < (s' - \lambda)^{-1} \log N'_h \\
&= (s' - \lambda)^{-1} \log N'_l \left(1 + \frac{\log(n'_{l+1} \cdots n'_h)}{\log N'_l}\right) \\
&\leq (s' - \lambda)^{-1} \log N'_l \left(1 + \frac{(h-l) \log C'_0}{l \log 2}\right) \\
&< (s' - \lambda)^{-1} \log N'_l \ (1 + \lambda) \leq (1 + \lambda)(s' - \lambda)^{-1} \log N_{k_i} \\
&= (1 + \lambda)(s' - \lambda)^{-1} \log N_{k_{i-2}} \left(1 + \frac{\log(n_{k_{i-2}+1} \cdots n_{k_i})}{\log N_{k_{i-2}}}\right) \\
&\leq (1 + \lambda)(s' + \lambda)^{-1} \log N_{k_{i-2}} \left(1 + \frac{(k_i - k_{i-2}) \log C_0}{k_{i-2} \log 2}\right) \\
&< (1 + \lambda)^2 (s' - \lambda)^{-1} \log N_{k_{i-2}} \\
&< (1 + \lambda)^2 (s' - \lambda)^{-1} (s + \lambda)(-\log \delta_{k_{i-2}}) \\
&\leq (1 + \lambda)^2 (s' - \lambda)^{-1} (s + \lambda)(-\log(y - x)) \\
&< (\frac{s}{s'} + \epsilon)(-\log(y - x))
\end{aligned}$$

Hence, for any $\epsilon > 0$, $f(y) - f(x) > (y - x)^{\frac{s}{s'} + \epsilon}$ holds for any $x < y$ in E such that $y - x$ is sufficiently small. Thus, for any $\epsilon > 0$, there exists C_2 such that

$$f(y) - f(x) > C_2 (y - x)^{\frac{s}{s'} + \epsilon}$$

for any $x < y$ in E.

Thus, we complete the proof.

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