Publ. Math. Debrecen
89/3 (2016), 263–275
DOI: 10.5486/PMD.2016.7586

On extensions of the generalized cosine functions from some large sets

By ANNA BAHYRYCZ (Kraków), JANUSZ BRZDĘK (Kraków) and ELIZA JABŁOŃSKA (Rzeszów)

Dedicated to Professor Zsolt Páles on the occasion of his 60th birthday

Abstract. Let $(G,+)$ be a commutative semigroup, τ be an endomorphism of G and involution, D be a nonempty subset of G, and P be a quadratically closed field with $\mathrm{char}P \neq 2$. We show that if the set $D \setminus g^{-1}(\{0\})$ is 'sufficiently large', then each function $g : D \to P$, satisfying the condition: $g(x+y) + g(x+\tau(y)) = 2g(x)g(y)$ for $x, y \in D$ with $x+y, x+\tau(y) \in D$, can be extended to a unique solution $f : G \to P$ of the functional equation $f(x+y) + f(x+\tau(y)) = 2f(x)f(y)$ for $x, y \in G$.

Let P be a field that is quadratically closed (i.e., for each $x \in P$ there is $y \in P$ with $y^2 = x$), $\mathrm{char}P \neq 2$, $(G,+)$ be a commutative semigroup, and D be a nonempty subset of G, unless explicitly stated otherwise. Let τ be an endomorphism of G and involution (cf. [4]), i.e., $\tau(x+y) = \tau x + \tau y$ and $\tau(\tau x) = x$ for $x, y \in G$, where $\tau x := \tau(x)$ for $x \in G$.

We say that a function $f : D \to P$ satisfies the functional equation

$$f_0(x+y) + f_0(x+\tau y) = 2f_0(x)f_0(y) \tag{1}$$

on the set D provided

$$f(x+y) + f(x+\tau y) = 2f(x)f(y), \quad x, y \in D, x+y, x+\tau y \in D. \tag{2}$$

Mathematics Subject Classification: 39B52, 39B82.
Key words and phrases: generalized cosine function, d'Alembert's functional equation, restricted domain, abelian semigroup, involution, filter, ideal.

Note that if G is a group, then (1) is a natural generalization of the well-known d'Alembert (cosine) equation

$$f(x+y)+f(x-y)=2f(x)f(y), \quad x,y \in G, \tag{3}$$

with $\tau x \equiv -x$. Therefore, every solution to (1) can be called a generalized cosine function. For some information and references on the d'Alembert equation (3) and recent examples of results concerning its solutions, see [3], [9], [12], [14], [15], [18], [21], [22], [23], [24], [25], [26], [27], [28], [29].

Solutions of (1) have been determined in [22] in the case where G is a commutative group (under some additional assumptions). Later, it has been proved in [20] that a similar result is valid also in the case where G is 'only' a commutative semigroup.

In this paper we study possibilities of extensions of functions satisfying equation (1) on D to a solution $g_0 : G \to P$ of (1). It is obvious that such an extension does not need to exist if D is 'meagre' (i.e., not large enough). Therefore, we have to find an assumption on D that makes it 'sufficiently large' to guarantee the above-mentioned property of extension.

Clearly, it is natural to assume that a set $D \subset G$ is large provided it belongs to some filter of subsets of G that is proper (i.e., different from 2^G). So, we assume that $D \in \mathcal{L}$, where $\mathcal{L}$ is a family of subsets of G such that the following three conditions are valid:

$$\mathcal{L} \neq 2^G, \tag{4}$$

$$B \in \mathcal{L}, \quad B \in 2^G, 2^B \cap \mathcal{L} \neq \emptyset, \tag{5}$$

$$\tau(B), B-x, B+x, A\cap B \in \mathcal{L}, \quad A,B \in \mathcal{L}, x \in G, \tag{6}$$

where

$$T+a := \{a+x : x \in T\}, \quad T-a := \{x \in G : x+a \in T\}$$

for $a \in G$ and $T \in 2^G$. Note that (4) and (5) imply that $\emptyset \notin \mathcal{L}$.

We show that then, for each function $g : D \to P$ satisfying (2), there exists a solution $g_0 : G \to P$ of (1) with $g(x) = g_0(x)$ for $x \in D$. This outcome corresponds in particular to [8, Theorems 1–4], [2, Theorem 1] and to some analogous results obtained for the equation of homomorphism in [1], [5], [6], [13], [16], [17] (see also [10, Theorem 4.1] or [19, Theorem 1.1, Ch. XVIII, p. 468]).

Remark 1. Let $\mathcal{I} \subset 2^G$ be an ideal (i.e., $2^B \subset \mathcal{I}$ and $B \cup C \in \mathcal{I}$ for every $B, C \in \mathcal{I}$) and

$$\mathcal{L} := \{A \subset G : G \setminus A \in \mathcal{I}\}.$$

Then it is easily seen that $\mathcal{L}$ is a filter, i.e., (5) holds and $A \cap B \in \mathcal{L}$ for every $A, B \in \mathcal{L}$. Moreover, if $\mathcal{I}$ has some additional suitable properties, then also conditions (4) and (6) are valid. Below we provide natural examples of such ideals $\mathcal{I} \subset 2^G$ having those suitable properties guaranteeing (4) and (6).

(a) G is cancellative and not of finite cardinality and $\mathcal{I} = \{A \subset G : \operatorname{card} A < \operatorname{card} G\}$.

(b) d is an invariant metric in G (i.e., $d(x+y, z+y) = d(x,z)$ for $x, y, z \in G$), $\sup_{x,y \in G} d(x,y) = \infty$, the set $\tau(B)$ is bounded (i.e., $\sup_{x,y \in \tau(B)} d(x,y) < \infty$) for each bounded set $B \in 2^G$, and $\mathcal{I}$ is the family of all bounded subsets of G.

(c) $G = \{z \in \mathbb{C} : \Re z > 0\}$ (with the usual addition of complex numbers), $\mathcal{I}$ is the family of all subsets A of G with $\sup_{z \in A} \Re z < \infty$ and $\tau z = \overline{z}$ for $z \in G$, where $\overline{z}$ is the complex conjugate and $\Re z$ denotes the real part of the complex number z.

(d) G is a topological group of the second category of Baire, $\mathcal{I}$ is the family of all first category subsets of G and τ is continuous (which actually means that τ is a homeomorphism, because $\tau^{-1} = \tau$).

(e) G is a locally compact topological group, μ is the Haar measure in G with $\mu(G) = \infty$, $\mathcal{I} = \{A \subset G : \mu(A) < \infty\}$ and τ is continuous.

(f) G is an abelian Polish group, $\mathcal{I}$ is the σ-ideal of Haar zero subsets of G (see [7]) and τ is continuous.

(g) G is an abelian Polish group, $\mathcal{I}$ is the σ-ideal of Christensen zero subsets of G (see [11]) and τ is continuous.

In this paper we also use the following notions:

$$T + S := \{x + y : x \in T, y \in S\}, \quad T - S := \{z \in G : (S + z) \cap T \neq \emptyset\},$$

for $S, T \subset G$. Clearly, if G is a group, then

$$T - S = \{x - y : x \in T, y \in S\}.$$

We start with some auxiliary lemmas. The first one is a very simple observation.

Lemma 1. *Assume that $S \in \mathcal{L}$. Then*

$$S - S = G. \tag{7}$$

Proof. Take $y \in G$. By (6),

$$S \cap (y + S) \neq \emptyset.$$

Hence there are $u, v \in S$ such that $u = y + v$, which means that $y \in S - S$. In this way we have shown that $G \subset S - S$, which completes the proof of (7). $\square$

Lemma 2. *Assume that $(H, +)$ is an abelian group, $S \in \mathcal{L}$ and $h_0 : S \to H$ satisfies*

$$h_0(x + y) = h_0(x) + h_0(y), \quad x, y \in S, x + y, x + \tau y \in S. \tag{8}$$

Then there exists a unique solution $h : G \to H$ of the equation

$$h(x + y) = h(x) + h(y), \quad x, y \in G \tag{9}$$

such that $h(x) = h_0(x)$ for $x \in S$.

Proof. Take $a, b, c, d \in S$ with $a + d = b + c$ and write

$$\begin{aligned} S_1 &:= (S - a) \cap (S - \tau a) \cap (S - (a + d)) \cap (S - (a + \tau d)), \\ S_2 &:= (S - \tau b) \cap (S - b) \cap (S - (b + c))) \cap (S - (b + \tau c)). \end{aligned}$$

Clearly, by (6), $S_1, S_2 \in \mathcal{L}$, whence $S_0 := S \cap S_1 \cap S_2 \neq \emptyset$.

Let $v \in S_0$. Then

$$v, v + a, v + \tau a, v + a + d, v + a + \tau d \in S,$$

$$v + b, v + \tau b, v + b + c, v + b + \tau c \in S.$$

Consequently, by (8),

$$\begin{aligned} h_0(v) + h_0(b) + h_0(c) &= h_0(v + b) + h_0(c) = h_0(v + b + c) \\ &= h_0(v + a + d) = h_0(v + a) + h_0(d) \\ &= h_0(v) + h_0(a) + h_0(d). \end{aligned}$$

Thus we have proved that

$$h_0(a) - h_0(b) = h_0(c) - h_0(d), \quad a, b, c, d \in S, a + d = c + b. \tag{10}$$

Note that if $z \in G$ and $z + b = a$, $z + d = c$ for some $a, b, c, d \in S$, then

$$a + d = z + b + d = b + z + d = b + c.$$

Therefore, in view of (7) and (10), we may define $h : G \to H$ by

$$h(z) := h_0(a) - h_0(b)$$

for every $z \in G$ and $a, b \in S$ such that $z + b = a$.

First, we show that $h(z) = h_0(z)$ for $z \in S$. To this end, take $z \in S$ and $u \in S \cap (S - \tau z) \cap (S - z)$. Then $u + z, u + \tau z \in S$ and, according to the definition of h and (8),

$$h(z) = h_0(z + u) - h_0(u) = h_0(z) + h_0(u) - h_0(u) = h_0(z).$$

Next, we prove that (9) holds. Let $z, w \in G$. According to (7), there exist $a, b, c, d \in S$ with $z + b = a$ and $w + d = c$ and, in view of the definition of h, $h(z) = h_0(a) - h_0(b)$ and $h(w) = h_0(c) - h_0(d)$. Write

$$S_a := (S - a) \cap (S - \tau a) \cap (S - (a + c)) \cap (S - (a + \tau c)),$$

$$S_b := (S - b) \cap (S - \tau b) \cap (S - (b + d)) \cap (S - (b + \tau d)).$$

Further, there is $u \in S \cap S_a \cap S_b \in \mathcal{L}$. Hence

$$\begin{aligned} h(z + w) &= h_0(u + a + c) - h_0(u + b + d) \\ &= h_0(u + a) + h_0(c) - (h_0(u + b) + h_0(d)) \\ &= h_0(u) + h_0(a) + h_0(c) - (h_0(u) + h_0(b) + h_0(d)) \\ &= h_0(a) - h_0(b) + h_0(c) - h_0(d) = h(z) + h(w). \end{aligned}$$

It remains to show the uniqueness of h. So, let $h_1 : G \to H$ be such that $h_1(x) = h_0(x)$ for $x \in S$ and

$$h_1(x + y) = h_1(x) + h_1(y), \quad x, y \in G.$$

Take $z \in G$ and $a, b \in S$ with $z + b = a$. Then

$$h_1(z) = h_1(a) - h_1(b) = h_0(a) - h_0(b) = h(z). \qquad \square$$

It is easily seen that Lemma 2 implies the following corollary (cf. [1], [5], [6], [13], [16], [17]; see also [10, Theorem 4.1] or [19, Theorem 1.1, Ch. XVIII, p. 468]).

Corollary 1. *Assume that $(H,+)$ is an abelian group, $S \in \mathcal{L}$ and $h_0 : S \to H$ satisfies*

$$h_0(x+y) = h_0(x) + h_0(y), \quad x, y \in S, x+y \in S.$$

Then there is a unique solution $h : G \to H$ of (9) such that $h(x) = h_0(x)$ for $x \in S$.

Lemma 3. *Let $D \in \mathcal{L}$ and $g : D \to P$ fulfil (2). Then there is $\widehat{D} \in \mathcal{L}$ with $\widehat{D} \subset D$, $\tau(\widehat{D}) = \widehat{D}$ and*

$$g(\tau x) = g(x), \quad x \in \widehat{D}. \tag{11}$$

Proof. If $g(x) = 0$ for each $x \in D$, then it is enough to take $\widehat{D} := D \cap \tau(D)$. So, assume now that there is $y \in D$ with $g(y) \neq 0$. Write

$$D_y := (D - y) \cap D \in \mathcal{L}, \quad \widehat{D} := D_y \cap \tau(D_y).$$

Clearly, $\tau(\widehat{D}) = \widehat{D}$ and $\widehat{D} \in \mathcal{L}$. Take $w \in \widehat{D}$. Then $\tau w \in \widehat{D}$. Moreover, $w + y, \tau w + y \in D$, and consequently,

$$\begin{aligned} 2g(y)g(w) &= g(y+w) + g(y+\tau w) \\ &= g(y+\tau w) + g(y+\tau(\tau w)) = 2g(y)g(\tau w), \end{aligned}$$

which yields $g(\tau w) = g(w)$. □

Lemma 4. *Let $D \in \mathcal{L}$, $g : D \to P$ fulfil (2) and $D_g := g^{-1}(\{0\}) \notin \mathcal{L}$. Then there exist $D_1 \in \mathcal{L}$ and a function $m : G \to P$ such that $D_1 \subset D$, $D_1 = \tau(D_1)$,*

$$m(x+y) = m(x)m(y), \quad x, y \in G, \tag{12}$$

$$g(x) = \frac{m(x) + m(\tau x)}{2}, \quad x \in D_1. \tag{13}$$

Proof. According to Lemma 3, there is $\widehat{D} \in \mathcal{L}$ such that (11) holds, $\widehat{D} \subset D$ and $\tau(\widehat{D}) = \widehat{D}$.

First, consider the case where

$$g(x+y) = g(x+\tau y), \quad x, y \in \widehat{D}, x+y, x+\tau y \in \widehat{D}. \tag{14}$$

Then, by (2),

$$g(x+y) = g(x)g(y), \quad x, y \in \widehat{D}, x+y, x+\tau y \in \widehat{D}.$$

We show that $0 \notin g(\widehat{D})$. For the proof by contradiction suppose that there is $y \in \widehat{D}$ with $g(y) = 0$. Let

$$D_y := \widehat{D} \cap (\widehat{D} - y) \cap (\widehat{D} - \tau y) \in \mathcal{L}, \quad D_0 := (D_y + y) \cap \widehat{D} \in \mathcal{L}.$$

Take $z \in D_0$. Then $z = x + y$ with some $x \in D_y$ and $x + y, x + \tau y \in \widehat{D}$. Hence

$$g(z) = g(x + y) = g(x)g(y) = 0.$$

Thus we have shown that $g(D_0) = \{0\}$, which is a contradiction, because $D_g \notin \mathcal{L}$.

So, $0 \notin g(\widehat{D})$. Consequently, in view of Lemma 2, there is $m : G \to P$ such that $g(x) = m(x)$ for $x \in \widehat{D}$ and (12) holds. It is easily seen that, by (11),

$$g(x) = \frac{1}{2}(g(x) + g(\tau x)) = \frac{1}{2}(m(x) + m(\tau x)), \quad x \in \widehat{D},$$

which means that (13) holds with $D_1 = \widehat{D}$.

Now, let us study the case when there exist $x_0, y_0 \in \widehat{D}$ such that

$$x_0 + y_0, x_0 + \tau y_0 \in \widehat{D}$$

and $g(x_0 + y_0) \neq g(x_0 + \tau y_0)$. Write $D_2 := \widehat{D} \cap (\widehat{D} - y_0) \cap (\widehat{D} - \tau y_0)$ and

$$f(x) := g(x + y_0) - g(x + \tau y_0), \quad x \in D_2.$$

It is easily seen that $\tau(D_2) = D_2$ and $x_0 \in D_2$, whence

$$f(x_0) \neq 0. \tag{15}$$

Since $D_2 \subset \widehat{D}$, (11) implies

$$\begin{aligned} f(\tau x) &= g(\tau x + y_0) - g(\tau x + \tau y_0) = g(x + \tau y_0) - g(x + y_0) \\ &= -(g(x + y_0) - g(x + \tau y_0)) = -f(x), \quad x \in D_2. \end{aligned} \tag{16}$$

Take $x, y \in D_2$ with $x + y, x + \tau y \in D_2$. Then

$$x + y + y_0, x + y + \tau y_0, x + \tau y + y_0, x + \tau y + \tau y_0 \in \widehat{D},$$

and consequently,

$$\begin{aligned} f(x + y) &= g(x + y + y_0) - g(x + y + \tau y_0), \\ f(x + \tau y) &= g(x + \tau y + y_0) - g(x + \tau y + \tau y_0). \end{aligned}$$

Adding those two equalities, we get

$$\begin{aligned} f(x+y)+f(x+\tau y) &= g(x+y+y_0)+g(x+\tau y+y_0) \\ &\quad - g(x+y+\tau y_0)-g(x+\tau y+\tau y_0) \\ &= 2g(x+y_0)g(y)-2g(x+\tau y_0)g(y)=2f(x)g(y). \end{aligned} \tag{17}$$

Next, $y+\tau x \in D_2$, and consequently, $y+\tau x+y_0, y+\tau x+\tau y_0 \in \widehat{D}$, because $\tau(D_2)=D_2$. So, analogously, we get

$$f(y+x)+f(y+\tau x)=2f(y)g(x). \tag{18}$$

Finally, conditions (16), (17) and (18) imply that

$$f(x+y)=f(x)g(y)+f(y)g(x), \quad x,y\in D_2, x+y, x+\tau y \in D_2. \tag{19}$$

Write

$$D_3 := D_2 \cap (D_2 - x_0) \cap (D_2 - \tau x_0).$$

Clearly,

$$\tau(D_3)=D_3. \tag{20}$$

Let $x,y\in D_3$ and $x+y, x+\tau y\in D_3$. Then

$$y+x_0, y+\tau x_0, x+y+x_0, x+y+\tau x_0, x+\tau y+\tau x_0 \in D_2,$$

and consequently, by (19),

$$\begin{aligned} f((x+y)+x_0) &= f(x+y)g(x_0)+f(x_0)g(x+y) \\ &= (f(x)g(y)+f(y)g(x))g(x_0)+f(x_0)g(x+y), \end{aligned}$$

$$\begin{aligned} f(x+(y+x_0)) &= f(x)g(y+x_0)+f(y+x_0)g(x) \\ &= f(x)g(y+x_0)+(f(y)g(x_0)+f(x_0)g(y))g(x), \end{aligned}$$

whence

$$(g(x+y)-g(x)g(y))f(x_0)=(g(y+x_0)-g(y)g(x_0))f(x),$$

which can be rewritten as

$$h(y)f(x)=g(x+y)-g(x)g(y),$$

where (see (15))

$$h(y) := \frac{g(y + x_0) - g(y)g(x_0)}{f(x_0)}.$$

Thus we have proved that

$$g(x + y) = h(y)f(x) + g(x)g(y), \quad x, y \in D_3, x + y, x + \tau y \in D_3. \tag{21}$$

If $f(x) = 0$ for every $x \in D_3$, then (21) yields

$$g(x + y) = g(x)g(y), \quad x, y \in D_3, x + y, x + \tau y \in D_3,$$

and we can argue as in the case of (14) (with $\widehat{D} = D_3$).

It remains to study the case where there is $x_1 \in D_3$ with $f(x_1) \neq 0$. It is easily seen that, by (20) and (21),

$$h(y)f(x) = g(x + y) - g(x)g(y) = h(x)f(y) \qquad x, y \in D_3, x + y, x + \tau y \in D_3. \tag{22}$$

Since P is quadratically closed, there is $a \in P$ such that

$$\frac{h(x_1)}{f(x_1)} = a^2,$$

and consequently,

$$h(y) = a^2 f(y), \quad y \in D_3, x_1 + y, x_1 + \tau y \in D_3.$$

This and (20) imply that

$$h(y) = a^2 f(y), \quad y \in D_4,$$

where $D_4 := D_3 \cap (D_3 - x_1) \cap (D_3 - \tau x_1) \in \mathcal{L}$. So, by (22), we can write that

$$g(x + y) = g(x)g(y) + a^2 f(x)f(y), \quad x, y \in D_4, x + y, x + \tau y \in D_4. \tag{23}$$

Clearly, (19) implies that

$$af(x + y) = af(x)g(y) + af(y)g(x), \quad x, y \in D_4, x + y, x + \tau y \in D_4. \tag{24}$$

Hence, adding (23) and (24), we obtain

$$\begin{aligned} g(x + y) + af(x + y) &= g(y)(g(x) + af(x)) + af(y)(g(x) + af(x)) \\ &= (g(x) + af(x))(g(y) + af(y)) \end{aligned} \tag{25}$$

for every $x, y \in D_4$ with $x+y, x+\tau y \in D_4$. Define $h_1, h_2 : D_4 \to P$ by

$$h_1(x) := g(x) + af(x), \quad h_2(x) := g(x) - af(x), \quad x \in D_4.$$

Then, by (11), (16) and the equality $\tau(D_4) = D_4$,

$$h_2(x) = g(x) - af(x) = g(\tau x) + af(\tau x) = h_1(\tau x), \quad x \in D_4,$$

whence

$$g(x) = \frac{h_1(x) + h_2(x)}{2} = \frac{h_2(\tau x) + h_2(x)}{2} = \frac{h_1(x) + h_1(\tau x)}{2}, \quad x \in D_4, \tag{26}$$

and (25) implies that

$$h_1(x+y) = h_1(x)h_1(y), \quad x, y \in D_4, x+y, x+\tau y \in D_4,$$

$$h_2(x+y) = h_1(\tau(x+y)) = h_1(\tau x)h_1(\tau y) = h_2(x)h_2(y),$$
$$x, y \in D_4, x+y, x+\tau y \in D_4.$$

Note that $h_1^{-1}(\{0\}) \notin \mathcal{L}$ or $h_2^{-1}(\{0\}) \notin \mathcal{L}$, because otherwise, by (26), we would get $D_g \in \mathcal{L}$. So, analogously, as in the case of (14), we show that $0 \notin h_1(D_4)$ or $0 \notin h_2(D_4)$. Hence, by Lemma 2, we deduce that there is $m : G \to P$ such that (12) holds and $h_1(x) = m(x)$ for $x \in D_4$ or $h_2(x) = m(x)$ for $x \in D_4$. This completes the proof (in view of (26)). □

Lemma 5. *Let $f, g : D \to P$ be solutions to* (2),

$$S := \{x \in D : g(x) = f(x)\} \in \mathcal{L}$$

and $D_g \notin \mathcal{L}$. Then $g(x) = f(x)$ for $x \in D$.

Proof. Fix $w \in D$. Clearly,

$$A := S \cap (S - w) \cap (S - \tau w) \in \mathcal{L}.$$

Take $s \in A \setminus D_g$. Then $s + w, s + \tau w \in S$, whence

$$\begin{aligned} 2g(s)g(w) &= g(s+w) + g(s+\tau w) \\ &= f(s+w) + f(s+\tau w) = 2f(s)f(w) = 2g(s)f(w). \end{aligned}$$

Consequently, $g(w) = f(w)$. □

Now we are in a position to prove the main result of this paper.

Theorem 1. *Let $D \in \mathcal{L}$, $g : D \to P$ satisfy* (2) *and $D_g \notin \mathcal{L}$. Then there exists a unique solution $f : G \to P$ of equation* (1) *such that $g(x) = f(x)$ for $x \in D$.*

Proof. On account of Lemma 4, there exist a set $D_1 \in \mathcal{L}$ and a function $m : G \to P$ such that $D_1 \subset D$, $D_1 = \tau(D_1)$, and conditions (12) and (13) are valid. Write

$$f(x) := \frac{m(x) + m(\tau x)}{2}, \quad x \in G.$$

It is easy to check that f is a solution to (1) and, by (13), $g(x) = f(x)$ for $x \in D_1$. Hence, by Lemma 5, $g(x) = f(x)$ for $x \in D$, because $D_g \notin \mathcal{L}$.

It remains to show the uniqueness of the function f. To this end, assume that $f_0 : G \to P$ is a solution to equation (1) with $g(x) = f_0(x)$ for $x \in D$. Let $F := \{y \in G : f(x) = 0\}$ and $F_0 := F \cap (G \setminus D)$. Clearly, $D \subset G \setminus F_0$, so $G \setminus F_0 \in \mathcal{L}$. Suppose that $F \in \mathcal{L}$. Then, by (6), $D_g = F \cap (G \setminus F_0) \in \mathcal{L}$, which is a contradiction.

Thus we have proved that $F \notin \mathcal{L}$. Consequently, from Lemma 5 (with $D := G$ and $g := f$) we deduce that $f_0 = f$. □

Remark 2. There arises a natural question if the assumption that $D_g \notin \mathcal{L}$ is really necessary in Theorem 1. In some particular cases this is not the case, e.g., when G is a group and $\tau x \equiv -x$. The following corollary shows this (we write $\frac{1}{2}T := \{y \in G : 2y \in T\}$ for $T \in 2^G$).

Corollary 2. *Let G be a group, $D \in \mathcal{L}$, $g : D \to P$ fulfil* (2) *with $\tau x \equiv -x$ and*

$$\frac{1}{2}T \in \mathcal{L}, \quad T \in \mathcal{L}. \tag{27}$$

Then there exists a unique solution $f : G \to P$ of equation (1) *such that $g(x) = f(x)$ for $x \in D$.*

Proof. First we show that $D_g \notin \mathcal{L}$ or $g(D) = \{0\}$. So, suppose that $D_g \in \mathcal{L}$ and take $v \in D$. Clearly, by (27),

$$D_v := D \cap (D_g - v) \cap \frac{1}{2}(D_g - v) \in \mathcal{L}.$$

Take $w \in D_v$. Then $2w + v, w + v \in D_g$, which means that

$$g(2w + v) = 0 = g(w + v),$$

whence

$$g(v) = g(v + w + w) + g(v + w - w) = 2g(w + v)g(w) = 0.$$

If $D_g \notin \mathcal{L}$, then we use Theorem 1. If $g(D) = \{0\}$, then it is enough to take $f(x) \equiv 0$.

It remains to show that $f(x) \equiv 0$ is the unique solution to (1) such that $f(D) = \{0\}$. So, take a solution $f_1 : G \to P$ of (1) with $f_1(D) = \{0\}$. Fix $y_0 \in G \setminus D$ and

$$z \in D \cap \frac{1}{2}(D - y_0) \in \mathcal{L}.$$

Then $z, y_0 + 2z \in D$, and consequently,

$$f_1(y_0) = f_1(y_0 + 2z) + f_1(y_0 + z - z) = 2f_1(y_0 + z)f_1(z) = 0. \qquad \square$$

References

[1] J. Aczél, J. A. Baker, D. Ž. Djoković, P. Kannappan and F. Radó, Extensions of certain homomorphisms of subsemigroups to homomorphisms of groups, *Aequationes Math.* **6** (1971), 263–271.

[2] I. Adamaszek, Almost trigonometric functions, *Glasnik Mat.* **19** (1984), 83–104.

[3] J. d'Alembert, Mémoire sur les principes de mécanique, *Hist. Acad. Sci., Paris* **1769**, 278–286.

[4] K. Baron, On additive involutions and Hamel bases, *Aequationes Math.* **87** (2014), 159–163.

[5] N. G. de Bruijn, On almost additive mappings, *Colloq. Math.* **15** (1966), 59–63.

[6] J. Brzdęk, On almost additive mappings, *Bull. Austral. Math. Soc.* **54** (1996), 281–290.

[7] J. P. R. Christensen, On sets of Haar measure zero in abelian Polish groups, *Israel J. Math.* **13** (1972), 255–260.

[8] I. Corovei, The extensions of the functions satisfying the cosine functional equation, *Mathematica (Cluj)* **18 (41)** (1976), 131–136.

[9] E. Y. Deeba and E. L. Koh, D'Alembert functional equations in distributions, *Proc. Amer. Math. Soc.* **116** (1992), 157–164.

[10] J. Dhombres, Some Aspects of Functional Equations, *Chulalongkorn University Press, Bangkok*, 1979.

[11] P. Fischer and Z. Słodkowski, Christensen zero sets and measurable convex functions, *Proc. Amer. Math. Soc.* **79** (1980), 449–453.

[12] P. Friis, D'Alembert's and Wilson's equations on Lie groups, *Aequationes Math.* **67** (2004), 12–25.

[13] R. Ger, Note on almost additive functions, *Aequationes Math.* **17** (1978), 73–76.

[14] A. Gessinger, On the construction of cosine operator function and semigroups on function spaces with generator $a(x)(d^2/dx^2) + b(x)(d/dx) + c(x)$; theory, *J. Comp. Anal. Appl.* **3** (2001), 1–31.

[15] S. Gudder, A generalization of d'Alembert's functional equation, *Proc. Amer. Math. Soc.* **115** (1992), 419–425.

[16] S. Hartman, A remark about Cauchy's equation, *Colloq. Math.* **8** (1961), 77–79.

[17] W. B. Jurkat, On Cauchy's functional equation, *Proc. Amer. Math. Soc.* **16** (1965), 683–696.

[18] P. Kannappan, The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ for groups, *Proc. Amer. Math. Soc.* **19** (1968), 69–74.

[19] M. Kuczma, An Introduction to the Theory of Functional Equations and Inequalities, *Państwowe Wydawnictwo Naukowe & Uniwersytet Śląski*, *Warszawa – Kraków – Katowice*, 1985.

[20] P. Sinopoulos, Functional equations on semigroups, *Aequationes Math.* **59** (2000), 255–261.

[21] H. Stetkær, D'Alembert's equation and spherical functions, *Aequationes Math.* **48** (1994), 220–227.

[22] H. Stetkær, Functional equation on abelian groups and involutions, *Aequationes Math.* **54** (1997), 144–172.

[23] H. Stetkær, D'Alembert's equation on metabelian groups, *Aequationes Math.* **59** (2000), 306–320.

[24] H. Stetkær, D'Alembert's and Wilson's functional equation on step 2 nilpotent groups, *Aequationes Math.* **67** (2004), 241–262.

[25] H. Stetkær, On operator-valued spherical functions, *J. Funct. Anal.* **224** (2005), 338–351.

[26] H. Stetkær, Functional Equations on Groups, *World Scientific Publishing Company*, *Singapore*, 2013.

[27] L. Székelyhidi, Almost periodicity and functional equations, *Aequationes Math.* **26** (1983), 163–175.

[28] V. V. Vasil'ev and S. I. Piskarev, Differential equations in Banach spaces II. Theory of cosine operator functions, *J. Math. Sci.* **122** (2004), 3055–3174.

[29] D. Yang, Factorization of cosine functions on compact connected groups, *Math. Z.* **254** (2006), 655–674.

ANNA BAHYRYCZ
FACULTY OF APPLIED MATHEMATICS
AGH UNIVERSITY OF SCIENCE
AND TECHNOLOGY
MICKIEWICZA 30
30-059 KRAKÓW
POLAND

E-mail: bahyrycza@gmail.com

JANUSZ BRZDĘK
DEPARTMENT OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
PODCHORĄŻYCH 2
30-084 KRAKÓW
POLAND

E-mail: jbrzdek@up.krakow.pl

ELIZA JABŁOŃSKA
DEPARTMENT OF DISCRETE MATHEMATICS
RZESZÓW UNIVERSITY OF TECHNOLOGY
POWSTAŃCÓW WARSZAWY 12
35-959 RZESZÓW
POLAND

E-mail: elizapie@prz.edu.pl

(Received December 30, 2015; revised April 12, 2016)