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Lipschitzian solutions to linear iterative equations

By KAROL BARON (Katowice) and JANUSZ MORAWIEC (Katowice)

Dedicated to Professor Zsolt Páles on his 60th birthday

Abstract. We study the problems of the existence, uniqueness and continuous dependence of Lipschitzian solutions φ of equations of the form

$$\varphi(x)=\int_\Omega g(\omega)\varphi\big(f(x,\omega)\big)\mu(d\omega)+F(x),$$

where μ is a measure on a σ-algebra of subsets of Ω.

1. Introduction

Fix a measure space $(\Omega,\mathcal{A},\mu)$ and a separable metric space (X,ρ).

Motivated by the appearance of the equation

$$\varphi(x)=\int_{A_1}\varphi\big(f(x,\omega)\big)\mu(d\omega)+c-\int_{A_2}\varphi\big(f(x,\omega)\big)\mu(d\omega)$$

with disjoint $A_1,A_2\in\mathcal{A}$ in the theory of perpetuities and of refinement equations, see section 3.4 of the survey paper [3], we consider problems of the existence, uniqueness and continuous dependence of Lipschitzian solutions φ to the equation

$$\varphi(x)=\int_\Omega g(\omega)\varphi\big(f(x,\omega)\big)\mu(d\omega)+F(x). \tag{1}$$

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Concerning the given functions f, g and F, we assume the following hypotheses in which $\mathcal{B}$ stands for the σ-algebra of all Borel subsets of X and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(H_1) Function f maps $X \times \Omega$ into X and for every $x \in X$ the function $f(x, \cdot)$ is $\mathcal{A}$-measurable, i.e.,

$$\big\{\omega \in \Omega : f(x, \omega) \in B\big\} \in \mathcal{A} \quad \text{for all } x \in X \text{ and } B \in \mathcal{B}.$$

(H_2) Function $g\colon \Omega \to \mathbb{K}$ is integrable,

$$\int_\Omega |g(\omega)| \rho\big(f(x, \omega), x\big) \mu(d\omega) < \infty \quad \text{for every } x \in X,$$

and

$$\int_\Omega |g(\omega)| \rho\big(f(x, \omega), f(z, \omega)\big) \mu(d\omega) \le \lambda \rho(x, z) \quad \text{for all } x, z \in X \tag{2}$$

with a $\lambda \in [0, 1)$.

(H_3) Function F maps X into a separable Banach space Y over $\mathbb{K}$ and

$$\|F(x) - F(z)\| \le L\rho(x, z) \quad \text{for all } x, z \in X \tag{3}$$

with an $L \in [0, +\infty)$.

As emphasized in [4, section 0.3] iteration is the fundamental technique for solving functional equations in a single variable, and iterates usually appear in the formulae for solutions. However, as it seems, Lipschitzian solutions are examined rather by the fixed-point method (cf. [4, section 7.2D]). We iterate the operator which transforms a Lipschitzian $F\colon X \to Y$ into $\int_\Omega g(\omega) F\big(f(x, \omega)\big) \mu(d\omega)$; cf. formulas (6) and (8) below. The special case where $g(\omega) = -1$ for every $\omega \in \Omega$ and $\mu(\Omega) = 1$ was examined in [2] on a base of iteration of random-valued functions.

Integrating vector functions we use the Bochner integral.

2. Existence and uniqueness

Putting

$$\gamma = \int_\Omega g(\omega) \mu(d\omega), \tag{4}$$

we start with two simple lemmas.

Lemma 2.1. *Assume* (H_1) *and let* $g\colon \Omega \to \mathbb{K}$ *be integrable with* $\gamma \neq 1$. *If* (2) *holds with a* $\lambda \in [0, 1)$, *then, for any* F *mapping* X *into a normed space* Y *over* $\mathbb{K}$, *equation* (1) *has at most one Lipschitzian solution* $\varphi\colon X \to Y$.

PROOF. Fix a function F mapping X into a normed space Y over $\mathbb{K}$, let $\varphi_1, \varphi_2 \colon X \to Y$ be Lipschitzian solutions of (1), and put $\varphi = \varphi_1 - \varphi_2$. Then φ is a Lipschitzian solution of (1) with $F = 0$, and denoting by L_φ the smallest Lipschitz constant for φ, by (2) for all $x, z \in X$, we have

$$\|\varphi(x) - \varphi(z)\| \le \int_\Omega |g(\omega)| \big\|\varphi\big(f(x,\omega)\big) - \varphi\big(f(z,\omega)\big)\big\| \mu(d\omega) \le L_\varphi \lambda \rho(x,z),$$

whence $L_\varphi = 0$ and φ is a constant function. Since γ defined by (4) is different from 1, the only constant solution of (1) with $F = 0$ is the zero function. $\square$

Lemma 2.2. *Under the assumptions* (H$_1$)–(H$_3$)*, for every $x \in X$ the function*

$$\omega \mapsto g(\omega) F\big(f(x,\omega)\big), \quad \omega \in \Omega,$$

is Bochner integrable, and

$$\left\| \int_\Omega g(\omega) F\big(f(x,\omega)\big) \mu(d\omega) - \int_\Omega g(\omega) F\big(f(z,\omega)\big) \mu(d\omega) \right\| \le L \lambda \rho(x,z) \tag{5}$$

for all $x, z \in X$.

PROOF. The function considered is $\mathcal{A}$-measurable, for every $\omega \in \Omega$ we have

$$\big\|g(\omega) F\big(f(x,\omega)\big)\big\| \le L|g(\omega)| \rho\big(f(x,\omega), x\big) + L|g(\omega)| \|F(x)\|,$$

and (5) holds for all $x, z \in X$. $\square$

Assuming (H$_1$)–(H$_3$) and applying Lemma 2.2, we define

$$F_0(x) = F(x), \quad F_n(x) = \int_\Omega g(\omega) F_{n-1}\big(f(x,\omega)\big) \mu(d\omega) \tag{6}$$

for all $x \in X$ and $n \in \mathbb{N}$, and we see that

$$\|F_n(x) - F_n(z)\| \le L \lambda^n \rho(x,z) \quad \text{for all } x, z \in X \text{ and } n \in \mathbb{N}. \tag{7}$$

Our main result reads.

Theorem 2.3. *Assume* (H$_1$)–(H$_3$)*. If $\gamma \ne 1$, then equation* (1) *has exactly one Lipschitzian solution $\varphi \colon X \to Y$; it is given by the formula*

$$\varphi(x) = \frac{1}{1-\gamma} \left(\sum_{n=1}^{\infty} \big(F_n(x) - \gamma F_{n-1}(x)\big) + F(x) \right) \quad \textit{for every } x \in X, \tag{8}$$

$$\|\varphi(x) - \varphi(z)\| \le \frac{L(1+|\gamma|)}{|1-\gamma|(1-\lambda)} \rho(x,z) \quad \textit{for all } x, z \in X, \tag{9}$$

and

$$\|\varphi(x)\| \le \frac{1}{|1-\gamma|} \left(\frac{L}{1-\lambda} \int_\Omega |g(\omega)| \rho\big(f(x,\omega), x\big) \mu(d\omega) + \|F(x)\| \right) \tag{10}$$

for every $x \in X$.

Proof. For the proof of the existence, observe first that by (4), (6) and (7) for all $x \in X$ and $n \in \mathbb{N}$ we have

$$\|F_n(x) - \gamma F_{n-1}(x)\| = \left\| \int_\Omega g(\omega)F_{n-1}\big(f(x,\omega)\big)\mu(d\omega) - \int_\Omega g(\omega)F_{n-1}(x)\mu(d\omega) \right\|$$
$$\leq L\lambda^{n-1} \int_\Omega |g(\omega)|\rho\big(f(x,\omega),x\big)\mu(d\omega). \tag{11}$$

Consequently, (8) defines a function $\varphi\colon X \to Y$. Routine calculations, (8), (7), (2) and (11) show that this function satisfies (9) and (10).

It remains to prove that φ solves (1). To this end, define $M\colon X \to [0,\infty)$ by

$$M(x) = L \int_\Omega |g(\omega)|\rho\big(f(x,\omega),x\big)\mu(d\omega) \tag{12}$$

and fix $x_0 \in X$. An obvious application of (12), (H_2), (10) and (3) gives

$$M(x) \leq c_1\rho(x,x_0) + c_2, \quad \|\varphi(x)\| \leq c_1\rho(x,x_0) + c_2 \quad \text{for every } x \in X \tag{13}$$

with some constants $c_1, c_2 \in [0,\infty)$.

Fix $x \in X$. According to Lemma 2.2, the function

$$\omega \longmapsto g(\omega)\varphi\big(f(x,\omega)\big), \quad \omega \in \Omega,$$

is Bochner integrable. Moreover, by (11)–(13),

$$\begin{aligned}
&\Big\| g(\omega)\Big(F_n\big(f(x,\omega)\big) - \gamma F_{n-1}\big(f(x,\omega)\big)\Big)\Big\| \\
&\quad \leq \lambda^{n-1}|g(\omega)|M\big(f(x,\omega)\big) \leq \lambda^{n-1}|g(\omega)|\big(c_1\rho(f(x,\omega),x_0) + c_2\big) \\
&\quad \leq \lambda^{n-1}|g(\omega)|\big(c_1\rho(f(x,\omega),x) + c_1\rho(x,x_0) + c_2\big)
\end{aligned}$$

for all $n \in \mathbb{N}$ and $\omega \in \Omega$. Hence, making use of (H_2), the dominated convergence theorem and (6), we see that

$$\begin{aligned}
&\int_\Omega \sum_{n=1}^\infty g(\omega)\Big(F_n\big(f(x,\omega)\big) - \gamma F_{n-1}\big(f(x,\omega)\big)\Big)\mu(d\omega) \\
&\quad = \sum_{n=1}^\infty \int_\Omega g(\omega)\Big(F_n\big(f(x,\omega)\big) - \gamma F_{n-1}\big(f(x,\omega)\big)\Big)\mu(d\omega) \\
&\quad = \sum_{n=1}^\infty \big(F_{n+1}(x) - \gamma F_n(x)\big).
\end{aligned} \tag{14}$$

Applying now (8), (14) and (6), we get

$$\begin{aligned}
&\int_\Omega g(\omega)\varphi\big(f(x,\omega)\big)\mu(d\omega)\\
&\quad=\frac{1}{1-\gamma}\int_\Omega\left[\sum_{n=1}^{\infty}g(\omega)\Big(F_n\big(f(x,\omega)\big)-\gamma F_{n-1}\big(f(x,\omega)\big)\Big)+g(\omega)F\big(f(x,\omega)\big)\right]\mu(d\omega)\\
&\quad=\frac{1}{1-\gamma}\left[\sum_{n=1}^{\infty}\Big(F_{n+1}(x)-\gamma F_n(x)\Big)+F_1(x)\right]\\
&\quad=\frac{1}{1-\gamma}\left[\sum_{n=1}^{\infty}\Big(F_n(x)-\gamma F_{n-1}(x)\Big)+\gamma F(x)\right]=\varphi(x)-F(x).
\end{aligned}$$

The proof is complete. □

3. Examples

Example 3.1. Given $\lambda\in(0,1)$ and an integrable $\xi\colon\Omega\to\mathbb{R}$, consider the equation

$$\varphi(x)=\lambda^2\int_\Omega\varphi\left(\frac{1}{\lambda}x+\xi(\omega)\right)\mu(d\omega)$$

with $\mu(\Omega)=1$. According to Lemma 2.1, the zero function is its only Lipschitzian solution $\varphi\colon\mathbb{R}\to\mathbb{R}$. Note, however, that if

$$\int_\Omega\xi(\omega)\mu(d\omega)=0\qquad\text{and}\qquad\int_\Omega\xi(\omega)^2\mu(d\omega)<\infty,$$

then this equation solves also the function $\varphi\colon\mathbb{R}\to\mathbb{R}$ given by

$$\varphi(x)=x^2+\frac{\lambda^2}{1-\lambda^2}\int_\Omega\xi(\omega)^2\mu(d\omega).$$

Example 3.2. Given $\lambda\in(0,1)$, consider the equation

$$\varphi(x)=2\varphi\big(\lambda\sqrt{x}+1-\lambda\big)+\log\frac{x}{(\lambda\sqrt{x}+1-\lambda)^2}.$$

According to Lemma 2.1 (in this case $f(x,\omega)=\lambda\sqrt{x}+1-\lambda$, $g(\omega)=2$ and $F(x)=\log\frac{x}{(\lambda\sqrt{x}+1-\lambda)^2}$ for all $x\in[1,\infty)$ and $\omega\in\Omega$, $\mu(\Omega)=1$), the logarithmic function restricted to $[1,\infty)$ is the only Lipschitzian solution $\varphi\colon[1,\infty)\to\mathbb{R}$ to this equation, and it is unbounded in spite of the fact that F is bounded.

Example 3.3. To see that assumptions (H_1)–(H_3) do not guarantee the existence of a *continuous* solution $\varphi\colon X \to Y$ to equation (1), given $\alpha \in (-1,1)$, a bounded and $\mathcal{A}$-measurable $\xi\colon \Omega \to \mathbb{R}$, and a Lipschitzian $F\colon \mathbb{R} \to [0,\infty)$ such that $F^{-1}(\{0\})$ is a singleton, consider the equation

$$\varphi(x) = \int_\Omega \varphi\big(\alpha x + \xi(\omega)\big)\mu(d\omega) + F(x) \tag{15}$$

with $\mu(\Omega) = 1$. Assume a continuous $\varphi\colon \mathbb{R} \to \mathbb{R}$ solves it. We shall see that then ξ is a.e. constant. To this end, fix an $M \in (0,\infty)$ such that $|\xi(\omega)| \le M$ for every $\omega \in \Omega$, and a real number $a \ge \frac{M}{1-|\alpha|}$ such that $F^{-1}(\{0\}) \subset [-a,a]$. Then

$$|\alpha x + \xi(\omega)| \le a \quad \text{for all } x \in [-a,a] \text{ and } \omega \in \Omega,$$

and so $\varphi|_{[-a,a]}$ is a continuous, hence also bounded, solution of (15). According to [1, Corollary 4.1(ii) and Example 4.1], it is possible only if ξ is a.e. constant.

4. Continuous dependence

Given a normed space $(Y, \|\cdot\|)$, consider now the linear space $\mathrm{Lip}(X,Y)$ of all Lipschitzian functions mapping X into Y, and its linear subspace $\mathrm{BL}(X,Y)$ of all Lipschitzian and bounded functions mapping X into Y. Fix $x_0 \in X$ and define $\|\cdot\|_{\mathrm{Lip}}\colon \mathrm{Lip}(X,Y) \to [0,\infty)$ by

$$\|u\|_{\mathrm{Lip}} = \|u(x_0)\| + \|u\|_L,$$

where $\|u\|_L$ stands for the smallest Lipschitz constant for u. Clearly, $\|\cdot\|_{\mathrm{Lip}}$ is a norm in $\mathrm{Lip}(X,Y)$. It depends on the given point x_0, but for different points such norms are equivalent. It is well known that if $(Y, \|\cdot\|)$ is Banach, then so is $(\mathrm{Lip}(X,Y), \|\cdot\|_{\mathrm{Lip}})$. In the linear space $\mathrm{BL}(X,Y)$ we consider the norm $\|\cdot\|_{\mathrm{BL}}$ given by

$$\|u\|_{\mathrm{BL}} = \sup\big\{\|u(x)\| : x \in X\big\} + \|u\|_L.$$

It is also well known that if $(Y, \|\cdot\|)$ is Banach, then so is $(\mathrm{BL}(X,Y), \|\cdot\|_{\mathrm{BL}})$.

Assume (H_1) and (H_2), $\gamma \ne 1$, and let Y be a separable Banach space over $\mathbb{K}$.

According to Theorem 2.3, for every $F \in \mathrm{Lip}(X,Y)$ the formula

$$\varphi^F(x) = \frac{1}{1-\gamma}\left(\sum_{n=1}^{\infty}\Big(F_n(x) - \gamma F_{n-1}(x)\Big) + F(x)\right) \quad \text{for every } x \in X \tag{16}$$

defines the only Lipschitzian solution φ^F of equation (1),

$$\|\varphi^F\|_L \leq \frac{1+|\gamma|}{|1-\gamma|(1-\lambda)}\|F\|_L \tag{17}$$

and

$$\|\varphi^F(x)\| \leq \frac{1}{|1-\gamma|}\left(\frac{\|F\|_L}{1-\lambda}\int_\Omega |g(\omega)|\rho\big(f(x,\omega),x\big)\mu(d\omega)+\|F(x)\|\right) \tag{18}$$

for every $x \in X$. Putting

$$c_0=\frac{1}{1-\lambda}\left(\int_\Omega |g(\omega)|\rho\big(f(x_0,\omega),x_0\big)\mu(d\omega)+1+|\gamma|\right), \quad c=\max\{1,c_0\}, \tag{19}$$

and applying (17) and (18), we see that if $F \in \mathrm{Lip}(X,Y)$, then

$$\begin{aligned}\|\varphi^F\|_{\mathrm{Lip}} &= \|\varphi^F(x_0)\|+\|\varphi^F\|_L \leq \frac{1}{|1-\gamma|}\big(c_0\|F\|_L+\|F(x_0)\|\big)\\ &\leq \frac{c}{|1-\gamma|}\|F\|_{\mathrm{Lip}}.\end{aligned}$$

Moreover, if d_0 defined by

$$d_0=\sup\left\{\int_\Omega |g(\omega)|\rho\big(f(x,\omega),x\big)\mu(d\omega) : x \in X\right\} \tag{20}$$

is finite, then putting

$$d=\max\left\{1,\frac{d_0+1+|\gamma|}{1-\lambda}\right\} \tag{21}$$

and applying (18) and (17), again we see also that if $F \in \mathrm{BL}(X,Y)$, then $\varphi^F \in \mathrm{BL}(X,Y)$ as well, and

$$\|\varphi^F\|_{\mathrm{BL}} \leq \frac{1}{|1-\gamma|}\left(\frac{d_0+1+|\gamma|}{1-\lambda}\|F\|_L+\sup\big\{\|F(x)\| : x \in X\big\}\right) \leq \frac{d}{|1-\gamma|}\|F\|_{\mathrm{BL}}.$$

Theorem 4.1. *Assume* (H_1), (H_2), *and let* γ *defined by* (4) *be different from* 1. *If* Y *is a separable Banach space over* $\mathbb{K}$, *then:*

(i) *for any* $F \in \mathrm{Lip}(X,Y)$, *the function* $\varphi^F : X \to Y$ *defined by* (16) *and* (6) *is the only Lipschitzian solution of* (1), *the operator*

$$F \mapsto \varphi^F, \quad F \in \mathrm{Lip}(X,Y), \tag{22}$$

is a linear homeomorphism of $(\mathrm{Lip}(X,Y), \|\cdot\|_{\mathrm{Lip}})$ *onto itself, and*

$$\|\varphi^F\|_{\mathrm{Lip}} \le \frac{c}{|1-\gamma|}\|F\|_{\mathrm{Lip}} \qquad \text{for every } F \in \mathrm{Lip}(X,Y),$$

with c *given by* (19)*;*

(ii) *if, additionally,* d_0 *defined by* (20) *is finite, then the restriction of the operator* (22) *to* $\mathrm{BL}(X,Y)$ *is a linear homeomorphism of* $(\mathrm{BL}(X,Y), \|\cdot\|_{\mathrm{BL}})$ *onto itself, and*

$$\|\varphi^F\|_{\mathrm{BL}} \le \frac{d}{|1-\gamma|}\|F\|_{\mathrm{BL}} \qquad \text{for every } F \in \mathrm{BL}(X,Y),$$

with d *given by* (21)*.*

Proof. By the above considerations and the Banach inverse mapping theorem, it remains to show that operator (22) is one-to-one, maps $\mathrm{Lip}(X,Y)$ onto $\mathrm{Lip}(X,Y)$, and $\mathrm{BL}(X,Y)$ onto $\mathrm{BL}(X,Y)$.

The first property follows from the fact that for any $F \in \mathrm{Lip}(X,Y)$ the function φ^F is a solution of (1): if $\varphi^F = 0$, then $F = 0$. To get the next two, observe that if $\psi \in \mathrm{Lip}(X,Y)$, then, by Lemma 2.2, the function $F\colon X \to Y$ given by

$$F(x) = \psi(x) - \int_\Omega g(\omega)\psi(f(x,\omega))\mu(d\omega)$$

belongs to $\mathrm{Lip}(X,Y)$, if ψ is also bounded, then so is F, and, since both ψ and φ^F solve (1), $\psi = \varphi^F$ by Lemma 2.1. $\square$

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KAROL BARON
INSTYTUT MATEMATYKI
UNIWERSYTET ŚLĄSKI
BANKOWA 14
PL-40-007 KATOWICE
POLAND

E-mail: baron@us.edu.pl

JANUSZ MORAWIEC
INSTYTUT MATEMATYKI
UNIWERSYTET ŚLĄSKI
BANKOWA 14
PL-40-007 KATOWICE
POLAND

E-mail: morawiec@math.us.edu.pl

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