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A regularity condition for quadratic functions involving the unit circle

By ZOLTÁN BOROS (Debrecen), WŁODZIMIERZ FECHNER (Łódz)
and PÉTER KUTAS (Budapest)

Dedicated to Professor Zsolt Páles on the occassion of his 60^{th} birthday

Abstract. We prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is additive or quadratic and the mapping $\mathbb{R}^2 \ni (x, y) \to f(x)f(y)$ is bounded on a non-degenerated arc of the unit circle, then f is continuous, i.e. $f(x) = cx$ for all $x \in \mathbb{R}$ or $f(x) = cx^2$ for all $x \in \mathbb{R}$, respectively, with some real coefficient c.

1. Introduction

Let $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the sets of all real numbers, rationals, and positive integers, respectively. Further, by S we denote the unit circle on the plane:

$$S = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}.$$

We call a function $f\colon \mathbb{R} \to \mathbb{R}$ *additive* if

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. The function f is called $\mathbb{Q}$-*homogeneous* if the equation $f(qx) = qf(x)$ is fulfilled by every $q \in \mathbb{Q}$ and $x \in \mathbb{R}$. As it is well-known (see M. Kuczma

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[11, Theorem 5.2.1]), if $f:\mathbb{R}\to\mathbb{R}$ is additive, then f is $\mathbb{Q}$-homogeneous as well. For more information concerning these notions, the reader is referred to the monograph [11].

A function $f\colon\mathbb{R}\to\mathbb{R}$ is called *quadratic* if it satisfies the Jordan-von Neumann functional equation:

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for every $x,y\in\mathbb{R}$. In what follows, we will apply the fact that f is quadratic if and only if there exists a bi-additive and symmetric functional $B\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that $f(x)=B(x,x)$ for $x\in\mathbb{R}$ (see e.g. J. Aczél, J. Dhombres [2, Chapter 11, Proposition 1]). Quadratic functions are also called generalized monomials of order 2. Further, additive functions are generalized monomials of order 1 and non-zero constants are generalized monomials of order 0. Generalized polynomials are defined as sums of generalized monomials of respective degrees. For more facts on generalized polynomials, the reader is referred to [11, Chapter 15.9] and L. Székelyhidi [17].

Among several problems in the theory of functional equations, J. Aczél [1] listed the following problem of I. Halperin: is every additive mapping $f\colon\mathbb{R}\to\mathbb{R}$, which satisfies

$$f\left(\frac{1}{x}\right)=\frac{1}{x^2}f(x)$$

for all $x\neq 0$, of the form $f(x)=f(1)x$ for all $x\in\mathbb{R}$ (i.e. linear)? Two independent affirmative answers to Halperin's question are due to S. Kurepa [12] and W. B. Jurkat [7]. These results were extended in various directions by several authors (see A. Grząślewicz [6], Pl. Kannappan and S. Kurepa [8], [9], S. Kurepa [13], c.f. [11, Theorem 14.3.3] and A. Nishiyama and S. Horinouchi [15]).

During the 27th International Symposium on Functional Equations, W. Benz formulated the following problem [3]. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is an additive function satisfying $yf(x)=xf(y)$ for every $(x,y)\in S$. Does it imply that f is linear? This question, together with a similar one for derivations, was answered in the affirmative by Z. Boros and P. Erdei [4].

Motivated by the question of W. Benz [3], Gy. Szabó [16] posed the following problem: Suppose that $f\colon\mathbb{R}\to\mathbb{R}$ is additive and $f(x)f(y)=0$ for all $(x,y)\in S$. Does it imply that f is identically equal to zero? The solution was published in the paper by Z. Kominek, L. Reich and J. Schwaiger [10], where the authors proved that the implication is true. Z. Boros and W. Fechner [5] extended this result to the case when f is a generalized polynomial function.

In this paper we investigate stability of the condition $f(x)f(y) = 0$ on the unit circle by replacing it with the assumption that $f(x)f(y)$ is bounded. Moreover, we require only the boundedness on an arc of S of positive length. Clearly, the assertion that $f = 0$ will be no longer true under these assumptions. Instead, we will prove the continuity of f in case f is additive or quadratic.

2. Conditions involving the unit circle

We will start with a technical lemma in which we do not need any functional equation to be satisfied by the investigated mapping.

Lemma 1. *Assume that $I \subset [0,1]$ is a non-degenerated interval, I_0 is a non-void open subinterval such that $\mathrm{cl}(I_0) \subset \mathrm{int}(I)$ and $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that:*

(a) *f is unbounded on I_0,*

(b) *the mapping*

$$I \ni x \to \Phi(x) := f(x)f(\sqrt{1-x^2}) \tag{1}$$

is bounded.

Further, let

$$\alpha_k := \frac{k^2-1}{k^2+1}, \quad \beta_k := \frac{2k}{k^2+1} \quad (k \in \mathbb{N}).$$

Then, there exist a positive integer k_0 and a sequence $(x_n)_{n\in\mathbb{N}} \subset I_0$ such that with $y_n := \sqrt{1-x_n^2}$ $(n \in \mathbb{N})$ for every integer $k \geq k_0$ we have:

(i) *$|f(x_n)| > n$ for all $n \in \mathbb{N}$,*

(ii) *$\lim\limits_{n\to\infty} f(y_n) = 0$,*

(iii) *the sequence $(f(\alpha_k x_n - \beta_k y_n) \cdot f(\beta_k x_n + \alpha_k y_n))_{n\in\mathbb{N}}$ is bounded.*

Proof. By assumption (a), there exists a sequence (x_n) in I_0 such that

$$|f(x_n)| > n \quad \text{for all} \quad n \in \mathbb{N}.$$

Let $y_n = \sqrt{1-x_n^2}$ for $n \in \mathbb{N}$. Then, due to assumption (b), the sequence $(f(x_n)f(y_n))_{n\in\mathbb{N}}$ is bounded, while $\lim\limits_{n\to\infty} |f(x_n)| = +\infty$, hence $\lim\limits_{n\to\infty} f(y_n) = 0$. Therefore, (i) and (ii) are proven.

It remains to prove (iii). Observe that $\alpha_k, \beta_k \in \mathbb{Q}$ and $\alpha_k^2 + \beta_k^2 = 1$ for all $k \in \mathbb{N}$. Moreover, we have $\lim\limits_{k\to\infty} \alpha_k = 1$ and $\lim\limits_{k\to\infty} \beta_k = 0$. Hence, there exists some $k_0 \in \mathbb{N}$ such that for all $k > k_0$ and all $x \in I_0$ with $y := \sqrt{1-x^2}$, we have $\alpha_k x - \beta_k y \in I$. Moreover, observe that

$$\beta_k x + \alpha_k y = \sqrt{1 - (\alpha_k x - \beta_k y)^2}.$$

Therefore, we have in particular that $\alpha_k x_n - \beta_k y_n \in I$ for all $n \in \mathbb{N}$. Consequently, the sequence

$$\big(f(\alpha_k x_n - \beta_k y_n) f(\beta_k x_n + \alpha_k y_n)\big)_{n\in\mathbb{N}}$$

is bounded. □

Now, we are ready to prove our first main result.

Theorem 1. *Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is additive. If there exists a non-degenerated interval $I \subset [0,1]$ such that the mapping (1) is bounded, then f is continuous, i.e. there exists some real coefficient c such that $f(x) = cx$ for $x \in \mathbb{R}$.*

Proof. For the contrary, suppose that f is discontinuous. Therefore, it must be unbounded on every non-void open subinterval I_0 (it follows, for example, from a general result of L. Székelyhidi [17, Theorem 4.3]). Therefore, assumptions (a) and (b) of Lemma 1 are satisfied. We will reach a contradiction using assertions (i), (ii) and (iii) of this lemma.

Note that f is $\mathbb{Q}$-linear, so, following notations of Lemma 1, we have

$$\begin{aligned} &f(\alpha_k x_n - \beta_k y_n) f(\beta_k x_n + \alpha_k y_n) \\ &\quad = (\alpha_k f(x_n) - \beta_k f(y_n))(\beta_k f(x_n) + \alpha_k f(y_n)) \\ &\quad = \alpha_k\beta_k f(x_n)^2 + \left(\alpha_k^2 - \beta_k^2\right) f(x_n) f(y_n) - \alpha_k \beta_k f(y_n)^2. \end{aligned}$$

Here the first term tends to $+\infty$ as n tends to $+\infty$, the second term is bounded, the third term tends to 0. Consequently, the sum tends to $+\infty$, contrary to Lemma 1 (iii). □

Remark 1. Let us note that the boundedness of the mapping (1) cannot be replaced by its boundedness from one direction (above or below) in Theorem 1. As a counterexample, let us consider a not identically zero derivation $d\colon \mathbb{R} \to \mathbb{R}$. This means that d is a discontinuous additive function that fulfils the additional equation $d(x^2) = 2xd(x)$ for all $x \in \mathbb{R}$. The existence of such a function is established, for instance, in [11, Theorem 14.2.2]. In particular, $d(1) = 2d(1)$, thus $d(1) = 0$. If I denotes the open unit interval, $x \in I$ implies $x > 0$, $y = \sqrt{1-x^2} > 0$,

$$0 = d(1) = d(x^2 + y^2) = d(x^2) + d(y^2) = 2xd(x) + 2yd(y),$$

and thus

$$d(x)d(y) = d(x)\left(-\frac{x}{y}d(x)\right) = -\frac{x}{y}\,(d(x))^2 \le 0.$$

If we replace I by $-I$ (i.e., we change the sign of x but we keep the formula and the sign of y), we obtain the reversed inequality.

In our next theorem we are going to prove an analogous statement for quadratic functions. The proof is, however, more complicated. We will need the following observation concerning two countable almost disjoint families of natural numbers.

Lemma 2. *Assume that we are given two families* $\{N_k\}_{k\in\mathbb{N}}$ *and* $\{M_k\}_{k\in\mathbb{N}}$ *of subsets of natural numbers such that for every two distinct positive integers* k, l*, the sets* $N_k \cap N_l$ *and* $M_k \cap M_l$ *are finite. Then the sets*

$$\mathbb{N} \setminus (N_k \cup M_k)$$

are infinite for all but at most two numbers $k \in \mathbb{N}$*.*

Proof. Let us suppose that $k, l \in \mathbb{N}$ are such that $k \neq l$, and the sets

$$\mathbb{N} \setminus (N_k \cup M_k) \quad \text{and} \quad \mathbb{N} \setminus (N_l \cup M_l)$$

are finite. Then the union

$$\big(\mathbb{N} \setminus (N_k \cup M_k)\big) \cup \big(\mathbb{N} \setminus (N_l \cup M_l)\big)$$

is also finite, so its complement

$$(N_k \cup M_k) \cap (N_l \cup M_l) = (N_k \cap N_l) \cup (M_k \cap N_l) \cup (N_k \cap M_l) \cup (M_k \cap M_l)$$

is co-finite in $\mathbb{N}$ (i.e., it contains all but finitely many positive integers). Since $N_k \cap N_l$ and $M_k \cap M_l$ are finite, the set

$$R_{k,l} := (M_k \cap N_l) \cup (N_k \cap M_l)$$

is also co-finite in $\mathbb{N}$.

Let $j \in \mathbb{N} \setminus \{k, l\}$ and observe that

$$N_j \cap R_{k,l} \subseteq (N_j \cap N_l) \cup (N_j \cap N_k).$$

According to our assumptions, the sets $N_j \cap N_l$ and $N_j \cap N_k$ are finite, hence $N_j \cap R_{k,l}$ is finite. Since the set $R_{k,l}$ is co-finite in $\mathbb{N}$, we get that N_j is finite. An analogous argument shows that M_j is finite as well. Therefore, the set $\mathbb{N}\setminus(N_j \cup M_j)$ is co-finite in $\mathbb{N}$. In particular, it is infinite. $\square$

Theorem 2. *Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is quadratic. If there exists a non-degenerated interval $I \subset [0,1]$ such that the mapping* (1) *is bounded, then f is continuous, i.e. there exists some real coefficient c such that $f(x) = cx^2$ for $x \in \mathbb{R}$.*

Proof. We will begin as in the proof of the previous theorem, by supposing that f is discontinuous. From [17, Theorem 4.3] we get that f is unbounded on every non-void open subinterval I_0. Therefore, assumptions of Lemma 1 are satisfied, and we will follow the notations from this lemma.

Since f is quadratic, then there exists a symmetric bi-additive (and thus bilinear over $\mathbb{Q}$) mapping $B\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = B(x,x) \quad \text{for all} \quad x \in \mathbb{R}.$$

So the product, which is bounded by Lemma 1 (iii), can be rewritten as

$$f(\alpha_k x_n - \beta_k y_n) f(\beta_k x_n + \alpha_k y_n) = \Big(\alpha_k^2 f(x_n) - 2\alpha_k\beta_k B(x_n,y_n) + \beta_k^2 f(y_n)\Big) \times \Big(\beta_k^2 f(x_n) + 2\alpha_k\beta_k B(x_n,y_n) + \alpha_k^2 f(y_n)\Big).$$

By Lemma 1 (ii), the last terms in both factors tend to 0 as n tends to $+\infty$.

We will prove that there exist some $k \in \mathbb{N}$ and a strictly increasing sequence $(p_n)_{n\in\mathbb{N}}$ of positive integers such that

$$\lim_{n\to\infty} |\alpha_k f(x_{p_n}) - 2\beta_k B\,(x_{p_n}, y_{p_n})| = +\infty$$

and

$$\lim_{n\to\infty} |\beta_k f(x_{p_n}) + 2\alpha_k B\,(x_{p_n}, y_{p_n})| = +\infty.$$

Clearly, as soon as this two equalities are proved, we reach a contradiction.

For $k \in \mathbb{N}$ let us consider the sets

$$N_k = \{n \in \mathbb{N}\colon\ |\alpha_k f(x_n) - 2\beta_k B\,(x_n, y_n)| \le \sqrt{n}\}$$

and

$$M_k = \{n \in \mathbb{N}\colon\ |\beta_k f(x_n) + 2\alpha_k B\,(x_n, y_n)| \le \sqrt{n}\}.$$

We will show that the families $\{M_k\}_{k\in\mathbb{N}}$ and $\{N_k\}_{k\in\mathbb{N}}$ satisfy the assumptions of Lemma 2. Fix two positive integers k, l such that $k \ne l$, and assume that $n \in N_k \cap N_l$. Then

$$|\alpha_k f(x_n) - 2\beta_k B(x_n,y_n)| \le \sqrt{n}, \qquad |-\alpha_l f(x_n) + 2\beta_l B(x_n,y_n)| \le \sqrt{n}.$$

This implies that:

$$|\alpha_k\beta_l f(x_n)-2\beta_k\beta_l B(x_n,y_n)| \le \sqrt{n}\beta_l, \quad |-\alpha_l\beta_k f(x_n)+2\beta_k\beta_l B(x_n,y_n)| \le \sqrt{n}\beta_k.$$

Let us add the two inequalities side-by-side to get

$$|\alpha_k\beta_l - \alpha_l\beta_k||f(x_n)| \le (\beta_l + \beta_k)\sqrt{n}.$$

Let us join this inequality with Lemma 1 (i) and with the definitions of the α's and the β's to get the following estimate:

$$\sqrt{n} < \frac{|f(x_n)|}{\sqrt{n}} \le \frac{\beta_l + \beta_k}{|\alpha_k\beta_l - \alpha_l\beta_k|} = \frac{k+l}{|k-l|}.$$

It is clear, since the right-hand-side does not depend upon n, that this inequality can be satisfied by at most finite number of positive integers n.

Now, fix $n \in M_k \cap M_l$. We have

$$|\beta_k f(x_n) + 2\alpha_k B(x_n,y_n)| \le \sqrt{n}, \quad |-\beta_l f(x_n) - 2\alpha_l B(x_n,y_n)| \le \sqrt{n}.$$

Then,

$$|\alpha_l\beta_k f(x_n)+2\alpha_l\alpha_k B(x_n,y_n)| \le \alpha_l\sqrt{n}, \quad |-\alpha_k\beta_l f(x_n)-2\alpha_k\alpha_l B(x_n,y_n)| \le \alpha_k\sqrt{n}.$$

And finally,

$$\sqrt{n} < \frac{|f(x_n)|}{\sqrt{n}} \le \frac{\alpha_k + \alpha_l}{|\alpha_l\beta_k - \alpha_k\beta_l|} = \frac{kl-1}{|k-l|},$$

which proves the finiteness of $M_k \cap M_l$.

Now, we can apply Lemma 2 in order to verify that the set $\mathbb{N} \setminus (N_k \cup M_k)$ is infinite for all k's except of at most two. In particular, we can find such k with the property $k \ge k_0$, where k_0 is postulated by Lemma 1. This ensures the existence of the sequence $(p_n)_{n\in\mathbb{N}}$ with the desired properties. □

Corollary 1. *Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized polynomial of degree at most 2 such that $f(0) = 0$. If the mapping*

$$(x,y) \to \Psi(x,y) = f(x)f(y)$$

is bounded on a set $S_0 \subseteq S$ which contains an arc of positive length and which is symmetric with respect to both co-ordinate axes, then f is continuous, i.e. $f(x) = ax^2 + bx$ for all $x \in \mathbb{R}$ with some real coefficients a, b.

Proof. Due to our assumptions, f can be represented in the form

$$f(x) = A(x) + G(x) \quad (x \in \mathbb{R}),$$

where A is additive (thus odd) and G is quadratic (thus even). Moreover, $(x, y) \in S_0$ implies $(\pm x, \pm y) \in S_0$, hence all four mappings $(x, y) \mapsto f(\pm x)f(\pm y)$ are bounded on S_0. Thus

$$A(x)A(y) = \frac{1}{4}\big(f(x) - f(-x)\big)\big(f(y) - f(-y)\big)$$

and

$$G(x)G(y) = \frac{1}{4}\big(f(x) + f(-x)\big)\big(f(y) + f(-y)\big)$$

are also bounded on S_0. To finish the proof, it is enough to take $I \subset [0, 1]$ as an interval such that for every $x \in I$ the point $(x, \sqrt{1 - x^2})$ belongs to the arc of S_0 of positive length on the first quarter of the plane, and apply Theorem 1 or 2, respectively. □

Remark 2. If in Corollary 1 function f is additive, then it is odd, and clearly, one-sided boundedness of the map Ψ implies its boundedness from both sides on S_0 (due to the symmetry assumptions concerning S_0). However, if f is quadratic, then it is possible that Ψ is bounded from below and f is discontinuous. If $a\colon \mathbb{R} \to \mathbb{R}$ is additive and discontinuous, then $f = a^2$ is a quadratic mapping. Clearly, f and Ψ are nonnegative.

3. Conditions involving hyperbolas

We will terminate the paper with some related problems which arise quite naturally. Another possible way to generalize Kominek *et al.*'s result from [10] would be the following. Let C be a curve in the plane (here by curve we mean the solution set of an irreducible polynomial equation in two variables). Let f be an additive function which satisfies the following: whenever $(x, y) \in C$, $f(x)f(y) = 0$. Does this imply that f is identically zero? To the best of our knowledge, the answer is not even known in the case of conics. However, many special cases are known. For example, the case of the parabola $y = x^2$ is already settled in [10]. The case $x^2 - ny^2 = 1$ is interesting (here n is a positive integer). The case $n = 1$ is settled explicitly in [10]. The case where n is the square of an integer is basically the same. Here we give a short proof for the case when n is square-free.

Theorem 3. *Let f be an additive function and C be the curve determined by the equation $x^2 - ny^2 = 1$ where n is a square-free integer. Assume that $f(x)f(y) = 0$ if $(x, y) \in C$. Then f is identically zero.*

Proof. First let $a, b \in \mathbb{Z}$ be such that $ab \neq 0$ and $a^2 - nb^2 = 1$. Such a, b do exist (actually infinitely many of them), a proof of this fact can be found e.g. in the monograph of L. J. Mordell [14, Chapter 8, Theorem 1]. Now let us consider the following matrix:

$$\begin{pmatrix} a & bn \\ b & a \end{pmatrix}$$

We prove that if $(x, y) \in C$, then if we multiply it by this matrix (from the left), the resulting point $(ax + bny, bx + ay)$ will lie on the curve as well. Indeed:

$$\begin{aligned}(ax + bny)^2 - n(bx + ay)^2 &= a^2x^2 + b^2n^2y^2 - n(b^2x^2 + a^2y^2) \\ &= (a^2 - nb^2)(x^2 - ny^2) = 1 \cdot 1 = 1.\end{aligned}$$

Now, applying Theorem 6 of [10], we get the desired result. □

So we know the answer for a large class of hyperbolas. However, the proof is sensitive to the parametrization of the curve. So, for example, the question for the hyperbola $y = \frac{1}{x}$ is still an open problem.

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ZOLTÁN BOROS
INSTITUTE OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H–4002 DEBRECEN
P.O.B. 400
HUNGARY

E-mail: zboros@science.unideb.hu
URL: http://math.unideb.hu/boros-zoltan/

WŁODZIMIERZ FECHNER
INSTITUTE OF MATHEMATICS
ŁÓDZ UNIVERSITY OF TECHNOLOGY
UL. WÓLCZAŃSKA 215
93-005 ŁÓDZ
POLAND

E-mail: wlodzimierz.fechner@gmail.com

PÉTER KUTAS
DEPARTMENT OF MATHEMATICS
AND ITS APPLICATIONS
CENTRAL EUROPEAN UNIVERSITY
NÁDOR U. 9
H–1051 BUDAPEST
HUNGARY

E-mail: Kutas_Peter@phd.ceu.edu

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