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The dilogarithm function and the Abel functional equation

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Dedicated to the 60th birthday of Professor Zsolt Páles

Abstract. In the literature, mostly the identities, the applications and the special values of the dilogarithm functions are investigated. In this note, we deal with the problem of the connection between a famous identity, namely the so-called Abel equation, and the dilogarithm functions, and show the close connection between the dilogarithm functions and the measurable solutions of the Abel equation.

1. Introduction

The complex dilogarithm function is defined by the power series

$$z \mapsto \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \qquad |z| < 1, z \in \mathbb{C} \text{ (the set of all complex numbers)}$$

and it has a unique analytic continuation to the domain $\mathbb{C} \setminus [1, +\infty[$. A real variant Li_2 of this function is defined by

$$Li_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \qquad x \in [-1, 1].$$

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This function and also its complex variant have appearances in several areas of mathematics. Some of its values can exactly be given (for example, in the real case)

$$Li_2(1) = \frac{\pi^2}{6} \quad \text{and} \quad Li_2(\omega) = \frac{\pi^2}{10} - \ln^2\left(\frac{\sqrt{5}+1}{2}\right) \tag{1.1}$$

where $\omega = \frac{\sqrt{5}-1}{2}$, and Li_2 satisfies several identities. For the details and the historical background, see ZAGIER [15] and LEWIN [9]. One of the numerous variants of the dilogarithm functions is the so-called ROGERS dilogarithm (see [15] and ROGERS [14]), defined on $I =]0,1[$ by

$$L(x) = Li_2(x) + \frac{1}{2}\ln(x)\ln(1-x). \tag{1.2}$$

In the third section, we shall prove that the function $F = L - L(\omega)$ satisfies the so-called Abel functional equation (see [1], DARÓCZY–KIESEWETTER [3])

$$F(u) + F(v) + F(1-uv) + F\left(\frac{1-u}{1-uv}\right) + F\left(\frac{1-v}{1-uv}\right) = 0 \tag{1.3}$$

for all $u, v \in I$, moreover, there we characterize it.

In this paper, firstly, we determine all the measurable solutions of (1.3). This result can also be found in [7] but we use another approach to find the differentiable solutions. We remark that, in the literature, mostly the identities satisfied by the dilogarithm function (or some of its variants) are dealt with. The only exceptional cases we found are in DARÓCZY–KIESEWETTER [3] and JÁRAI [7], where the connection between the solutions of (1.3) and the dilogarithm function was investigated also in the reverse direction, and all the Lebesgue integrable solutions $F : [0,1] \to \mathbb{R}$ (the set of all real numbers) and all the Lebesgue measurable solutions, respectively were determined supposing that (1.3) holds for all $u, v \in [0,1[$ and $u, v \in]0,1[$ respectively.

2. Preliminaries

The regularity improvement results have a very important role in the theory of functional equations. With their help – supposing only weak regularity (say, measurability in the Lebesgue sense) on the solution of the considered functional equation – one can often prove that the solution is infinitely many times differentiable, and the functional equation can be reduced to ordinary or partial differential equation. Nowadays, there can be found a lot of these kind of results in the literature. Some of them refer to the functional equation of the form

$$f(t) = h(t, y, f(y), f(g_1(t, y)), \dots, f(g_n(t, y)))$$

(see Járai [5], [6]), but there are regularity improvement methods also for functional equations containing composite unknown functions (see Páles [11], [12], [13], Gilányi–Páles [4]).

In what follows, we would use the expression "infinitely many times differentiable" frequently. Instead of this, we write that the function is $\mathcal{C}^\infty$, which means that the function is real-valued, it is defined on a non-void open subset of $\mathbb{R}$ or $\mathbb{R}^2$ and it is infinitely many times differentiable.

The following lemma is proved in [7]:

Lemma 2.1. *Suppose that the function $F : I \to \mathbb{R}$ is a measurable solution of* (1.3). *Then F is $\mathcal{C}^\infty$.*

The following observation can be found in [3] (see Satz 1). For the sake of completeness, we present a little bit modified version of it, together with the proof.

Lemma 2.2. *Let the function $F : I \to \mathbb{R}$ be a $\mathcal{C}^\infty$ solution of* (1.3) *and*

$$f(x) = x(1-x)F'(x) \qquad (x \in I). \tag{2.1}$$

Then the function f is $\mathcal{C}^\infty$ and

$$f(1-x) + (1-x)f\left(\frac{y}{1-x}\right) = f(1-y) + (1-y)f\left(\frac{x}{1-y}\right) \tag{2.2}$$

holds for all $(x, y) \in \Delta$, where $\Delta = \{(x, y) : x, y \in I, x + y < 1\}$.

Proof. It is obvious that the function f is $\mathcal{C}^\infty$. Differentiating both sides of (1.3) with respect to u, we get that

$$F'(u) - vF'(1-uv) - \frac{1-v}{(1-uv)^2}F'\left(\frac{1-u}{1-uv}\right) + \frac{v(1-v)}{(1-uv)^2}F'\left(\frac{1-v}{1-uv}\right) = 0 \tag{2.3}$$

holds for all $u, v \in I$. Taking into consideration (2.1), we find that

$$f(u) + uf\left(\frac{1-v}{1-uv}\right) = f\left(\frac{1-u}{1-uv}\right) + \frac{1-u}{1-uv}f(1-uv) = 0 \quad (u, v \in I). \tag{2.4}$$

Let now $(x, y) \in \Delta$ and

$$u = 1 - x, \quad v = \frac{1-x-y}{(1-x)(1-y)}$$

in (2.4). Then we have (2.2). $\square$

The lemma above and the next lemma show that there is a close connection between the $\mathcal{C}^\infty$ solutions of (1.3) and (2.2).

Lemma 2.3. *Let the function* $f : I \to \mathbb{R}$ *be a* $\mathcal{C}^\infty$ *solution of* (2.2) *and*

$$F(x) = \int_\omega^x \frac{f(t)}{t(1-t)} dt \qquad (x \in I). \tag{2.5}$$

Then the function F *is a* $\mathcal{C}^\infty$ *solution of* (1.3).

Proof. It is obvious that the function F is $\mathcal{C}^\infty$ and (2.1) holds. Therefore, (2.2) implies that

$$\begin{aligned} F'(1-x) + \frac{y(1-x-y)}{x(1-x)^2} F'\left(\frac{y}{1-x}\right) \\ = \frac{y(1-y)}{x(1-x)} F'(1-y) + \frac{1-x-y}{(1-x)(1-y)} F'\left(\frac{x}{1-y}\right) \end{aligned}$$

for all $(x,y) \in \Delta$. Let now $u, v \in I$ and

$$x = 1-u, \qquad y = \frac{u(1-v)}{1-uv}.$$

Then $(x,y) \in \Delta$ and – after some calculation – the equation above implies that (2.3) holds for all $u, v \in I$. Define the function Φ on I^2 by

$$\Phi(u,v) = F(u) + F(v) + F(1-uv) + F\left(\frac{1-u}{1-uv}\right) + F\left(\frac{1-v}{1-uv}\right).$$

We shall prove that Φ is identically zero. Indeed, Φ is $\mathcal{C}^\infty$, furthermore,

$$\Phi(u,v) = \Phi(v,u) \quad \text{and} \quad \Phi(\omega,\omega) = 0 \qquad (u, v \in I). \tag{2.6}$$

Here, the first identity is trivial, while the second equality follows from (2.5) and the fact that $\omega^2 + \omega - 1 = 0$. Furthermore, (2.3) implies that $\partial_1 \Phi(u,v) = 0$, that is, the partial derivative of Φ with respect to its first variable is identically zero. Therefore, $\Phi(u,v) = g(v)$ for all $u, v \in I$ and for some $g : I \to \mathbb{R}$. Because of the first part of (2.6) g is constant, and, by the second part of (2.6), this constant must be zero. $\square$

3. The main results

Functional equation (2.2) is very similar to the fundamental equation of information (see Aczél–Daróczy [2]) and its general solution is known (see Maksa [10]). However, here we use an elementary method to solve it by using that the solutions are $\mathcal{C}^\infty$ functions (in fact, we use only that the solutions are twice differentiable), and the idea in [2, pp. 94–102]. From now on, we denote the set of all positive real numbers by $\mathbb{R}_+$.

Theorem 3.1. *Let the function $f : I \to \mathbb{R}$ be $\mathcal{C}^\infty$. Then f is a solution of* (2.2) *if and only if there exist $a, b \in \mathbb{R}$ such that*

$$f(v) = a(v\ln(v) + (1-v)\ln(1-v)) + b(v-2) \qquad (v \in I). \tag{3.1}$$

Proof. Suppose first that the function f is $\mathcal{C}^\infty$ and it is a solution of (2.2). Define the function H on $\mathbb{R}_+^2$ by

$$H(u,v) = (u+v)f\left(\frac{v}{u+v}\right). \tag{3.2}$$

Then H is a $\mathcal{C}^\infty$ function again and

$$H(tu,tv) = tH(u,v) \qquad (t,u,v \in \mathbb{R}_+), \tag{3.3}$$

$$H(w,u+v) + H(u,v) = H(v,u+w) + H(u,w) \qquad (u,v,w \in \mathbb{R}_+). \tag{3.4}$$

Here (3.3) is an obvious consequence of (3.2), while (3.4) follows from (2.2) and (3.2) with the substitutions

$$x = \frac{w}{u+v+w}, \qquad y = \frac{v}{u+v+w}.$$

Differentiate both sides of (3.4) with respect to u and with respect to w, respectively we obtain that

$$\partial_2 H(w,u+v) + \partial_1 H(u,v) = \partial_2 H(v,u+w) + \partial_1 H(u,w)$$

and

$$\partial_1 H(w,u+v) = \partial_2 H(v,u+w) + \partial_2 H(u,w)$$

hold, respectively for all $u,v,w \in \mathbb{R}_+$. (Here $\partial_i H$ denotes the partial derivative function of H with respect to its i-th variable.) Combining these equations, we get that

$$\partial_1 H(u,v) = \partial_1 H(w,u+v) - \partial_2 H(w,u+w) - \partial_2 H(u,w) + \partial_1 H(u,w) \tag{3.5}$$

holds for all $u, v, w \in \mathbb{R}_+$. Substituting $w = 1$ in (3.5) and integrating both sides of the equation so obtained with respect to u, we get that H must be of the form

$$H(u,v) = \alpha(u+v) + \beta(u) + \gamma(v) \qquad (u, v \in \mathbb{R}_+) \tag{3.6}$$

with some $\mathcal{C}^\infty$ functions $\alpha, \beta, \gamma : \mathbb{R}_+ \to \mathbb{R}$. This form of H and (3.4) imply that

$$\gamma(u+v)+\alpha(u+v) = \gamma(u+w)+\alpha(u+w)+\beta(v)-\gamma(v)+\gamma(w)-\beta(w) \quad (u, v, w \in \mathbb{R}_+).$$

Differentiate both sides of this equation with respect to v, we obtain that

$$\gamma'(u+v) + \alpha'(u+v) = \beta'(v) - \gamma'(v) \qquad (u, v \in \mathbb{R}_+). \tag{3.7}$$

Because of the symmetry in u and v of the left hand side of (3.7), we get that

$$(\beta - \gamma)'(v) = c_1 \qquad (v \in \mathbb{R}_+) \tag{3.8}$$

for some $c_1 \in \mathbb{R}$. Hence (3.7) implies that

$$(\alpha + \gamma)'(t) = c_1 \qquad (t \in \mathbb{R}_+). \tag{3.9}$$

Finally, it follows from (3.9), (3.8) and (3.6) that

$$H(u,v) = \alpha(u+v) - \alpha(u) - \alpha(v) + c_1(2u+v) + c_2 \qquad (u, v \in \mathbb{R}_+) \tag{3.10}$$

holds for all $u, v \in \mathbb{R}_+$ and for some $c_2 \in \mathbb{R}$. At this point, we use the homogeneity (3.3) of H to have that

$$\alpha(t(u+v)) - \alpha(tu) - \alpha(tv) + c_2 = t(\alpha(u+v) - \alpha(u) - \alpha(v)) + c_2 t$$

holds for all $t, u, v \in \mathbb{R}_+$. Differentiating both sides of this equation with respect to u and then the equation so obtained with respect to v, we get that

$$\alpha''(t(u+v)) = \frac{1}{t}\alpha''(u+v) \qquad (t, u, v \in \mathbb{R}_+),$$

which, with the substitution $u = v = \frac{1}{2}$, implies that $\alpha''(t) = \frac{1}{t}\alpha''(1)$ holds for all $t \in \mathbb{R}_+$. Solving this differential equation, we find that

$$\alpha(t) = \alpha''(1) t \ln(t) + d_1 t + d_2 \qquad (t \in \mathbb{R}_+) \tag{3.11}$$

for some $d_1, d_2 \in \mathbb{R}$. Taking into consideration (3.2), (3.10) and (3.11), we obtain that

$$f(v) = a(v\ln(v) + (1-v)\ln(1-v)) + b(v-2) + c \qquad (v \in I)$$

for some $a, b, c \in \mathbb{R}$. Finally, substituting this form of f into (2.2), we get that it is satisfied by f if and only if $c = 0$. Thus we obtain (3.1). $\square$

The following theorem is an easy consequence of Lemma 2.1, Lemma 2.2, Theorem 3.1 and Lemma 2.3, see also [7].

Theorem 3.2. *Let $F : I \to \mathbb{R}$ be a measurable function. Then F is a solution of Abel functional equation* (1.3) *if and only if there exist $a, b \in \mathbb{R}$ such that*

$$F(u) = a \int_{\omega}^{u} \left(\frac{\ln(t)}{1-t} + \frac{\ln(1-t)}{t} \right) dt + b \ln\left(\frac{u^2}{1-u} \right) \qquad (u \in I). \tag{3.12}$$

In the next theorem, we list some properties of the Rogers dilogarithm function L defined in (1.2).

Theorem 3.3. *The Rogers dilogarithm function L has the following properties.*

(a) *L is measurable and non-constant,*

(b) *$L - L(\omega)$ is a solution of* (1.3), *and*

(c) *$L(t) + L(1-t) = \frac{\pi^2}{6}$ for all $t \in I$.*

Proof. (a) is obvious. To prove (b), first compute the derivative function L' for all $t \in I$:

$$\begin{aligned} L'(t) &= \sum_{n=1}^{\infty} \frac{t^{n-1}}{n} + \frac{1}{2}\frac{\ln(1-t)}{t} - \frac{1}{2}\frac{\ln(t)}{1-t} = \frac{1}{t}(-\ln(1-t)) + \frac{1}{2}\frac{\ln(1-t)}{t} - \frac{1}{2}\frac{\ln(t)}{1-t} \\ &= -\frac{1}{2}\left(\frac{\ln(t)}{1-t} + \frac{\ln(1-t)}{t} \right). \end{aligned}$$

This implies that

$$L(x) - L(\omega) = -\frac{1}{2} \int_{\omega}^{x} \left(\frac{\ln(t)}{1-t} + \frac{\ln(1-t)}{t} \right) dt \qquad (x \in I). \tag{3.13}$$

Hence, by Theorem 3.2, we get (b). On the other hand, since $L'(t) = L'(1-t)$ for all $t \in I$, we obtain that the function $t \mapsto L(t) + L(1-t)$, $t \in I$ must be constant. However,

$$\lim_{t \to 0} L(t) = 0 \quad \text{and, by (1.1),} \quad \lim_{t \to 1} L(t) = Li_2(1) = \frac{\pi^2}{6}.$$

Thus we have (c). □

The next theorem together with Theorem 3.2. shows that the properties $(a) - (b) - (c)$ characterize the Rogers dilogarithm.

Theorem 3.4. *Suppose that the function $G : I \to \mathbb{R}$ has the following properties*

(i) *G is measurable and non-constant,*

(ii) *$G - Li_2(\omega) - \frac{1}{2}\ln(\omega)\ln(1-\omega)$ is a solution of (1.3), and*

(iii) *$G(t) + G(1-t) = \frac{\pi^2}{6}$ for all $t \in I$.*

Then G is identical with the Rogers dilogarithm L.

Proof. Because of (i) and (ii), Theorem 3.2. implies that there exist $a, b \in \mathbb{R}$ such that

$$G(u) - Li_2(\omega) - \frac{1}{2}\ln(\omega)\ln(1-\omega) = a\int_\omega^u \left(\frac{\ln(t)}{1-t} + \frac{\ln(1-t)}{t}\right) dt + b\ln\left(\frac{u^2}{1-u}\right)$$

holds for all $u \in I$. Therefore,

$$G'(u) = a\left(\frac{\ln(u)}{1-u} + \frac{\ln(1-u)}{u}\right) + b\left(\frac{2-u}{u(1-u)}\right) \qquad (u \in I). \tag{3.14}$$

Taking into consideration (iii), we have that $G'(u) = G'(1-u)$ for all $u \in I$. Therefore, by (3.14), we obtain that $b = 0$ and so

$$G'(u) = a\left(\frac{\ln(u)}{1-u} + \frac{\ln(1-u)}{u}\right) \qquad (u \in I).$$

Thus

$$G(t) - G(\omega) = a\int_\omega^t \left(\frac{\ln(u)}{1-u} + \frac{\ln(1-u)}{u}\right) du \qquad (t \in I).$$

But every solution of (1.3) vanishes at ω. Thus, by (ii), $G(\omega) = L(\omega)$, so

$$G(t) = L(\omega) + a\int_\omega^t \left(\frac{\ln(u)}{1-u} + \frac{\ln(1-u)}{u}\right) du \qquad (t \in I),$$

whence, by (3.13), we get that

$$G(t) + 2aL(t) = (1+2a)L(\omega) \qquad (t \in I). \tag{3.15}$$

Applying now (iii) and (c), we have that $(2a+1)\frac{\pi^2}{6} = 2(2a+1)L(\omega)$. If $a \neq -\frac{1}{2}$, then it would follow that

$$\frac{\pi^2}{6} = 2L(\omega) = \frac{\pi^2}{5} - 2\ln^2\left(\frac{\sqrt{5}+1}{2}\right) + \ln(\omega)\ln(1-\omega),$$

which, by the definition of ω, is equivalent with $\ln(\omega)\ln\left(\frac{\omega^2}{1-\omega}\right) = \frac{\pi^2}{3}$, which is a contradiction, since $\ln\left(\frac{\omega^2}{1-\omega}\right) = 0$. Thus $a = -\frac{1}{2}$ and (3.15) implies that $G = L$. $\square$

Finally, we make a remark on the non-measurable solutions of (1.3).

Remark. It is easy to see that if F is a solution of (1.3) and $A:\mathbb{R}\to\mathbb{R}$ is any additive function, that is, A satisfies the additive Cauchy functional equation

$$A(x+y)=A(x)+A(y) \qquad (x,y\in\mathbb{R}),$$

then the composite function $A\circ F$ is also a solution of (1.3). It is well-known (see e.g. Kuczma [8]) that there are non-measurable additive functions. Thus, if $A:\mathbb{R}\to\mathbb{R}$ is a non-measurable additive function and $\ell:\mathbb{R}_+\to\mathbb{R}$ is a logarithmic function (which means that ℓ satisfies the logarithmic Cauchy functional equation $\ell(xy)=\ell(x)+\ell(y)\,(x,y\in\mathbb{R}_+)$), then among the functions $F:I\to\mathbb{R}$ defined by

$$F(u)=A\left(\int_{\omega}^{u}\left(\frac{\ln(t)}{1-t}+\frac{\ln(1-t)}{t}\right)dt\right)+\ell\left(\frac{u^2}{1-u}\right) \qquad (u\in I)$$

there are non-measurable solutions of (1.3).

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