Publ. Math. Debrecen
89/3 (2016), 347–353
DOI: 10.5486/PMD.2016.7673

Mazur's type problem for convexity of higher orders

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Dedicated to Professor Zsolt Páles on the occasion of his 60th birthday

Abstract. I. Labuda and R. D. Mauldin [4] have solved in affirmative the following S. Mazur's problem posed about 1935 (see [6]):

"In a space E of type (B), there is given an additive functional $F(x)$ with the following property: if $x(t)$ is a continuous function in $0 \leq t \leq 1$ with values in E, then $F(x(t))$ is a measurable function. Is $F(x)$ continuous?"

In [1], we showed that the same remains true in the case where F is a Jensen-convex functional on an open and convex subdomain of a real Banach space. In the present paper, we study the possibilities of an extension of this result to convexity of higher orders.

Some eighty years ago Stanisław Mazur asked the following question (see "Problem 24" from the famous "Scottish Book" [6]):

> "In a space E of type (B), there is given an additive functional $F(x)$ with the following property: if $x(t)$ is a continuous function in $0 \leq t \leq 1$ with values in E, then $F(x(t))$ is a measurable function. Is $F(x)$ continuous?"

The solution, in the affirmative, was given by I. Labuda and R. D. Mauldin [4]. As a matter of fact, they have proved a more general theorem: instead of functionals, they considered additive operators from a Banach space into a Hausdorff topological vector space. Fairly soon afterwards, this result was generalized by Z. Lipecki [5] to the case where the domain and range of the additive transformations considered are suitable Abelian topological groups.

Mathematics Subject Classification: 26B25, 39B62, 46A19.
Key words and phrases: Mazur's problem, Jensen-convexity of n-th order, n-convex functionals, measurability implies continuity, core topology.

It is widely known that the regularity behaviour of Jensen-convex functionals is, in general, very similar to that of additive ones. Therefore, it was natural to look after the following generalization of Mazur's Problem 24 from the famous "Scottish Book":

Assume that we are given a nonempty open and convex subdomain D of a real Banach space $(E, \|\cdot\|)$, and that $F : D \longrightarrow \mathbb{R}$ is a Jensen-convex functional on D, i.e. F is a solution to the functional inequality

$$F\left(\frac{u+v}{2}\right) \leq \frac{F(u)+F(v)}{2}, \quad u, v \in D. \tag{1}$$

Suppose that the superposition $F \circ x$ is Lebesgue measurable for every continuous map $x : [0,1] \longrightarrow D$. Does it force F to be continuous?

In 1995, a positive answer to that question was given by the present author in [1].

What about the convexity of higher orders? More exactly, the question reads as follows:

Given a nonempty open and convex subdomain D of a real Banach space $(E, \|\cdot\|)$ endowed with a cone C of positive elements with $C \cap (-C) = \{0\}$, assume that F is an n-th order Jensen-convex functional on D, i.e. F is a solution to the conditional functional inequality

$$v - u \in C \Longrightarrow \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} F\left((1 - \frac{j}{n+1})u + \frac{j}{n+1} v \right) \geq 0, \qquad u, v \in D.^{1} \tag{2}$$

Does the requirement that the superposition $F \circ x$ is Lebesgue measurable for every continuous map $x : [0,1] \longrightarrow D$, force F to be continuous? To give a partial answer to that question, we need to recall some facts.

(1) Any continuous Jensen-convex functional $F : D \longrightarrow \mathbb{R}$ (a solution to (1)) is convex in the usual sense, i.e. it satisfies the inequality

$$F(\lambda x + (1-\lambda)y) \leq \lambda F(x) + (1-\lambda)F(y)$$

for all vectors x, y from D and all scalars λ from the unit interval $[0,1]$.

[1] Usually (see e.g. M. Kuczma's book [3]), this definition is formulated equivalently with the aid of iterations of the difference operator Δ_h, but (2) is by far more convenient in the case where the domain of the function in question yields a proper convex subset of the space.

(2) Any continuous n-th order Jensen-convex functional $F : D \longrightarrow \mathbb{R}$ (a solution to (2)) is n-convex in the sense of Tiberiu Popoviciu (see e.g. M. Kuczma [3]), i.e. it satisfies the conditional inequality

$$v-u \in C \Longrightarrow \sum_{j=0}^{n+1} (-1)^{n+1-j} V(\lambda_0, \lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots \lambda_n, \lambda_{n+1}) \times F((1-\lambda_j)u + \lambda_j v) \geq 0 \quad (3)$$

for every choice of real numbers $0 = \lambda_0 < \lambda_1 < \dots < \lambda_n < \lambda_{n+1} = 1$, where V stands for the Vandermonde's determinant of the variables considered (a higher order counterpart of the standard convexity and reducing to it in the case $n = 1$).

(3) The linear structure of a real linear space E induces the so-called *core*-topology in E as follows: given a set $G \subset E$, denote by $\operatorname{core} G$ the set of all points $y \in G$ enjoying the property that for every $x \in E$ one may find an $\varepsilon > 0$ such that $y + tx \in G$ for all $t \in (-\varepsilon, \varepsilon)$. Then G is said to be algebraically open provided that $G = \operatorname{core} G$. The family of all algebraically open sets forms a topology in E which is just termed *core*-topology. Unfortunately, in general, a linear space with its *core*-topology fails to be a topological linear space: instead of having the joint continuity of the map

$$E \times E \times \mathbb{R} \ni (x, y, \lambda) \longmapsto \lambda x + y \in E,$$

we get merely the separate continuity of it (semilinearity). The *core*-topology of a linear space E turns out to be the finest possible semilinear topology in E. There exist numerous sources in mathematical literature where these topologies are studied. In what follows, the knowledge of Chapter I in Z. Kominek's dissertation [2] is utterly sufficient.

Theorem 1. *Let* $(E, \|\cdot\|), D$ *and* C *have the meaning described above. Then any solution* $F : D \longrightarrow \mathbb{R}$ *of* (2) *having the property that the superposition* $F \circ x$ *is Lebesgue measurable for every affine map* $x : [0,1] \longrightarrow D$*, is* n*-convex. In the case where the order relation generated by the cone* C *in question is linear (i.e.* $C \cup (-C) = E$*),* F *is also continuous in the core-topology of* E*.*

Proof. Fix arbitrarily elements u and v from D with $v - u \in C$. It is well known (see, for instance, M. Kuczma [3]) that inequality (2) is equivalent to (3) for every choice of *rationals* $0 = \lambda_0 < \lambda_1 < \dots < \lambda_n < \lambda_{n+1} = 1$.

Since the domain D is supposed to be open, there exists an $\varepsilon > 0$ such that $x(t) := (1-t)u + tv$ belongs to D provided that $t \in I_\varepsilon := (-\varepsilon, 1+\varepsilon)$.

Putting $f(t) := F \circ x(t), t \in I_\varepsilon$, we deduce that for every choice of rationals $0 = \lambda_0 < \lambda_1 < \dots < \lambda_n < \lambda_{n+1} = 1$, and for every pair $s, t \in (-\varepsilon, 1+\varepsilon), s < t$, in view of (3) one has

$$\begin{aligned}
&\sum_{j=0}^{n+1}(-1)^{n+1-j}V(\lambda_0,\lambda_1,\dots,\lambda_{j-1},\lambda_{j+1},\dots\lambda_n,\lambda_{n+1})f((1-\lambda_j)s+\lambda_j t)\\
&=\sum_{j=0}^{n+1}(-1)^{n+1-j}V(\lambda_0,\lambda_1,\dots,\lambda_{j-1},\lambda_{j+1},\dots\lambda_n,\lambda_{n+1})F(x((1-\lambda_j)s+\lambda_j t))\\
&=\sum_{j=0}^{n+1}(-1)^{n+1-j}V(\lambda_0,\lambda_1,\dots,\lambda_{j-1},\lambda_{j+1},\dots\lambda_n,\lambda_{n+1})\\
&\quad\times F((1-\lambda_j)x(s)+\lambda_j x(t))) \ge 0, \qquad (4)
\end{aligned}$$

provided that $x(t) - x(s) \in C$. Consequently, taking any $s, t \in (-\varepsilon, 1+\varepsilon)$ and any $u, v \in D$ such that $v - u \in C$, we infer that

$$s < t \Longrightarrow x(t) - x(s) = (t-s)(v-u) \in C,$$

and, by means of the continuity of f resulting from a *measurability implies continuity* type theorem for n-convex functions (see e.g. M. Kuczma [3])), we get

$$\sum_{j=0}^{n+1}(-1)^{n+1-j}V(\lambda_0,\lambda_1,\dots,\lambda_{j-1},\lambda_{j+1},\dots\lambda_n,\lambda_{n+1})f((1-\lambda_j)s+\lambda_j t) \ge 0$$

for every choice of *real* numbers $0 = \lambda_0 < \lambda_1 < \dots < \lambda_n < \lambda_{n+1} = 1$. Therefore, since the first equality in (4) is valid for all real numbers $0 = \lambda_0 < \lambda_1 < \dots < \lambda_n < \lambda_{n+1} = 1$ as well, we infer that the conditional implication (3) holds true for any choice of real numbers $0 = \lambda_0 < \lambda_1 < \dots < \lambda_n < \lambda_{n+1} = 1$. This, in turn, forces F to be n-convex in the sense of Popoviciu.

To prove the remaining part of the assertion, it suffices to show that F is both upper and lower semicontinuous, i.e. that for each real number α the sets

$$A := \{x \in D : F(x) < \alpha\} \quad \text{and} \quad B := \{x \in D : F(x) > \alpha\}$$

are algebraically open. To show that that is really the case, we shall follow in Z. Kominek's footsteps while proving Lemma 3.8 in his dissertation [2]. Fix arbitrarily a $y \in A$ and an $x \in E$. Since $y \in D$ and D is open, one may find a positive real number δ such that

$$y + tx \in D \quad \text{for all} \quad t \in I := (-\delta, \delta).$$

Due to the linearity of the order, without loss of generality, we may assume that $x \in C$. On setting $g(t) := F(y + tx)$, $t \in I$, we define an n-convex function $g : I \longrightarrow \mathbb{R}$. Indeed, we have proved already that relation (3) holds true for all $u, v \in D$ and for every choice of real numbers $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda_{n+1} = 1$ (the n-convexity of F). We are going to show that g is n-convex. To this end, fix arbitrarily $s, t \in I, s < t$, and real numbers $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda_{n+1} = 1$; then

$$\begin{aligned}
&\sum_{j=0}^{n+1}(-1)^{n+1-j}V(\lambda_0,\lambda_1,\cdots,\lambda_{j-1},\lambda_{j+1},\cdots\lambda_n,\lambda_{n+1})g((1-\lambda_j)s+\lambda_j t)\\
&=\sum_{j=0}^{n+1}(-1)^{n+1-j}V(\lambda_0,\lambda_1,\cdots,\lambda_{j-1},\lambda_{j+1},\cdots\lambda_n,\lambda_{n+1})F(y+((1-\lambda_j)s+\lambda_j t)x)\\
&=\sum_{j=0}^{n+1}(-1)^{n+1-j}V(\lambda_0,\lambda_1,\cdots,\lambda_{j-1},\lambda_{j+1},\cdots\lambda_n,\lambda_{n+1})\\
&\quad\times F((1-\lambda_j)(y+sx)+\lambda_j(y+tx))\geq 0,
\end{aligned}$$

because $(y + tx) - (y + sx) = (t - s)x \in C$.

It is known that n-convex functions from an open subinterval of $\mathbb{R}$ into $\mathbb{R}$ are continuous (as a matter of fact, even of class C^{n-1}) in the natural topology of $\mathbb{R}$. Therefore, since $g(0) = F(y) < \alpha$, there exists an $\varepsilon > 0$ such that $g(t) < \alpha$ for all $t \in (-\varepsilon, \varepsilon)$, i.e. $F(y+tx) < \alpha$ for all $t \in (-\varepsilon, \varepsilon)$, which states that $y+tx \in A$ for all $t \in (-\varepsilon, \varepsilon)$, and finishes the proof of the algebraic openness of A. The proof of the algebraic openness of B is pretty much the same. This ends the proof of the theorem. □

In what follows, we offer a modified version of the proof of Theorem 1.

Alternative Proof. Fix arbitrarily elements u and v from D with $u - v \in C$, and take an $\varepsilon > 0$ such that $x(t) := tu + (1 - t)v$ belongs to D provided that $t \in I_\varepsilon := (-\varepsilon, 1 + \varepsilon)$. We shall show that a function $f : I_\varepsilon :\longrightarrow \mathbb{R}$ given by the formula $f(t) := F(tu + (1 - t)v)$, $t \in I_\varepsilon$, is n-convex. To this end, let us first observe that the following equality

$$\begin{aligned}
&\left[\left(1-\frac{j}{n+1}\right)s+\frac{j}{n+1}t\right]u+\left[1-\left(\left(1-\frac{j}{n+1}\right)s+\frac{j}{n+1}t\right)\right]v\\
&=\left(1-\frac{j}{n+1}\right)(su+(1-s)v)+\frac{j}{n+1}(tu+(1-t)v) \qquad (5)
\end{aligned}$$

holds true for all $s, t \in I_\varepsilon$ and every $j \in \{0, 1, ..., n+1\}$ (a simple calculation). Plainly, because of the convexity of the domain D, both the vectors

$$a := su + (1-s)v \quad \text{and} \quad b := tu + (1-t)v$$

fall into D. Moreover,

$$b - a = (t-s)(u-v) \in C \tag{6}$$

whenever $s < t$. Thus, on account of (5) and the definitions of a and b, we conclude that for any two members s, t from the interval I_ε one has

$$\begin{aligned} s < t \Longrightarrow & \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} f\left((1 - \frac{j}{n+1})s + \frac{j}{n+1}t \right) \\ &= \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} F\left(\left[\left(1 - \frac{j}{n+1}\right) s + \frac{j}{n+1}t \right] u \right. \\ &\quad \left. + \left[1 - \left(\left(1 - \frac{j}{n+1}\right) s + \frac{j}{n+1}t \right) \right] v \right) \\ &= \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} F\left(\left(1 - \frac{j}{n+1}\right) a + \frac{j}{n+1} b \right) \geq 0, \end{aligned}$$

because of (6) and the n-th order Jensen-convexity of F. This states nothing else but the n-th order Jensen-convexity of f. Along the same lines as in the previous proof, we obtain the continuity of f which gives the implication (3) for every choice of *real* numbers $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda_{n+1} = 1$. The rest of the proof remains unchanged. □

Remark 1. Noteworthy is the fact that the family of all *continuous* "testing functions" $x : [0,1] \longrightarrow D$ used in the statement of Mazur's problem has been restricted *merely* to affine functions.

Remark 2. The question whether or not F is continuous in the *norm* topology of the space $(E, \|\cdot\|)$ considered remains open, even under the Mazur's assumption that the superposition $F \circ x$ is Lebesque measurable for *every* continuous map $x : [0,1] \longrightarrow E$.

However, since in finite dimensional spaces n-convex functions on open and convex domains are automatically continuous (see e.g. M. Kuczma [3]), a positive answer to suitable Mazur's question results immediately from Theorem 1. Namely, we have the following:

Theorem 2. *Given a non-empty open and convex subdomain D of a finite dimensional real Banach space endowed with a cone C of positive elements with $C \cap (-C) = \{0\}$, assume that F is an n-th order Jensen-convex functional on D, i.e. F is a solution to the conditional functional inequality*

$$v - u \in C \Longrightarrow \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} F\left(\left(1 - \frac{j}{n+1}\right) u + \frac{j}{n+1} v\right) \geq 0, \qquad u, v \in D.$$

Suppose that the superposition $F \circ x$ is Lebesgue measurable for every affine map $x : [0,1] \longrightarrow D$. Then F is continuous.

Remark 3. Without any changes in the proof one might replace the assumption that *the superposition $F \circ x$ is Lebesgue measurable for every affine map $x : [0,1] \longrightarrow D$* by the requirement that *for every affine map $x : [0,1] \longrightarrow D$, the superposition $F \circ x$ is bounded on a second category Baire subset of D, that may depend upon x*, or any other alternative assumption forcing an n-th order Jensen-convex functional on an open subinterval of the real axis to be continuous. A number of such sufficient conditions may be found in M. Kuczma's monograph [3], for instance.

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(Received April 9, 2016; revised July 14, 2016)