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Separation by convex and strongly convex stochastic processes

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Dedicated to Professor Zsolt Páles on his 60^{th} birthday

Abstract. Characterizations of pairs of stochastic processes that can be separated by a convex or strongly convex stochastic process are presented. As consequences, stability results of the Hyers–Ulam type are obtained.

1. Introduction

Separation theorems, that is theorems providing conditions under which two given functions can be separated by a function from some special class, play an important role in many fields of mathematics and have various applications. In the literature one can find numerous results of this type (see, for instance [1], [2], [3], [5], [9], [10], [12], [13], [14], [15], [16], [18] and the references therein). In [1], the following theorem about separation by convex functions was obtained:

Theorem BMN. *Let $I \subset \mathbb{R}$ be an interval and $f, g : I \to \mathbb{R}$. There exists a convex function $h : I \to \mathbb{R}$ such that $f \le h \le g$ on I if and only if*

$$f\big(\lambda x + (1-\lambda)y\big) \leqslant \lambda g(x) + (1-\lambda)g(y), \tag{1}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

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The "only if" part of this theorem is a straightforward calculation, but in the proof of the "if" part the classical Carathéodory theorem play a crucial role.

The aim of the present note is to give some counterparts of the above theorem for convex and strongly convex stochastic processes. As a consequence, we obtain also Hyers–Ulam-type stability results for such classes of processes. Since in our settings the Carathéodory theorem can not be used, the proof of our main result is different from the proof of Theorem BMN.

Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space and $I \subset \mathbb{R}$ be an interval. A function $C : \Omega \to \mathbb{R}$ is called a *random variable* if it is $\mathcal{A}$-measurable. A function $X : I \times \Omega \to \mathbb{R}$ is called a *stochastic process* if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

Let $C : \Omega \to \mathbb{R}$ be a positive random variable. Recall that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is said to be *strongly convex with modulus* $C(\cdot)$ if the inequality

$$X\big(\lambda t_1+(1-\lambda)t_2,\cdot\big) \leqslant \lambda X(t_1,\cdot)+(1-\lambda)X(t_2,\cdot)-C(\cdot)\lambda\big(1-\lambda\big)\big(t_1-t_2\big)^2 \quad \text{(a.e.)} \quad (2)$$

is satisfied for all $t_1, t_2 \in I$ and $\lambda \in [0,1]$ (cf. [7]). By omitting the term $C(\cdot)\lambda(1-\lambda)(t_1 - t_2)^2$ in the inequality (2), we get the definition of a *convex stochastic process* introduced in 1980 in [11]. Many properties of convex and strongly convex stochastic processes can be found in [7], [8], [11], [17].

At the end of this section, let us recall the definition of the essential infimum of a collection of functions. We will use this notion as a basic tool in the proof of our main theorem. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\mathcal{S}$ be a collection of measurable functions $f : \Omega \to \mathbb{R}$. On $\mathbb{R}$ the Borel σ-algebra is used. If $\mathcal{S}$ is a countable set, then we may define the pointwise infimum of the functions from $\mathcal{S}$, which is measurable itself. If $\mathcal{S}$ is uncountable, then the pointwise infimum need not be measurable. In this case, the essential infimum can be used. The *essential infimum* of $\mathcal{S}$, written as $\operatorname{ess\,inf} \mathcal{S}$, if it exists, is a measurable function $f : \Omega \to \mathbb{R}$ satisfying the following two axioms:

- $f \leqslant g$ almost everywhere, for any $g \in \mathcal{S}$,
- if $h : \Omega \to \mathbb{R}$ is measurable and $h \leqslant g$ almost everywhere for every $g \in \mathcal{S}$, then $h \leqslant f$ almost everywhere.

It can be shown that for a σ-finite measure μ, the essential infimum of $\mathcal{S}$ do exists, whenever $\mathcal{S}$ is a family of measurable functions jointly bounded from below. For more details, we refer the reader to [4].

2. Separation by convex processes

Now we present the main result of this paper. It gives a condition under which two given stochastic processes can be separated by a convex stochastic process.

Theorem 1. *Let $X, Y : I \times \Omega \to \mathbb{R}$ be stochastic processes. There exists a convex stochastic process $Z : I \times \Omega \to \mathbb{R}$ such that*

$$X(t,\cdot) \leqslant Z(t,\cdot) \leqslant Y(t,\cdot) \quad \text{(a.e.)}$$

for all $t \in I$, if and only if

$$X\left(\sum_{i=1}^{n} \lambda_i t_i, \cdot\right) \leqslant \sum_{i=1}^{n} \lambda_i Y(t_i,\cdot) \quad \text{(a.e.)} \tag{3}$$

for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in I$ and $\lambda_1, \dots, \lambda_n \geqslant 0$ with $\lambda_1 + \dots + \lambda_n = 1$.

Proof. The "only if" part follows by the Jensen inequality for convex stochastic processes (see [7]):

$$X\left(\sum_{i=1}^{n} \lambda_i t_i, \cdot\right) \leqslant Z\left(\sum_{i=1}^{n} \lambda_i t_i, \cdot\right) \leqslant \sum_{i=1}^{n} \lambda_i Z(t_i,\cdot) \leqslant \sum_{i=1}^{n} \lambda_i Y(t_i,\cdot) \quad \text{(a.e.)}.$$

To prove the "if" part, fix $t \in I$ and define the process Z by

$$Z(t,\cdot) = \operatorname{ess\,inf} \left\{ \sum_{i=1}^{n} \lambda_i Y(t_i,\cdot) : \quad n \in \mathbb{N}, t_1, \dots, t_n \in I, \lambda_1, \dots, \lambda_n \in [0,1] \right.$$
$$\left. \text{such that } \lambda_1 + \dots + \lambda_n = 1 \text{ and } t = \lambda_1 t_1 + \dots + \lambda_n t_n \right\}.$$

By (3) and the definition of essential infimum, we have

$$X(t,\cdot) \leqslant Z(t,\cdot) \quad \text{(a.e.)}, \quad t \in I.$$

By the definition of Z (taking $n = 1$, $\lambda_1 = 1$ and $t_1 = t$) we get also

$$Z(t,\cdot) \leqslant Y(t,\cdot) \quad \text{(a.e.)}, \quad t \in I.$$

To prove that Z is convex, fix $t_1, t_2 \in I$ and $\lambda \in [0,1]$. Take arbitrary $u_1, \dots, u_n \in I$, $\alpha_1, \dots, \alpha_n \in [0,1]$ and $v_1, \dots, v_m \in I$, $\beta_1, \dots, \beta_m \in I$ such that $\alpha_1 + \dots + \alpha_n = 1$, $\beta_1 + \dots + \beta_m = 1$ and $t_1 = \alpha_1 u_1 + \dots + \alpha_n u_n$, $t_2 = \beta_1 v_1 + \dots + \beta_m v_m$. Since

$$\sum_{i=1}^{n} \lambda \alpha_i + \sum_{j=1}^{m} (1-\lambda)\beta_j = 1,$$

the point $\lambda t_1 + (1-\lambda)t_2$ is a convex combination of $u_1, \dots, u_n, v_1, \dots, v_m$, and

$$\lambda t_1 + (1-\lambda)t_2 = \lambda \sum_{i=1}^{n} \alpha_i u_i + \big(1-\lambda\big) \sum_{j=1}^{m} \beta_j v_j.$$

Therefore, by the definition of Z we have

$$Z\big(\lambda t_1 + (1-\lambda)t_2, \cdot\big) \leqslant \lambda \sum_{i=1}^{n} \alpha_i Y(u_i, \cdot) + (1-\lambda) \sum_{j=1}^{m} \beta_j Y(v_j, \cdot) \quad \text{(a.e.)}. \tag{4}$$

This inequality holds for every $n \in \mathbb{N}$, $u_1, \dots, u_n \in I$ and $\alpha_1, \dots, \alpha_n \in [0,1]$ such that $\alpha_1 + \dots + \alpha_n = 1$ and $\alpha_1 u_1 + \dots + \alpha_n u_n = t_1$, as well as for all $m \in \mathbb{N}$, $v_1, \dots, v_m \in I$ and $\beta_1, \dots, \beta_m \in [0,1]$ such that $\beta_1 + \dots + \beta_m = 1$ and $\beta_1 v_1 + \dots + \beta_m v_m = t_2$. Therefore, taking the essential infimum in the first term of the right hand side of (4) and next in the second term and using the second axiom of the definition of essential infimum, we get

$$Z\big(\lambda t_1 + (1-\lambda)t_2, \cdot\big) \leqslant \lambda Z(t_1, \cdot) + (1-\lambda) Z(t_2, \cdot) \quad \text{(a.e.)}.$$

This shows that Z is convex and finishes the proof. □

Remark 2. In Theorem BMN, to get the function h that separates f and g, it was enough to assume inequality (1) only for convex combinations of two points. In our Theorem 1 we assume that the corresponding inequality (3) holds for arbitrarily long convex combination. It is caused by the fact that we can not use the Carathéodory theorem in the proof. It is an open problem whether Theorem 1 remains true if we assume (3) only for $n = 2$.

As an immediate consequence of the above theorem, we obtain the following Hyers–Ulam-type stability results for convex stochastic processes. For the classical Hyers–Ulam theorem, see [6].

Let ε be a positive constant. We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is ε- *convex* if

$$X\left(\sum_{i=1}^{n} \lambda_i t_i, \cdot\right) \leqslant \sum_{i=1}^{n} \lambda_i X(t_i, \cdot) + \varepsilon \quad \text{(a.e.)} \tag{5}$$

for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in I$ and $\lambda_1, \dots, \lambda_n \geqslant 0$ with $\lambda_1 + \dots + \lambda_n = 1$.

Corollary 3. *If a stochastic process $X : I \times \Omega \to \mathbb{R}$ is ε-convex, then there exists a convex stochastic process Z such that*

$$|X(t, \cdot) - Z(t, \cdot)| \leqslant \frac{\varepsilon}{2} \quad \text{(a.e.)} \tag{6}$$

for all $t \in I$.

Proof. Define $Y(t,\cdot) = X(t,\cdot) + \varepsilon,\ \ t \in I$. In view of (5), the processes X and Y satisfy (3). Therefore, by Theorem 1, there exists a convex process $Z_1 : I \times \Omega \to \mathbb{R}$, such that $X(t,\cdot) \leqslant Z_1(t,\cdot) \leqslant X(t,\cdot) + \varepsilon$ (a.e.), for all $t \in I$. Putting $Z(t,\cdot) = Z_1(t,\cdot) - \frac{\varepsilon}{2}$, we get (6). This completes the proof. □

3. Separation by strongly convex processes

The following theorem characterizes pairs of stochastic processes which can be separated by a strongly convex stochastic process.

Theorem 4. *Let $X, Y : I \times \Omega \to \mathbb{R}$ be stochastic processes. There exists a stochastic process $Z : I \times \Omega \to \mathbb{R}$ strongly convex with modulus $C(\cdot)$ such that*

$$X(t,\cdot) \leqslant Z(t,\cdot) \leqslant Y(t,\cdot) \quad \text{(a.e.)}$$

for all $t \in I$, if and only if

$$X\left(\sum_{i=1}^{n} \lambda_i t_i, \cdot\right) \leqslant \sum_{i=1}^{n} \lambda_i Y(t_i,\cdot) - C(\cdot)\sum_{i=1}^{n} \lambda_i (t_i - m)^2 \quad \text{(a.e.)} \tag{7}$$

for all $n \in \mathbb{N}$, $t_1, \ldots, t_n \in I$, $\lambda_1, \ldots, \lambda_n \geqslant 0$ with $\lambda_1 + \cdots + \lambda_n = 1$ and $m = \sum_{i=1}^{n} \lambda_i t_i$.

Proof. To prove the "only if" part, assume that there exists a strongly convex stochastic process Z with modulus C, such that $X(t,\cdot) \leqslant Z(t,\cdot) \leqslant Y(t,\cdot)$ (a.e.), for every $t \in I$. The Jensen inequality for strongly convex stochastic processes (see [7]) implies that

$$\begin{aligned} X\left(\sum_{i=1}^{n} \lambda_i t_i, \cdot\right) &\leqslant Z\left(\sum_{i=1}^{n} \lambda_i t_i, \cdot\right) \leqslant \sum_{i=1}^{n} \lambda_i Z(t_i,\cdot) - C(\cdot)\sum_{i=1}^{n} \lambda_i (t_i - m)^2 \\ &\leqslant \sum_{i=1}^{n} \lambda_i Y(t_i,\cdot) - C(\cdot)\sum_{i=1}^{n} \lambda_i (t_i - m)^2 \quad \text{(a.e.)}. \end{aligned}$$

To prove the "if" part, assume that X and Y satisfy (7). Consider the stochastic processes X_1 and Y_1 defined by $X_1(t,\cdot) = X(t,\cdot) - C(\cdot)t^2$ and $Y_1(t,\cdot) = Y(t,\cdot) - C(\cdot)t^2$, for all $t \in I$.

By (7) and the following equality

$$\sum_{i=1}^{n} \lambda_i (t_i - m)^2 = \sum_{i=1}^{n} \lambda_i t_i^2 - m^2$$

we have

$$\begin{aligned}
X_1\left(\sum_{i=1}^{n}\lambda_i t_i,\cdot\right) &= X\left(\sum_{i=1}^{n}\lambda_i t_i,\cdot\right) - C(\cdot)\left(\sum_{i=1}^{n}\lambda_i t_i\right)^2 \\
&\leqslant \sum_{i=1}^{n}\lambda_i Y(t_i,\cdot) - C(\cdot)\sum_{i=1}^{n}\lambda_i (t_i-m)^2 - C(\cdot)\left(\sum_{i=1}^{n}\lambda_i t_i\right)^2 \\
&= \sum_{i=1}^{n}\lambda_i Y(t_i,\cdot) - C(\cdot)\left(\sum_{i=1}^{n}\lambda_i t_i^2 - m^2\right) - C(\cdot)m^2 \\
&= \sum_{i=1}^{n}\lambda_i (Y(t_i,\cdot) - C(\cdot)t_i^2) = \sum_{i=1}^{n}\lambda_i Y_1(t_i,\cdot) \quad \text{(a.e.)}.
\end{aligned}$$

By Theorem 1, there exists a convex stochastic process $Z_1 : I\times\Omega\to\mathbb{R}$ such that for all $t\in I$ holds

$$X_1(t,\cdot)\leqslant Z_1(t,\cdot)\leqslant Y_1(t,\cdot) \quad \text{(a.e.)}.$$

Take $Z(t,\cdot) = Z_1(t,\cdot) + C(\cdot)t^2$. The process Z is strongly convex with modulus $C(\cdot)$ (see [8, Lemma 2.1]). Moreover, the inequality

$$X(t,\cdot)\leqslant Z(t,\cdot)\leqslant Y(t,\cdot) \quad \text{(a.e.)}$$

holds for every $t\in I$. □

At the end of this section we present Hyers–Ulam type stability result for strongly convex stochastic processes. We say that a stochastic process $X : I\times\Omega\to\mathbb{R}$ is ε- *strongly convex with modulus* $C(\cdot)$ if

$$X\left(\sum_{i=1}^{n}\lambda_i t_i,\cdot\right)\leqslant \sum_{i=1}^{n}\lambda_i X(t_i,\cdot) - C(\cdot)\sum_{i=1}^{n}\lambda_i (t_i-m)^2 + \varepsilon \quad \text{(a.e.)} \tag{8}$$

for all $t_1,...,t_n\in I$, $\lambda_1,\dots,\lambda_n\geqslant 0$, with $\lambda_1+\dots+\lambda_n=1$ and $m=\lambda_1 t_1+\dots+\lambda_n t_n$.

Corollary 5. *If a stochastic process* $X : I\times\Omega\to\mathbb{R}$ *is* ε-strongly convex with modulus $C(\cdot)$, *then there exists a strongly convex stochastic process* $Z : I\times\Omega\to\mathbb{R}$ *with modulus* $C(\cdot)$ *such that*

$$|X(t,\cdot) - Z(t,\cdot)|\leqslant \frac{\varepsilon}{2} \quad \text{(a.e.)} \tag{9}$$

for all $t\in I$.

PROOF. To prove Corollary 5, we define $Y(t,\cdot) = X(t,\cdot) + \varepsilon$ for all $t \in I$. By (8) we have

$$\begin{aligned} X\left(\sum_{i=1}^n \lambda_i t_i, \cdot\right) &\leqslant \sum_{i=1}^n \lambda_i X(t_i,\cdot) - C(\cdot)\sum_{i=1}^n \lambda_i (t_i - m)^2 + \varepsilon \\ &= \sum_{i=1}^n \lambda_i X(t_i,\cdot) + \varepsilon \sum_{i=1}^n \lambda_i - C(\cdot)\sum_{i=1}^n \lambda_i (t_i - m)^2 \\ &= \sum_{i=1}^n \lambda_i (X(t_i,\cdot) + \varepsilon) - C(\cdot)\sum_{i=1}^n \lambda_i (t_i - m)^2 \\ &= \sum_{i=1}^n \lambda_i Y(t_i,\cdot) - C(\cdot)\sum_{i=1}^n \lambda_i (t_i - m)^2 \quad \text{(a.e.)}. \end{aligned}$$

We apply Theorem 4 to the processes X and Y. There exists a strongly convex stochastic process Z_1 such that $X(t,\cdot) \leqslant Z_1(t,\cdot) \leqslant Y(t,\cdot)$ (a.e.). Putting $Z(t,\cdot) = Z_1(t,\cdot) - \frac{\varepsilon}{2}$, we get (9). □

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