Publ. Math. Debrecen
89/3 (2016), 373–387
DOI: 10.5486/PMD.2016.7441

Invariant means related to classical weighted means

By PETER KAHLIG (Vienna) and JANUSZ MATKOWSKI (Zielona Góra)

This paper is dedicated to Professor Zsolt Páles on the occasion of his 60th birthday

Abstract. Let A_t, H_t, and G_t denote, respectively, the two-variable weighted arithmetic, harmonic and geometric means with the weight $t \in (0,1)$. Fixing arbitrarily $s, t \in (0,1)$, and choosing for K one of these three means of weight s, and for M another mean of weight t, we examine when the function N satisfying the equality $K \circ (M,N) = K$ is a mean, that is when the mean K is (M,N)-invariant. The convergence of the iterates of (M,N) is considered. The obtained results are applied to find the invariant functions with respect to the suitable mean-type mappings.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and let $K, M, N : I^2 \to I$ be means. The mean K is called *invariant with respect to the mean-type mapping* $(M,N) : I^2 \to I^2$, briefly (M,N)-*invariant, if* $K \circ (M,N) = K$ ([13]). In the case when K is unique, it is called Gauss composition of the means M and N (cf. Z. Daróczy and Zs. Páles [4]). If K is a unique (M,N)-invariant mean, we say that N is a complementary mean to M with respect to K (briefly, N is K-complementary to M).

Recall that if M, N are continuous and strict means, then there exists a unique (M,N)-invariant mean (cf. [12], [16]).

Mathematics Subject Classification: Primary: 26E30, 39B12, 39B22.
Key words and phrases: invariant mean, complementary means, invariant function, mean-type mappings, iteration, convergence of iterates, functional equation.

Denote by $A:\mathbb{R}^2\to\mathbb{R}$, $G:(0,\infty)^2\to(0,\infty)$, $H\to(0,\infty)^2\to(0,\infty)$ the classical arithmetic, geometric and harmonic means

$$A(x,y)=\frac{x+y}{2},\quad G(x,y)=\sqrt{xy},\quad H(x,y)=\frac{2xy}{x+y}.$$

Since $G\circ(A,H)=G$, the geometric mean G is (A,H)-invariant, and the arithmetic and harmonic means are mutually complementary with respect to the geometric mean. Note that also $A\circ(H,N)=A$ where $N:(0,\infty)^2\to(0,\infty)$ given by

$$N(x,y):=\frac{x^2+y^2}{x+y}$$

is the *contra-harmonic mean* ([1], [2]); so the harmonic and contra-harmonic means are complementary with respect to the arithmetic mean A.

For $t\in(0,1)$ denote by $A_t:\mathbb{R}^2\to\mathbb{R}$, $H_t:(0,\infty)^2\to(0,\infty)$ and $G_t:(0,\infty)^2\to(0,\infty)$, the weighted arithmetic, harmonic and geometric means of weight t, given by

$$A_t(x,y):=tx+(1-t)y,\quad H_t(x,y):=\frac{xy}{(1-t)x+ty},\quad G_t(x,y):=x^ty^{1-t}.$$

In the present paper, fixing arbitrarily the weights $s,t\in(0,1)$, and choosing for K one of these three means of weight s, and for M a mean of weight t, we determine the functions N such that $K\circ(M,N)=K$ and examine when $N=N_{s,t}$ is a mean. In Sections 3 and 4 the cases when $K=H_s$ and $K=G_s$ are, respectively, considered. Some properties of the means $N_{s,t}$, and the convergence of the iterates of the suitable mean-type mappings are considered ([12], [15]). In each of these sections, we apply the obtained result to determine all functions (continuous on the diagonal) which are invariant with respect to the suitable mean-type mappings (see [8]).

It is interesting that the invariance problem is a special case of the equality considered by Z. Daróczy and J. Dascăl [3]:

$$L_\varphi^{(p,q)}=L_\varphi^{(r,1-r)},$$

where $L_\varphi^{(p,q)}$ is the *L-conjugate mean* $L_\varphi^{(p,q)}$ on I. This was introduced by Z. Daróczy and Zs. Páles in [5] and defined as follows: given a mean $L:I^2\to I$, a strictly monotonic continuous function $\varphi:I\to\mathbb{R}$, and the numbers $p,q\in(0,1)$,

$$L_\varphi^{(p,q)}(x,y):=\varphi^{-1}\left(p\varphi(x)+q\varphi(y)+(1-p-q)L(x,y)\right),\quad x,y\in I.$$

2. The case when the arithmetic mean is invariant

Theorem 1. *Let $s, t \in (0,1)$. Then*

(i) *a function $N : (0,\infty)^2 \to (0,\infty)$ satisfies the equation*

$$A_s \circ (A_t, N) = A_s$$

if, and only if, $N = N_{s,t}$ where

$$N_{s,t}(x,y) := \frac{s(1-t)}{1-s}x + \left(1 - \frac{s(1-t)}{1-s}\right)y, \quad x, y \in (0,\infty);$$

(ii) *the function $N_{s,t}$ is a mean if, and only if, $(s,t) \in W$, where*

$$W := \left(0, \frac{1}{2}\right] \times (0,1) \cup \left\{(s,t) : \frac{1}{2} < s < 1 \text{ and } \frac{2s-1}{s} \leq t\right\}; \tag{1}$$

(iii) *if $(s,t) \in W$, then $N_{s,t} = A_{\frac{s(1-t)}{1-s}}$ is a complementary mean to A_t with respect to A_s, and, moreover, the sequence $\left(A_t, A_{\frac{s(1-t)}{1-s}}\right)^n$ of iterates of the mean-type mapping $\left(A_t, A_{\frac{s(1-t)}{1-s}}\right)$ converges pointwise to the mean-type mapping (A_s, A_s):*

$$\lim_{n\to\infty}\left(A_t, A_{\frac{s(1-t)}{1-s}}\right)^n(x,y) = (A_s, A_s)(x,y), \quad x, y \in (0,\infty).$$

Proof. By the definition of the weighted arithmetic means, the equality $A_s \circ (A_t, N) = A_s$ is equivalent to

$$N = \frac{s(1-t)}{1-s}x + \left(1 - \frac{s(1-t)}{1-s}\right)y, \quad x, y \in (0,\infty),$$

which proves (i). The function N is a mean iff $0 < \frac{s(1-t)}{1-s} < 1$, which holds true iff either $s \in \left(0, \frac{1}{2}\right]$ and $t \in (0,1)$ or $s \in \left(\frac{1}{2}, 1\right)$ and $t \in \left(\frac{2s-1}{s}, 1\right)$. Thus $N_{s,t}$ is a mean iff $(s,t) \in W$, and clearly, $N_{s,t} = A_{\frac{s(1-t)}{1-s}}$.

Result (iii) follows from the definition of a complementary mean and Theorem 1 in [12]. □

Theorem 2. *Let* $s,t \in (0,1)$. *Then*

(i) *a function* $N : (0,\infty)^2 \to (0,\infty)$ *satisfies the equation*

$$A_s \circ (H_t, N) = A_s$$

if, and only if, $N = N_{s,t}$, *where*

$$N_{s,t}(x,y) = \frac{1}{1-s}\left[sx + (1-s)\,y - \frac{sxy}{(1-t)\,x + ty}\right], \quad x,y \in (0,\infty); \qquad (2)$$

(ii) *the function* $N_{s,t}$ *is a mean if, and only if,*

$$(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1);$$

(iii) *if* $(s,t) \in \left(0,\frac{1}{2}\right] \times (0,1)$, *then* $N_{s,t}$ *is complementary to* H_t *with respect to* A_s, *and, moreover, the sequence* $(H_t, N_{s,t})^n$ *of iterates of the mean-type mapping* $(H_t, N_{s,t})$ *converges pointwise to the mean-type mapping* (A_s, A_s)*:*

$$\lim_{n\to\infty} (H_t, N_{s,t})^n (x,y) = (A_s, A_s)(x,y), \quad x,y \in (0,\infty).$$

Proof. Part (i) is easy to verify as N satisfies equality $A_s \circ (H_t, N) = A_s$ iff N has form (2). Of course, we have

$$N(x,y) = \frac{1}{1-s}\left[y - s(y-x) - \frac{sxy}{x + t\,(y-x)}\right], \quad x,y \in (0,\infty).$$

The function $N = N_{s,t}$ is a mean iff for arbitrary $x,y \in (0,\infty)$,

$$\min(x,y) \le N(x,y) \le \max(x,y). \qquad (3)$$

Assume first that $x = \min(x,y) < \max(x,y) = y$. In this case, after some careful calculations, the first of these inequalities can be written in the form

$$(y-x)\,[t\,(1-s)\,y - (2s-1)\,(1-t)\,x] \ge 0.$$

Since $y-x$ is positive, this inequality is equivalent to

$$(2s-1)\,(1-t)\,x \le t\,(1-s)\,y,$$

and, obviously, it holds true if

$$s \in \left(0, \frac{1}{2}\right] \quad \text{and} \quad t \in (0,1)$$

(as then the left-hand side is non-positive and the right-hand side is positive). If $s > \frac{1}{2}$, taking into account that x and $y, x < y$, can be arbitrarily close, this inequality holds true for all $t \in (0,1)$ satisfying the inequality $(2s-1)(1-t) \leq t(1-s)$, that is, such that

$$s \in \left(\frac{1}{2}, 1\right) \quad \text{and} \quad t \geq \frac{2s-1}{s}.$$

The second of inequalities (3) holds true for all $s, t \in (0,1)$, as it is equivalent to the obvious inequality

$$x \leq H_t(x,y).$$

Now assume that $y = \min(x,y) < \max(x,y) = x$. In this case the first of inequalities (3) holds true for all $s, t \in (0,1)$, as it reduces to the obvious inequality

$$H_t(x,y) \leq x.$$

The second of inequalities (3) can be written in the form

$$(x-y)\{(2s-1)(1-t)x - t(1-s)y\} \leq 0,$$

and, as $x - y > 0$, it is equivalent to

$$(2s-1)(1-t)x \leq t(1-s)y.$$

Clearly, this inequality holds for all positive x, y, $x > y$, iff $s \leq \frac{1}{2}$, that is, iff

$$s \in \left(0, \frac{1}{2}\right] \quad \text{and} \quad t \in (0,1).$$

Summing up, the function $N_{s,t}$ is a mean if, and only if, $(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1)$, which completes the proof of (ii). From the definition of the complementary mean and Theorem 1 in [12] we get (iii). □

Theorem 3. *Let $s, t \in (0,1)$. Then*

(i) *a function $N : (0,\infty)^2 \to (0,\infty)$ satisfies the equation*

$$A_s \circ (G_t, N) = A_s$$

if, and only if, $N = N_{s,t}$, where

$$N_{s,t}(x,y) = \frac{1}{1-s}\left[sx + (1-s)y - sx^t y^{1-t}\right], \quad x, y \in (0,\infty); \tag{4}$$

(ii) *the function $N_{s,t}$ is a mean if, and only if,*

$$(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1);$$

(iii) *if $(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1)$, then $N_{s,t}$ is complementary to G_t with respect to A_s, and, moreover, the sequence $(G_t, N_{s,t})^n$ of iterates of the mean-type mapping $(H_t, N_{s,t})$ converges pointwise to the mean-type mapping (A_s, A_s):*

$$\lim_{n\to\infty} (G_t, N_{s,t})^n (x,y) = (A_s, A_s)(x,y), \quad x,y \in (0,\infty).$$

Proof. A simple calculation proves (i). For the proof of (ii), write the function (4) in the form

$$N(x,y) = \frac{1}{1-s}\left[y - s(y-x) - sx^t y^{1-t}\right], \quad x,y \in (0,\infty),$$

and take arbitrary $x,y \in (0,\infty)$. Assume that $x = \min(x,y) < \max(x,y) = y$. In this case, the first of inequalities (3) can be written in the form

$$sx^t\left(y^{1-t} - x^{1-t}\right) \le (1-s)(y-x),$$

or, equivalently, as $y - x > 0$,

$$sx^t \frac{y^{1-t} - x^{1-t}}{y-x} \le 1-s.$$

Since $t \in (0,1)$, the power function $y \to y^{1-t}$ is concave. It follows that

$$\sup\left\{\frac{y^{1-t} - x^{1-t}}{y-x} : y \in (x,\infty)\right\} = (1-t)x^{-t},$$

and, consequently, this inequality is satisfied iff

$$s(1-t) = sx^t(1-t)x^{-t} \le 1-s,$$

that is iff

$$t \ge \frac{2s-1}{s}. \tag{5}$$

The second inequality of (3) holds true for all $s,t \in (0,1)$, as it reduces to the following obvious inequality

$$\frac{x}{y} \le \left(\frac{x}{y}\right)^t.$$

Now assume that $y = \min(x,y) < \max(x,y) = x$. In this case the first of inequalities (3) is obvious, and the second one can be written in the form

$$sx^t\left(x^{1-t} - y^{1-t}\right) \le (1-s)(x-y),$$

or, equivalently, as $x > y$, in the form

$$sx^t\frac{x^{1-t} - y^{1-t}}{x-y} \le 1 - s.$$

The concavity of the power function $y \to y^{1-t}$ implies that

$$\sup\left\{\frac{x^{1-t} - y^{1-t}}{x-y} : y \in (0,x)\right\} = \lim_{y\to 0+}\frac{x^{1-t} - y^{1-t}}{x-y} = x^{-t},$$

whence, the above inequality holds true, iff

$$sx^t x^{-t} \le 1 - s,$$

that is iff $s \le \frac{1}{2}$.

Since $\frac{2s-1}{s} \le 0$ for all $s \le \frac{1}{2}$, we have $t \ge \frac{2s-1}{s}$, so inequality (5) is fulfilled for all $t \in (0,1)$. This concludes the proof of (ii). To prove (iii), we argue similarly as in previous results. □

Corollary 1. *Under the assumptions of Theorem 1, 2, 3, the complementary mean is symmetric iff* $s = t = \frac{1}{2}$.

Remark 1. In each of the three cases the complementary mean is homogeneous.

Remark 2. The complementary mean belongs to the class of quasi-arithmetic means only in the case of Theorem 1. This fact is also a consequence of some more general results, see [9], [10], [6] (cf. also [17], [18], [13], [4]).

2.1. Arithmetic means and invariant functions. For an interval $I \subset \mathbb{R}$ denote by $\Delta(I^2)$ the diagonal $\{(x,x) \in I : x \in I\}$.

Applying the above results and Theorem 1 on convergence of iterates of mean-type mapping ([16], [12], [14]), similarly as the suitable results in [11], we determine a broader class of invariant functions (cf. [8]). Namely, we prove the following

Theorem 4. *Let $s, t \in (0,1)$ be fixed.*

(i) *Suppose that (s,t) belongs to the set W defined by (1). Then a function $\Phi : \mathbb{R}^2 \to \mathbb{R}$, continuous on the diagonal $\Delta\left(\mathbb{R}^2\right)$, is invariant with respect to the mean-type mapping $\left(A_t, A_{\frac{s(1-t)}{1-s}}\right)$, i.e., satisfies the functional equation*

$$\Phi \circ \left(A_t, A_{\frac{s(1-t)}{1-s}}\right) = \Phi, \tag{6}$$

if, and only if, there exists a continuous function of a single variable $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\Phi = \varphi \circ A_s$.

(ii) *Suppose that $(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1)$ and let $N_{s,t}$ be a complementary mean to H_t with respect to A_s. Then a function $\Phi : (0,\infty)^2 \to (0,\infty)$, continuous on the diagonal, is invariant with respect to the mean-type mapping $(H_t, N_{s,t})$, i.e., satisfies the functional equation*

$$\Phi \circ (H_t, N_{s,t}) = \Phi,$$

if, and only if, there exists a continuous function of a single variable $\varphi : (0,\infty) \to (0,\infty)$ such that $\Phi = \varphi \circ A_s$.

(iii) *Suppose that $(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1)$ and let $N_{s,t}$ be a complementary mean to G_t with respect to A_s. Then a function $\Phi : (0,\infty)^2 \to (0,\infty)$, continuous on the diagonal, is invariant with respect to the mean-type mapping $(G_t, N_{s,t})$, i.e., satisfies the functional equation*

$$\Phi \circ (G_t, N_{s,t}) = \Phi,$$

if, and only if, there exists a continuous function of a single variable $\varphi : (0,\infty) \to (0,\infty)$ such that $\Phi = \varphi \circ A_s$.

Proof. To prove (i), take $(s,t) \in W$, where the set W is defined by (1).

First assume that a function $\Phi : I^k \to \mathbb{R}$ is continuous on the diagonal $\Delta(\mathbb{R}^2)$ and satisfies equation (6). Then, by induction, we have

$$\Phi = \Phi \circ \left(A_t, A_{\frac{s(1-t)}{1-s}}\right)^n, \quad n \in \mathbb{N},$$

where $\left(A_t, A_{\frac{s(1-t)}{1-s}}\right)^n$ denotes the n-th iterate of mean-type mapping $\left(A_t, A_{\frac{s(1-t)}{1-s}}\right)$. In view of Theorem 1, and Theorem 1 of [16], for all $x, y \in \mathbb{R}$,

$$\lim_{n\to\infty} \left(A_t, A_{\frac{s(1-t)}{1-s}}\right)^n (x,y) = \left(A_s\left(x,y\right), A_s\left(x,y\right)\right).$$

Hence, taking into account the continuity of Φ on the diagonal $\Delta(\mathbb{R}^2)$, and setting

$$\varphi(t) := \Phi\left(t,t\right), \quad \text{for} \quad t \in \mathbb{R},$$

we obtain, for all $x, y \in \mathbb{R}$,

$$\begin{aligned}\Phi\left(x,y\right) &= \lim_{n\to\infty} \Phi \circ \left(A_t, A_{\frac{s(1-t)}{1-s}}\right)^n (x,y) = \Phi\left(A_s\left(x,y\right), A_s\left(x,y\right)\right) \\ &= \varphi\left(A_s\left(x,y\right)\right) = \varphi \circ \left(A_s\right)\left(x,y\right),\end{aligned}$$

that is $\Phi = \varphi \circ A_s$, which completes the "only if" part of the theorem.

To prove the "if" part, take an arbitrary function $\varphi : \mathbb{R} \to \mathbb{R}$, and put $\Phi = \varphi \circ A_s$. Making use of the associativity of compositions and the invariance identity $A_s \circ \left(A_t, A_{\frac{s(1-t)}{1-s}}\right) = A_s$, guaranteed by Theorem 1, we have

$$\Phi \circ \left(A_t, A_{\frac{s(1-t)}{1-s}}\right) = (\varphi \circ A_s) \circ \left(A_t, A_{\frac{s(1-t)}{1-s}}\right) = \varphi \circ \left(A_s \circ \left(A_t, A_{\frac{s(1-t)}{1-s}}\right)\right) = \varphi \circ A_s = \Phi,$$

which was to be shown.

Since the proofs of results (ii) and (iii) are similar, we omit them. □

Remark 3. The proof of the "if" part of this theorem shows that any function of the form $\varphi \circ A_s$, even if φ is not continuous, is an invariant function with respect to the suitable mean-type mapping. The problem to determine all solutions of equation (6) is open.

3. The case when the harmonic mean is invariant

Arguing similarly as in Section 2, one can prove the following.

Theorem 5. *Let $s, t \in (0,1)$. Then*

(i) *a function $N : (0,\infty)^2 \to (0,\infty)$ satisfies the equation*

$$H_s \circ (A_t, N) = H_s$$

if, and only if, $N = N_{s,t}$, where

$$N_{s,t}\left(x,y\right) := \frac{(1-s)\,xy\,[tx + (1-t)\,y]}{[(1-s)\,x + sy]\,[tx + (1-t)\,y] - sxy}, \quad x, y \in (0,\infty); \tag{7}$$

(ii) *the function $N_{s,t}$ is a mean if, and only if,*

$$(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1);$$

(iii) *if* $(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1)$, *then* $N_{s,t}$ *is the complementary mean to* A_t *with respect to* H_s, *and, moreover, the sequence* $(A_t, N_{s,t})^n$ *of iterates of the mean-type mapping* $(A_t, N_{s,t})$ *converges pointwise to the mean-type mapping* (H_s, H_s):

$$\lim_{n\to\infty} (A_t, N_{s,t})^n (x,y) = (H_s, H_s)(x,y), \quad x,y \in (0,\infty).$$

Theorem 6. *Let* $s,t \in (0,1)$. *Then*

(i) *a function* $N : (0,\infty)^2 \to (0,\infty)$ *satisfies the equation*

$$H_s \circ (H_t, N) = H_s$$

if, and only if, $N = N_{s,t}$, *where*

$$N_{s,t}(x,y) := \frac{xy}{\left(1 - \frac{s(1-t)}{1-s}\right)x + \frac{s(1-t)}{1-s}y}, \quad x,y \in (0,\infty);$$

(ii) *the function* $N_{s,t}$ *is a mean if, and only if,* $N_{s,t} = H_{\frac{s(1-t)}{1-s}}$, *that is, if, and only if,* $(s,t) \in W$, *where* W *is defined by* (1);

(iii) *if* $(s,t) \in W$, *then* $H_{\frac{s(1-t)}{1-s}}$ *is the complementary mean to* H_t *with respect to* H_s, *and, moreover, the sequence* $\left(H_t, H_{\frac{s(1-t)}{1-s}}\right)^n$ *of iterates of the mean-type mapping* $\left(H_t, H_{\frac{s(1-t)}{1-s}}\right)$ *converges pointwise to the mean-type mapping* (H_s, H_s)

$$\lim_{n\to\infty} \left(H_t, H_{\frac{s(1-t)}{1-s}}\right)^n (x,y) = (H_s, H_s)(x,y), \quad x,y \in (0,\infty).$$

Theorem 7. *Let* $s,t \in (0,1)$. *Then*

(i) *a function* $N : (0,\infty)^2 \to (0,\infty)$ *satisfies the equation*

$$H_s \circ (G_t, N) = H_s$$

if, and only if, $N = N_{s,t}$, *where*

$$N_{s,t}(x,y) = \frac{(1-s)xy}{(1-s)x + sy - sx^{1-t}y^t}, \quad x,y \in (0,\infty); \tag{8}$$

(ii) *the function* $N_{s,t}$ *is a mean if, and only if,*

$$(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1);$$

(iii) *if* $(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1)$, *then* $N_{s,t}$ *is complementary to* G_t *with respect to* H_s, *and, moreover, the sequence* $(G_t, N_{s,t})^n$ *of iterates of the mean-type mapping* $(G_t, N_{s,t})$ *converges pointwise to the mean-type mapping* (H_s, H_s):

$$\lim_{n\to\infty} (G_t, N_{s,t})^n (x,y) = (H_s, H_s)(x,y), \quad x,y \in (0,\infty).$$

3.1. Harmonic means and invariant functions. Applying the results of the previous section, similarly as in Theorem 4, we obtain the following

Theorem 8. *Let $s, t \in (0,1)$ be fixed.*

(i) *Suppose that $(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1)$ and let $N_{s,t}$ be a complementary mean to A_t with respect to H_s (cf. formula (7)). Then a function $\Phi : (0,\infty)^2 \to \mathbb{R}$, continuous on the diagonal $\Delta\left((0,\infty)^2\right)$, is invariant with respect to the mean-type mapping $(A_t, N_{s,t})$, i.e.,*

$$\Phi \circ (A_t, N_{s,t}) = \Phi,$$

if, and only if, there exists a continuous function of a single variable $\varphi : (0,\infty) \to \mathbb{R}$ such that $\Phi = \varphi \circ H_s$.

(ii) *Suppose that (s,t) belongs to the set W defined by (1). Then a function $\Phi : (0,\infty)^2 \to (0,\infty)$, continuous on the diagonal, is invariant with respect to the mean-type mapping $\left(H_t, H_{\frac{s(1-t)}{1-s}}\right)$, i.e.,*

$$\Phi \circ \left(H_t, H_{\frac{s(1-t)}{1-s}}\right) = \Phi,$$

if, and only if, there exists a continuous function of a single variable $\varphi : (0,\infty) \to (0,\infty)$ such that $\Phi = \varphi \circ H_s$.

(iii) *Suppose that $(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1)$ and let $N_{s,t}$ be a complementary mean to G_t with respect to H_s (cf. formula (8)). Then a function $\Phi : (0,\infty)^2 \to (0,\infty)$, continuous on the diagonal, is invariant with respect to the mean-type mapping $(G_t, N_{s,t})$, i.e., satisfies the functional equation*

$$\Phi \circ (G_t, N_{s,t}) = \Phi,$$

if, and only if, there exists a continuous function of a single variable $\varphi : (0,\infty) \to (0,\infty)$ such that $\Phi = \varphi \circ H_s$.

4. The case when the geometric mean is invariant

Theorem 9. *Let $s, t \in (0,1)$ be fixed. Then*

(i) *a function $N : (0,\infty)^2 \to (0,\infty)$ satisfies the equation*

$$G_s \circ (A_t, N) = G_s$$

if, and only if, $N = N_{s,t}$, where

$$N_{s,t}(x,y) = \left(\frac{x}{tx + (1-t)y}\right)^{\frac{s}{1-s}} y, \quad x, y > 0; \tag{9}$$

(ii) *the function $N_{s,t}$ is a mean if, and only if,*

$$(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1);$$

(iii) *if $(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1)$, then $N_{s,t}$ is complementary to A_t with respect to G_s, and, moreover, the sequence $(A_t, N_{s,t})^n$ of iterates of the mean-type mapping $(A_t, N_{s,t})$ converges pointwise to the mean-type mapping (G_s, G_s)*

$$\lim_{n\to\infty} (A_t, N_{s,t})^n (x,y) = (G_s, G_s)(x,y), \quad x,y \in (0,\infty).$$

Remark 4. This result extends the classical harmony invariance. Taking here $s = t = \frac{1}{2}$, we obtain $N_{s,t} = H$ and the equality $G_s \circ (A_t, N_{s,t}) = G_s$ becomes the Pythagorean harmony proportion $G \circ (A, H) = G$ mentioned in the Introduction.

Theorem 10. *Let $s,t \in (0,1)$. Then*

(i) *a function $N : (0,\infty)^2 \to (0,\infty)$ satisfies the equation*

$$G_s \circ (H_t, N) = G_s$$

if, and only if, $N = N_{s,t}$, where

$$N_{s,t}(x,y) = [(1-t)x + ty]^{\frac{s}{1-s}} y^{1-\frac{s}{1-s}}, \quad x,y \in (0,\infty); \tag{10}$$

(ii) *the function $N_{s,t}$ is a mean if, and only if,*

$$(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1);$$

(iii) *if $(s,t) \in \left(0, \frac{1}{2}\right] \times (0,1)$, then $N_{s,t}$ is complementary to H_t with respect to G_s, and, moreover, the sequence $(H_t, N_{s,t})^n$ of iterates of the mean-type mapping $(H_t, N_{s,t})$ converges pointwise to the mean-type mapping (G_s, G_s)*

$$\lim_{n\to\infty} (H_t, N_{s,t})^n (x,y) = (G_s, G_s)(x,y), \quad x,y \in (0,\infty).$$

Remark 5. From formula (10) we obtain

$$N_{s,t} = G_{\frac{s}{1-s}} \circ (A_{1-t}, P),$$

where $P : (0,\infty)^2 \to (0,\infty)$ given by $P(x,y) = y$ is a projective mean.

Theorem 11. *Let $s,t \in (0,1)$. Then*

(i) *a function $N:(0,\infty)^2 \to (0,\infty)$ satisfies the equation*

$$G_s \circ (G_t, N) = G_s$$

if, and only if,$N = N_{s,t}$, where

$$N_{s,t}(x,y) = x^{\frac{s(1-t)}{1-s}} y^{1-\frac{s(1-t)}{1-s}}, \quad x,y \in (0,\infty);$$

(ii) *the function $N_{s,t}$ is a mean if, and only if, $N_{s,t} = G_{\frac{s(1-t)}{1-s}}$ and $(s,t) \in W$, where W is defined by (1);*

(iii) *if $(s,t) \in W$, then the sequence $(G_t, N_{s,t})^n$ of iterates of the mean-type mapping $(G_t, N_{s,t})$ converges pointwise to the mean-type mapping (G_s, HG_s)*

$$\lim_{n\to\infty} (G_t, N_{s,t})^n (x,y) = (G_s, G_s)(x,y), \quad x,y \in (0,\infty).$$

4.1. Geometric means and invariant functions. Analogously as in Theorems 4 and 8, we obtain the following

Theorem 12. *Let $s,t \in (0,1)$ be fixed.*

(i) *Suppose that $(s,t) \in \left(0,\frac{1}{2}\right] \times (0,1)$ and let $N_{s,t}$ be a complementary mean to A_t with respect to G_s (cf. formula (9)). Then a function $\Phi:(0,\infty)^2 \to \mathbb{R}$, continuous on the diagonal $\Delta\left((0,\infty)^2\right)$, is invariant with respect to the mean-type mapping $(A_t, N_{s,t})$, i.e.*

$$\Phi \circ (A_t, N_{s,t}) = \Phi,$$

if, and only if, there exists a continuous function of a single variable $\varphi : (0,\infty) \to \mathbb{R}$ such that $\Phi = \varphi \circ G_s$.

(ii) *Suppose that $(s,t) \in \left(0,\frac{1}{2}\right] \times (0,1)$ and let $N_{s,t}$ be a complementary mean to H_t with respect to G_s (cf. formula (10)). Then a function $\Phi:(0,\infty)^2 \to (0,\infty)$, continuous on the diagonal, is invariant with respect to the mean-type mapping $(H_t, N_{s,t})$, i.e., satisfies the functional equation*

$$\Phi \circ (H_t, N_{s,t}) = \Phi,$$

if, and only if, there exists a continuous function of a single variable $\varphi : (0,\infty) \to (0,\infty)$ such that $\Phi = \varphi \circ G_s$.

(iii) *Suppose that* (s, t) *belongs to the set* W *defined by* (1). *Then a function* $\Phi : (0, \infty)^2 \to (0, \infty)$, *continuous on the diagonal, is invariant with respect to the mean-type mapping* $\left(G_t, G_{\frac{s(1-t)}{1-s}}\right)$, *i.e.,*

$$\Phi \circ \left(G_t, G_{\frac{s(1-t)}{1-s}}\right) = \Phi,$$

if, and only if, there exists a continuous function of a single variable $\varphi : (0, \infty) \to (0, \infty)$ *such that* $\Phi = \varphi \circ G_s$.

Acknowledgements. The authors are indebted to the referees for calling attention to some strictly related bibliography items.

References

[1] P. S. Bullen, Handbook of Means and Their Inequalities, *Kluwer Academic Publishers, Dordrecht – Boston – London*, 2003.

[2] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, Means and Their Inequalities, Mathematics and its Applications (East European Series), Vol. **36**, *D. Reidel Publishing Company, Dordrecht – Boston – Lancaster – Tokyo*, 1988.

[3] Z. Daróczy and J. Dascăl, On the equality problem of conjugate means, *Results. Math.* **58** (2010), 69–79.

[4] Z. Daróczy and Zs. Páles, On functional equations involving means, *Publ. Math. Debrecen* **62** (2003), 363–377.

[5] Z. Daróczy and Zs. Páles, Generalized convexity and comparison of mean values, *Acta Sci. Math. (Szeged)* **71** (2005), 105–116.

[6] D. Głazowska, W. Jarczyk and J. Matkowski, Arithmetic mean as a linear combination of two quasi-arithmetic means, *Publ. Math. Debrecen* **61** (2002), 455–467.

[7] D. Głazowska and J. Matkowski, An invariance of the geometric mean with respect to Lagrangean means, *J. Math. Anal. Appl.* **31** (2007), 1187–1199.

[8] H. Izumi and J. Matkowski, Invariant translative mappings and a functional equation, *Acta Math. Hung.* **143** (2014), 220–231.

[9] J. Jarczyk and J. Matkowski, Invariance in the class of weighted quasi-arithmetic means, *Ann. Polon. Math.* **88** (2006), 39–51.

[10] J. Jarczyk, Invariance of weighted quasi-arithmetic means with continuous generators, *Publ. Math. Debrecen* **71** (2007), 279–294.

[11] P. Kahlig and J. Matkowski, Elliptic means and their generalizations, *Anz. Österreich. Akad. Wiss. Math.-Natur. Kl.* **139** (2003), 7–15.

[12] J. Matkowski, Iterations of mean-type mappings and invariant means, *Ann. Math. Siles.* **13** (1999), 211–226.

[13] J. Matkowski, Mean value property and associated functional equation, *Aequationes Math.* **58** (1999), 46–59.

[14] J. Matkowski, On iterations of means and functional equations, (ECIT'04), *Grazer Math. Ber.* **350** (2006), 184–197.

[15] J. Matkowski, Iterations of the mean-type mappings, (ECIT'08), A. N. Sharkovsky, I. M. Sushko (Eds.), *Grazer Math. Ber.* **354** (2009), 158–179.

[16] J. Matkowski, Iterations of the mean-type mappings and uniqueness of invariant means, *Ann. Univ. Sci. Budapest. Sect. Comput.* **41** ((2013)), 145–158.

[17] O. Sutô, Studies on some functional equations I, *Tôhoku Math. J.* **6** (1914), 1–15.

[18] O. Sutô, Studies on some functional equations II, *Tôhoku Math. J.* **6** (1914), 82–101.

PETER KAHLIG
SCIENCE POOL VIENNA
SECT. HYDROMETEOROLOGY
ALTMANNSDORFER STR. 21/5/2
A 1120 VIENNA
AUSTRIA

E-mail: peter.kahlig@gmx.at

JANUSZ MATKOWSKI
FACULTY OF MATHEMATICS
COMPUTER SCIENCE AND ECONOMETRICS
UNIVERSITY OF ZIELONA GÓRA
SZAFRANA 4A
PL 65-516, ZIELONA GÓRA
POLAND

E-mail: j.matkowski@wmie.uz.zgora.pl

(Received July 18, 2015; revised January 26, 2016)