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A minimal set of cancellation violating sequences for finite two-dimensional non-additive measurement

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This paper is dedicated to Professor Zsolt Páles on the occasion of his 60th birthday

Abstract. A weak order $\precsim$ on a finite two-dimensional Cartesian product set $X = X_1 \times X_2$ has an additive real-valued representation if and only if it satisfies a sequence of cancellation conditions $C(2), C(3), \ldots$. Given fixed cardinalities m and n for X_1 and X_2, there is a largest K, denoted by $f(m,n)$, such that some $\precsim$ on X satisfies $C(2)$ to $C(K-1)$ but violates $C(K)$. In 2001, Fishburn presented several open problems, including the exact values of $f(m,n)$ for some small (m,n). Recently, by giving a minimal chain of cancellation violating sequences adequate for the detection of all non-additively representable weak orders for $(m,n) = (3,3)$, $(3,4)$ and $(3,5)$, Ng shows that $f(3,5) = 4$. This article is a continuation of the above work for $(m,n) = (3,6)$.

1. Introduction

A binary relation $\precsim$ on the Cartesian product $X = X_1 \times X_2$ is said to be additively representable (or additive) if there exist real-valued functions u_i on X_i $(i = 1,2)$ such that, for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X,

$$x \precsim y \quad \text{iff} \quad u_1(x_1) + u_2(x_2) \leq u_1(y_1) + u_2(y_2).$$

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A function $f : X \to \mathbb{R}$ is said to be additive if there exist functions $u_i : X_i \to \mathbb{R}$ such that

$$f(x) = \sum_{i=1}^{2} u_i(x_i), \quad \forall x = (x_1, x_2) \in X.$$

A utility function representing $\precsim$ is any function $u : X \to \mathbb{R}$ such that

$$x \precsim y \quad \text{iff} \quad u(x) \leq u(y), \quad \forall x, y \in X.$$

It is clear that $\precsim$ is a weak order by additive representability. On a finite set it is equivalent to the presence of a utility representation.

Definition 1.1. Given $X = X_1 \times X_2$, integer $K \geq 2$, a sequence of distinct pairs $(x^1, y^1), (x^2, y^2), \ldots, (x^K, y^K) \in X \times X$, and positive integers $a_1, a_2, \ldots, a_K$, the expression

$$\sum_k a_k(x^k, y^k) \in \mathcal{C}_K$$

means that for the two coordinates $i = 1, 2$, the sequence $a_1 x_i^1$ [i.e. x_i^1 repeated a_1 times], $a_2 x_i^2, \ldots, a_K x_i^K$ is a permutation of the sequence $a_1 y_i^1, a_2 y_i^2, \ldots, a_K y_i^K$ in the coordinate set X_i. We shall say that the sequence $\sum_k a_k(x^k, y^k)$ is balanced.

We call $\sum_k a_k(y^k, x^k)$ the co-sequence of $\sum_k a_k(x^k, y^k)$ and it is clear that a sequence is balanced if and only if its co-sequence is balanced.

The K-th order cancellation condition on $\precsim$ is $C(K)$: for all $\sum_k a_k(x^k, y^k) \in \mathcal{C}_K$ it is false that $x^1 \precsim y^1, x^2 \precsim y^2, \ldots, x^K \precsim y^K$ and $x^k \prec y^k$ for at least one k.

In [6], $\sum_k a_k$ is called the cardinality of the sequence and K is called its width. Krantz *et al.* [4] show that every additively representable order satisfies $C(K)$ for all K whence a non-additive order violates $C(K)$ for some K. By the definition of $C(2)$, when it holds, it induces well-defined weak orders $\precsim_i$ on the coordinate sets X_i by $x_i \precsim_i y_i$ if $x \precsim y$ whenever $x_j = y_j$ for all $j \neq i$. The associated indifference relations $x_i \sim_i y_i$, defined by $x_i \precsim_i y_i$ and $y_i \precsim_i x_i$, are equivalence relations on X_i. The quotient spaces $X_i^* = X_i / \sim_i$ are then linearly ordered. We henceforth assume that the coordinate sets X_i are linearly ordered under the induced orders.

Definition 1.2. Let $X = X_1 \times X_2$ be of finite size m by n ($|X_1| = m$ and $|X_2| = n$), where $m, n \geq 2$. The unique $K \geq 2$ such that (i) every weak order on X that satisfies $C(2)$, $C(3)$, ..., $C(K)$ is additively representable and (ii) some weak order on X that satisfies all cancellation conditions (if any) prior to $C(K)$ is not additively representable will be denoted by $f(m, n)$.

It is obtained in [4] that $f(2, n) = 2$ and $f(3, 3) \geq 3$. In his pivotal papers, Fishburn gives many significant results, including the conclusions $f(3, 3) = 3$, $f(3, 4) = f(4, 4) = 4$, and $4 \leq f(3, 5) \leq 7$. Recently, by determining a minimal set of cancellation violating sequences adequate for the detection of all non-additive weak orders on a product of size 3 by 5, NG in [2] gets the exact value of $f(3, 5)$. In this paper, we continue the study for $(m, n) = (3, 6)$ and make explicit a minimal set of cancellation violating sequences to detect all non-additively representable weak orders on a 3 by 6 product.

2. The canonical representation and the diagrams for $C(K)$ violations

Let $\precsim$ be a weak order on a 3 by 6 product satisfying $C(2)$. We can map X_1 to $\{1, 2, 3\}$ and X_2 to $\{1, 2, 3, 4, 5, 6\}$ in such a way that the induced linear orders on X_i coincide with the natural order of the integers. In this sense we identify X_1 with $\{1, 2, 3\}$ and X_2 with $\{1, 2, 3, 4, 5, 6\}$. The weak order $\precsim$ is then representable by a utility function $u : \{1, 2, 3\} \times \{1, 2, 3, 4, 5, 6\} \to \mathbb{R}$. Then u is strictly increasing in its two variables. We may choose to use consecutive integer values and set $u(1, 1) = 1$. Such choice will make u unique for the given weak order and will be called the canonical representation of the weak order. If the order is linear, then there will be 18 distinct values and the canonical representation will start with $u(1, 1) = 1$, and end with $u(3, 6) = 18$. Each function u, canonical or otherwise, can be presented as a 6 by 3 array

$$A = \begin{bmatrix} u(1,6) & u(2,6) & u(3,6) \\ u(1,5) & u(2,5) & u(3,5) \\ u(1,4) & u(2,4) & u(3,4) \\ u(1,3) & u(2,3) & u(3,3) \\ u(1,2) & u(2,2) & u(3,2) \\ u(1,1) & u(2,1) & u(3,1) \end{bmatrix}.$$

Any array with strictly increasing rows and columns, not necessarily canonical, still represents a unique weak order.

From the known results in [2], NG shows that non-additive 3 by 3, 4 by 3 or 5 by 3 arrays are detected by a specific chain of sets of balanced sequences presented through diagrams. For instance, Figure 1, Figure 2 and Figure 3 detect all 5 by 3 non-additive arrays, where $\oplus$ and $\ominus$ signs mark the opposite sides in the comparison.

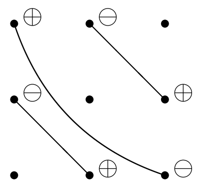
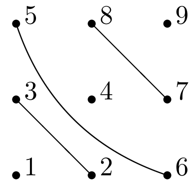


Figure 1.



An order detected by Fig. 1.

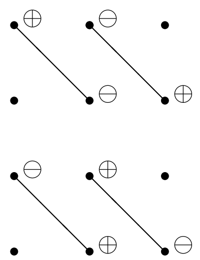


Figure 2.

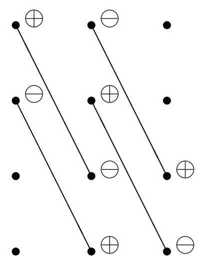


Figure 3.

Figure 1 carries two sequences: a balanced sequence $(x^1, y^1) = ((1,2),(2,1))$, $(x^2, y^2) = ((3,1),(1,3))$, $(x^3, y^3) = ((2,3),(3,2))$ (and with implicit multiplicities $a_1 = a_2 = a_3 = 1$) and its co-sequence. The illustration next to Figure 1 conveys the statement that the array with the nine indicated values is non-additive because it fails $C(3)$ by a balanced sequence carried in Figure 1. In this sense we say Figure 1 detects (the non-additivity of) the (weak order represented by the) array.

It is clear that if any 5 by 3 subarray is non-additive, the 6 by 3 array itself is non-additive, and so will be detected by Figures 1–3. Hence, we need to pay attention to those 6 by 3 arrays that are non-additive and have all 5 by 3 subarrays additive. We call them the critical-to-inspect ones.

Step 1. Using the Maple software we obtain all weak orders in canonical arrays. With input argument ([1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18]) in a procedure, our worksheet returns all (87516) linear orders (cf. [5]). By the same worksheet, we also collect non-linear orders. For example, with input argument ([1,2,2,3,3,3,4,4,4,5,5,5,6,6,7,8,9,10]) it returns all (8) arrays that come with a double 2, a triple 3, a triple 4, a triple 5 and a double 6.

Step 2. We obtain the sublist of (63903) linear orders which are non-additive.

Among them, (40) are found critical-to-inspect. They are

$$\left[\begin{bmatrix} 12 & 17 & 18 \\ 11 & 15 & 16 \\ 10 & 13 & 14 \\ 4 & 7 & 9 \\ 2 & 5 & 8 \\ 1 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 12 & 17 & 18 \\ 11 & 15 & 16 \\ 9 & 13 & 14 \\ 4 & 7 & 10 \\ 2 & 5 & 8 \\ 1 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 12 & 17 & 18 \\ 11 & 15 & 16 \\ 7 & 13 & 14 \\ 4 & 9 & 10 \\ 2 & 5 & 8 \\ 1 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 12 & 17 & 18 \\ 11 & 15 & 16 \\ 8 & 13 & 14 \\ 4 & 7 & 10 \\ 2 & 5 & 9 \\ 1 & 3 & 6 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 12 & 17 & 18 \\ 11 & 15 & 16 \\ 7 & 13 & 14 \\ 4 & 8 & 10 \\ 2 & 5 & 9 \\ 1 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 12 & 17 & 18 \\ 10 & 15 & 16 \\ 7 & 13 & 14 \\ 4 & 8 & 11 \\ 2 & 5 & 9 \\ 1 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 11 & 17 & 18 \\ 8 & 15 & 16 \\ 5 & 12 & 14 \\ 4 & 9 & 13 \\ 2 & 6 & 10 \\ 1 & 3 & 7 \end{bmatrix}, \begin{bmatrix} 12 & 17 & 18 \\ 11 & 14 & 16 \\ 8 & 13 & 15 \\ 4 & 7 & 10 \\ 2 & 5 & 9 \\ 1 & 3 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 9 & 17 & 18 \\ 6 & 14 & 16 \\ 5 & 11 & 15 \\ 3 & 10 & 13 \\ 2 & 7 & 12 \\ 1 & 4 & 8 \end{bmatrix}, \begin{bmatrix} 13 & 16 & 18 \\ 10 & 15 & 17 \\ 8 & 12 & 14 \\ 3 & 9 & 11 \\ 2 & 6 & 7 \\ 1 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 13 & 16 & 18 \\ 11 & 14 & 17 \\ 10 & 12 & 15 \\ 3 & 8 & 9 \\ 2 & 6 & 7 \\ 1 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 13 & 16 & 18 \\ 11 & 14 & 17 \\ 10 & 12 & 15 \\ 5 & 6 & 9 \\ 3 & 4 & 8 \\ 1 & 2 & 7 \end{bmatrix},$$

$$\begin{bmatrix} 13 & 16 & 18 \\ 11 & 14 & 17 \\ 9 & 12 & 15 \\ 3 & 8 & 10 \\ 2 & 6 & 7 \\ 1 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 13 & 16 & 18 \\ 11 & 14 & 17 \\ 9 & 12 & 15 \\ 5 & 6 & 10 \\ 3 & 4 & 8 \\ 1 & 2 & 7 \end{bmatrix}, \begin{bmatrix} 13 & 16 & 18 \\ 10 & 14 & 17 \\ 9 & 12 & 15 \\ 3 & 8 & 11 \\ 2 & 6 & 7 \\ 1 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 13 & 16 & 18 \\ 10 & 14 & 17 \\ 9 & 12 & 15 \\ 3 & 7 & 11 \\ 2 & 6 & 8 \\ 1 & 4 & 5 \end{bmatrix},$$

$$\begin{bmatrix} 13 & 16 & 18 \\ 10 & 14 & 17 \\ 9 & 12 & 15 \\ 5 & 6 & 11 \\ 3 & 4 & 8 \\ 1 & 2 & 7 \end{bmatrix}, \begin{bmatrix} 13 & 16 & 18 \\ 10 & 14 & 17 \\ 9 & 12 & 15 \\ 4 & 6 & 11 \\ 3 & 5 & 8 \\ 1 & 2 & 7 \end{bmatrix}, \begin{bmatrix} 13 & 16 & 18 \\ 10 & 14 & 17 \\ 8 & 12 & 15 \\ 3 & 9 & 11 \\ 2 & 6 & 7 \\ 1 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 13 & 16 & 18 \\ 10 & 14 & 17 \\ 7 & 12 & 15 \\ 3 & 9 & 11 \\ 2 & 6 & 8 \\ 1 & 4 & 5 \end{bmatrix},$$

$$\begin{bmatrix} 13 & 16 & 18 \\ 11 & 14 & 17 \\ 9 & 10 & 15 \\ 5 & 6 & 12 \\ 3 & 4 & 8 \\ 1 & 2 & 7 \end{bmatrix}, \begin{bmatrix} 13 & 16 & 18 \\ 10 & 14 & 17 \\ 9 & 11 & 15 \\ 5 & 6 & 12 \\ 3 & 4 & 8 \\ 1 & 2 & 7 \end{bmatrix}, \begin{bmatrix} 13 & 16 & 18 \\ 10 & 14 & 17 \\ 8 & 11 & 15 \\ 5 & 6 & 12 \\ 3 & 4 & 9 \\ 1 & 2 & 7 \end{bmatrix}, \begin{bmatrix} 12 & 16 & 18 \\ 8 & 14 & 17 \\ 5 & 11 & 15 \\ 3 & 10 & 13 \\ 2 & 7 & 9 \\ 1 & 4 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 12 & 16 & 18 \\ 9 & 13 & 17 \\ 6 & 10 & 15 \\ 5 & 7 & 14 \\ 3 & 4 & 11 \\ 1 & 2 & 8 \end{bmatrix}, \begin{bmatrix} 10 & 16 & 18 \\ 6 & 13 & 17 \\ 4 & 12 & 15 \\ 3 & 9 & 14 \\ 2 & 8 & 11 \\ 1 & 5 & 7 \end{bmatrix}, \begin{bmatrix} 7 & 16 & 18 \\ 6 & 13 & 17 \\ 4 & 12 & 15 \\ 3 & 9 & 14 \\ 2 & 8 & 11 \\ 1 & 5 & 10 \end{bmatrix}, \begin{bmatrix} 14 & 15 & 18 \\ 12 & 13 & 17 \\ 10 & 11 & 16 \\ 4 & 7 & 9 \\ 2 & 5 & 8 \\ 1 & 3 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 14 & 15 & 18 \\ 12 & 13 & 17 \\ 9 & 11 & 16 \\ 4 & 7 & 10 \\ 2 & 5 & 8 \\ 1 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 14 & 15 & 18 \\ 12 & 13 & 17 \\ 8 & 11 & 16 \\ 4 & 7 & 10 \\ 2 & 5 & 9 \\ 1 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 14 & 15 & 18 \\ 12 & 13 & 17 \\ 8 & 10 & 16 \\ 5 & 7 & 11 \\ 2 & 4 & 9 \\ 1 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 14 & 15 & 18 \\ 12 & 13 & 17 \\ 8 & 10 & 16 \\ 4 & 7 & 11 \\ 2 & 5 & 9 \\ 1 & 3 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 14 & 15 & 18 \\ 11 & 13 & 17 \\ 8 & 12 & 16 \\ 4 & 7 & 10 \\ 2 & 5 & 9 \\ 1 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 14 & 15 & 18 \\ 11 & 13 & 17 \\ 8 & 10 & 16 \\ 4 & 7 & 12 \\ 2 & 5 & 9 \\ 1 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 8 & 15 & 18 \\ 5 & 14 & 17 \\ 4 & 11 & 16 \\ 3 & 10 & 13 \\ 2 & 7 & 12 \\ 1 & 6 & 9 \end{bmatrix}, \begin{bmatrix} 13 & 15 & 18 \\ 10 & 12 & 17 \\ 6 & 9 & 16 \\ 4 & 8 & 14 \\ 2 & 5 & 11 \\ 1 & 3 & 7 \end{bmatrix},$$

$$\begin{bmatrix} 11 & 15 & 18 \\ 7 & 12 & 17 \\ 6 & 9 & 16 \\ 4 & 8 & 14 \\ 3 & 5 & 13 \\ 1 & 2 & 10 \end{bmatrix}, \begin{bmatrix} 12 & 14 & 18 \\ 8 & 11 & 17 \\ 5 & 10 & 16 \\ 4 & 7 & 15 \\ 2 & 6 & 13 \\ 1 & 3 & 9 \end{bmatrix}, \begin{bmatrix} 9 & 14 & 18 \\ 8 & 11 & 17 \\ 5 & 10 & 16 \\ 4 & 7 & 15 \\ 2 & 6 & 13 \\ 1 & 3 & 12 \end{bmatrix}, \begin{bmatrix} 10 & 13 & 18 \\ 7 & 12 & 17 \\ 6 & 9 & 16 \\ 3 & 8 & 15 \\ 2 & 5 & 14 \\ 1 & 4 & 11 \end{bmatrix}\Bigg].$$

None of the (8) arrays that come with a double 2, a triple 3, a triple 4, a triple 5 and a double 6 are critical-to-inspect.

Step 3. We find that Figures 4–11 detect all critical-to-inspect 3 by 6 arrays.

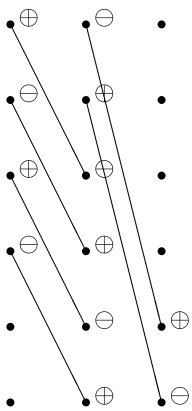


Figure 4.

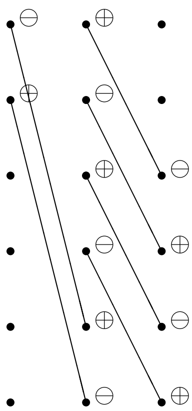


Figure 5.

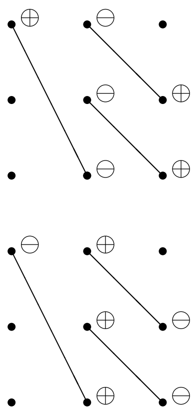


Figure 6.

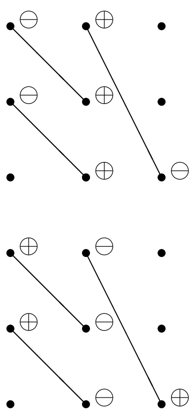


Figure 7.

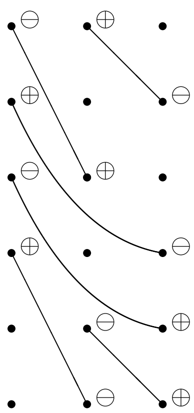


Figure 8.

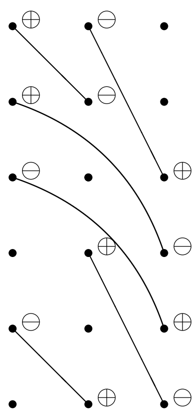


Figure 9.

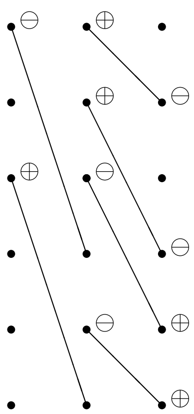


Figure 10.

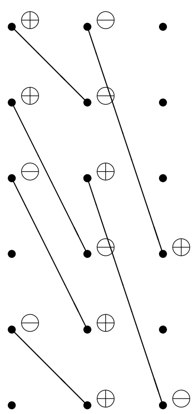


Figure 11.

Note: some arrays are detected by more than one figure, e.g. the fourth array is detected by both Figure 6 and Figure 8. Each figure detects at least one array which is not detected by the others – in this sense the family is a minimal set.

Steps 1 to 3 lead us to a conclusion – that the set of cancellation violating sequences presented by Figures 1–11 is a minimal collection detecting all non-additive 3 by 6 weak orders.

The widths of the sequences do not exceed six and so $f(3,6) \leq 6$ follows. On the other hand, a theorem of FISHBURN ([1, Theorem 3.1]) on lower bound is found applicable to the sequences in Figures 4–11 and we get $f(3,6) \geq 6$. Hence $f(3,6) = 6$.

3. Further works

Our determination of $f(3,6)$ assisted by the Maple software is very time demanding. For example, there are (87516) linear orders and (12441) arrays with a double 2 and a double 3. Time taken to handle all non-linear orders exceeds that for the linear ones by far. The following conjecture by Fishburn remains open.

(P1): if we define the function $f^*(m,n)$ by restricting Definition 1.2 to linear orders, is it true that $f(m,n) = f^*(m,n)$? (see Conjecture 1 in [3] and Problem 2 in [6]).

We have only confirmed here that $f(3,6) = f^*(3,6)$.

4. Supporting worksheets and data

All of our Maple Worksheets and data can be downloaded from Portal http://hdl.handle.net/10864/11822.

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