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J-tangent affine hyperspheres with an involutive contact distribution

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Abstract. In this paper we study J-tangent affine hyperspheres. The main purpose of this paper is to give a local characterization of J-tangent affine hyperspheres of arbitrary dimension with an involutive contact distribution. Some new examples of such hyperspheres are also given.

1. Introduction

Centro-affine real hypersurfaces with a J-tangent transversal vector field were first studied by V. Cruceanu in [2]. He proved that such hypersurfaces $f\colon M^{2n+1} \to \mathbb{C}^{n+1}$ can be locally expressed in the form

$$f(x_1, \dots, x_{2n}, z) = Jg(x_1, \dots, x_{2n}) \cos z + g(x_1, \dots, x_{2n}) \sin z,$$

where g is some smooth function defined on an open subset of $\mathbb{R}^{2n}$. He also showed that if the induced almost contact structure is Sasakian, then a hypersurface must be a hyperquadric. The latter result was generalized in [5] to arbitrary hypersurfaces with a J-tangent transversal vector field.

Since the class of centro-affine hypersurfaces with a J-tangent transversal vector field is quite large, the question arises whether there are affine hyperspheres with a J-tangent Blaschke field. A local characterization of 3-dimensional J-tangent affine hyperspheres with an involutive contact distribution was given

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in [7]. The main purpose of this paper is to generalize the results from [7] to an arbitrary dimension. That is, we give a local characterization of J-tangent affine hyperspheres of arbitrary dimension with an involutive contact distribution.

2. Preliminaries

We briefly recall the basic formulas of affine differential geometry. For more details, we refer to [4].

Let $f\colon M \to \mathbb{R}^{n+1}$ be an orientable connected differentiable n-dimensional hypersurface, immersed in the affine space $\mathbb{R}^{n+1}$, equipped with its usual flat connection D. Then for any transversal vector field C we have

$$\mathrm{D}_X f_* Y = f_*(\nabla_X Y) + h(X,Y)C \tag{1}$$

and

$$\mathrm{D}_X C = -f_*(SX) + \tau(X)C, \tag{2}$$

where X, Y are vector fields tangent to M. It is known that ∇ is a torsion-free connection, h is a symmetric bilinear form on M, called *the second fundamental form*, S is a tensor of type $(1,1)$, called *the shape operator*, and τ is a 1-form, called *the transversal connection form.* Recall that formula (1) is known as formula of Gauss, and formula (2) is known as formula of Weingarten.

For a hypersurface immersion $f\colon M \to \mathbb{R}^{n+1}$, a transversal vector field C is said to be *equiaffine* (resp. *locally equiaffine*) if $\tau = 0$ (resp. $d\tau = 0$). For an affine hypersurface $f\colon M \to \mathbb{R}^{n+1}$ with a transversal vector field C, we consider the following volume element on M:

$$\Theta(X_1, \ldots, X_n) = \det[f_*X_1, \ldots, f_*X_n, C]$$

for all $X_1, \ldots, X_n \in \mathcal{X}(M)$. We call Θ *the induced volume element* on M. Immersion $f\colon M \to \mathbb{R}^{n+1}$ is said to be a *centro-affine hypersurface* if the position vector x (from origin o) for each point $x \in M$ is transversal to the tangent plane of M at x. In this case $S = I$ and $\tau = 0$. If h is nondegenerate (that is h defines a semi-Riemannian metric on M), then we say that the hypersurface or the hypersurface immersion is *nondegenerate.* In this paper we assume that f is always nondegenerate. We have the following

Theorem 2.1 ([4], Fundamental equations)**.** *For an arbitrary transversal vector field C the induced connection ∇, the second fundamental form h, the shape operator S, and the 1-form τ satisfy the following equations:*

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \tag{3}$$

$$(\nabla_X h)(Y, Z) + \tau(X)h(Y, Z) = (\nabla_Y h)(X, Z) + \tau(Y)h(X, Z), \tag{4}$$

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \tag{5}$$

$$h(X, SY) - h(SX, Y) = 2d\tau(X, Y). \tag{6}$$

The equations (3), (4), (5), and (6) are called the equations of Gauss, Codazzi for h, Codazzi for S, and Ricci, respectively.

When f is nondegenerate, there exists a canonical transversal vector field C, called *the affine normal* (or *the Blaschke field*). The affine normal is uniquely determined up to sign by the following conditions:

(1) the metric volume form ω_h of h is ∇-parallel,

(2) ω_h coincides with the induced volume form Θ.

Recall that ω_h is defined by

$$\omega_h(X_1, \ldots, X_n) = |\det[h(X_i, X_j)]|^{1/2},$$

where $\{X_1, \ldots, X_n\}$ is any positively oriented basis relative to the induced volume form Θ. The affine immersion f with a Blaschke field C is called *a Blaschke hypersurface*. In this case, fundamental equations can be rewritten as follows

Theorem 2.2 ([4], Fundamental equations)**.** *For a Blaschke hypersurface f, we have the following fundamental equations:*

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY, \qquad (\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z),$$

$$(\nabla_X S)(Y) = (\nabla_Y S)(X), \qquad h(X, SY) = h(SX, Y).$$

A Blaschke hypersurface is called *an affine hypersphere* if $S = \lambda I$, where $\lambda = \text{const}$.

If $\lambda = 0$, f is called *an improper affine hypersphere*, if $\lambda \neq 0$, a hypersurface f is called *a proper affine hypersphere.*

Now, we will recall a notion of complex affine hypersurfaces, for details, we refer to [3]. We always assume that $\mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}$ is endowed with the standard complex structure J. That is

$$J(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}) = (-y_1, \ldots, -y_{n+1}, x_1, \ldots, x_{n+1}).$$

Let $g\colon M \to \mathbb{R}^{2n+2}$ be a *complex hypersurface* of the complex affine space $\mathbb{R}^{2n+2}$, that is for each point p of M we have $J(T_pM) = T_pM$. The complex structure J induces a complex structure on M, which we will also denote by J. Let

$\zeta\colon M \to T\mathbb{R}^{2n+2}$ be a local transversal vector field on M. Then $\zeta(p), J\zeta(p)$ and T_pM together span $T_p\mathbb{R}^{2n+2}$. Consequently, for all tangent vector fields X and Y to M, we can decompose $\mathrm{D}_X\, g_*Y$ and $\mathrm{D}_X\,\zeta$ into a component tangent to M and into a component lying in the plane spanned by ζ and $J\zeta$:

$$\mathrm{D}_X\, g_*Y = g_*(\widetilde{\nabla}_X Y) + h_1(X,Y)\zeta + h_2(X,Y)J\zeta \quad \text{(formula of Gauss)},$$

$$\mathrm{D}_X\,\zeta = -g_*(\widetilde{S}X) + \tau_1(X)\zeta + \tau_2(X)J\zeta \quad \text{(formula of Weingarten)},$$

where $\widetilde{\nabla}$ is a torsion free affine connection on M, h_1 and h_2 are symmetric bilinear forms on M, $\widetilde{S}$ is a $(1,1)$-tensor field on M, and τ_1 and τ_2 are 1-forms on M. We have the following relations between h_1 and h_2.

Lemma 2.3 ([3])**.**

$$h_1(X,JY) = h_1(JX,Y) = -h_2(X,Y), \qquad h_2(X,JY) = h_2(JX,Y) = h_1(X,Y).$$

On manifold M we define the volume form θ_ζ by

$$\theta_\zeta(X_1,\dots,X_{2n}) = \det(g_*X_1,\dots,g_*X_{2n},\zeta,J\zeta)$$

for tangent vectors X_i (i=1,...,2n). Then, consider the function H_ζ on M defined by

$$H_\zeta = \det[h_1(X_i,X_j)]_{i,j=1\dots 2n},$$

where $X_1,\dots,X_{2n}$ is a local basis in TM such that $\theta_\zeta(X_1,\dots,X_{2n}) = 1$. This definition is independent of the choice of basis. We say that a hypersurface is *nondegenerate* if h_1 (and in consequence h_2) is nondegenerate. When g is nondegenerate, there exist transversal vector fields ζ satisfying the following two conditions:

$$|H_\zeta| = 1, \qquad \tau_1 = 0.$$

Such vector fields are called affine normal vector fields. First condition is a kind of normalization and the second condition implies that $\widetilde{\nabla}\theta_\zeta = 0$. We observe that any transversal vector field $\widetilde{\zeta}$ can be written as

$$\widetilde{\zeta} = \varphi\zeta + \psi J\zeta + Z,$$

where φ and ψ are functions on M such that $\varphi^2+\psi^2 \neq 0$, and where Z is tangent to M.

A nondegenerate complex hypersurface is said to be a *proper complex affine hypersphere* if there exists an affine normal vector field ζ such that $S = \alpha I$, where $\alpha \in \mathbb{R}\setminus\{0\}$ and $\tau_2 = 0$. If there exists an affine normal vector field ζ such that $S = 0$ and $\tau_2 = 0$, we talk about an *improper affine hypersphere*.

To simplify the writing, sometimes we will omit g_* and/or f_* in front of vector fields.

3. Induced almost contact structures

Let $\dim M = 2n+1$ and $f\colon (M,g) \to (\mathbb{R}^{2n+2}, \tilde{g})$ be a nondegenerate isometric immersion, where $\tilde{g}$ is the standard inner product on $\mathbb{R}^{2n+2}$. Let C be a transversal vector field on M. We say that C is *J-tangent* if $JC_x \in f_*(T_xM)$ for every $x \in M$. We also define a distribution $\mathcal{D}$ on M as the biggest J invariant distribution on M, that is,

$$\mathcal{D}_x = f_*^{-1}(f_*(T_xM) \cap J(f_*(T_xM)))$$

for every $x \in M$. It is clear that $\dim \mathcal{D} = 2n$. A vector field X is called a *$\mathcal{D}$-field* if $X_x \in \mathcal{D}_x$ for every $x \in M$. We use the notation $X \in \mathcal{D}$ for vectors as well as for $\mathcal{D}$-fields. We say that the distribution $\mathcal{D}$ is nondegenerate if h is nondegenerate on $\mathcal{D}$.

First, recall [1] that a $(2n+1)$-dimensional manifold M is said to have an *almost contact structure* if there exist on M a tensor field φ of type (1,1), a vector field ξ and a 1-form η which satisfy

$$\varphi^2(X) = -X + \eta(X)\xi, \qquad \eta(\xi) = 1$$

for every $X \in TM$.

Let $f\colon M \to \mathbb{R}^{2n+2}$ be a nondegenerate hypersurface with a J-tangent transversal vector field C. Then we can define a vector field ξ, a 1-form η and a tensor field φ of type (1,1) as follows:

$$\xi := JC, \qquad \eta|_{\mathcal{D}} = 0 \text{ and } \eta(\xi) = 1, \qquad \varphi|_{\mathcal{D}} = J|_{\mathcal{D}} \text{ and } \varphi(\xi) = 0.$$

It is easy to see that (φ, ξ, η) is an almost contact structure on M. This structure is called *the almost contact structure on M induced by C* (or simply *induced almost contact structure*).

For an induced almost contact structure we have the following theorem

Theorem 3.1 ([5])**.** *If (φ, ξ, η) is an induced almost contact structure on M, then the following equations hold:*

$$\begin{aligned}
\eta(\nabla_X Y) &= -h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X),\\
\varphi(\nabla_X Y) &= \nabla_X \varphi Y + \eta(Y)SX - h(X,Y)\xi,\\
\eta([X,Y]) &= -h(X,\varphi Y) + h(Y,\varphi X) + X(\eta(Y)) - Y(\eta(X)) + \eta(Y)\tau(X) - \eta(X)\tau(Y),\\
\varphi([X,Y]) &= \nabla_X \varphi Y - \nabla_Y \varphi X - \eta(X)SY + \eta(Y)SX,\\
\eta(\nabla_X \xi) &= \tau(X),\\
\eta(SX) &= h(X,\xi)
\end{aligned}$$

for every $X, Y \in \mathcal{X}(M)$.

The next theorem characterizes hypersurfaces with a centro-affine J-tangent transversal vector field and with an involutive distribution $\mathcal{D}$.

Theorem 3.2 ([6]). *Let $f\colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a centro-affine J-tangent vector field. The distribution $\mathcal{D}$ is involutive if and only if for every $x \in M$ there exists a Kählerian immersion $g\colon V \to \mathbb{R}^{2n+2}$ defined on an open subset $V \subset \mathbb{R}^{2n}$ such that f can be expressed in the neighborhood of x in the form*

$$f(x_1, \dots, x_{2n}, y) = Jg(x_1, \dots, x_{2n}) \cos y + g(x_1, \dots, x_{2n}) \sin y.$$

An affine hypersphere with a transversal J-tangent Blaschke field we call a *J-tangent affine hypersphere.* We have the following

Theorem 3.3 ([7]). *There are no improper J-tangent affine hyperspheres.*

4. Main results

In this section the main results of this paper are provided. Namely, we shall prove the following

Theorem 4.1. *Let $f\colon M \to \mathbb{R}^{2n+2}$ be a J-tangent affine hypersphere with an involutive distribution $\mathcal{D}$. Then f can be locally expressed in the form:*

$$f(x_1, \dots, x_{2n}, z) = Jg(x_1, \dots, x_{2n}) \cos z + g(x_1, \dots, x_{2n}) \sin z, \tag{7}$$

where g is a proper complex affine hypersphere. Moreover, the converse is also true in the sense that if g is a proper complex affine hypersphere, then f given by the formula (7) is a J-tangent affine hypersphere with an involutive distribution $\mathcal{D}$.

Proof. ($\Rightarrow$) First note that due to Theorem 3.3 f must be a proper affine hypersphere. Let C be a J-tangent affine normal field. There exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $C = -\lambda f$. Since C is J-tangent and transversal, the same is $\frac{1}{\lambda}C = -f$. Thus f satisfies assumptions of Theorem 3.2. From Theorem 3.2, there exists a Kählerian immersion g from open subset $U \subset \mathbb{R}^{2n}$ into $\mathbb{R}^{2n+2}$, and there exists an open interval I such that f can be locally expressed in the form

$$f(x_1, \dots, x_{2n}, z) = Jg(x_1, \dots, x_{2n}) \cos z + g(x_1, \dots, x_{2n}) \sin z$$

for $(x_1, \dots, x_{2n}) \in U$ and $z \in I$. Now, we shall prove that g is a proper complex affine hypersphere.

Let $\zeta := |\lambda|^{\frac{2n+3}{2n+4}} g$. Assume that there exist functions α^i, γ, δ from U into $\mathbb{R}$ such that

$$\alpha^i g_{x_i} + \gamma g + \delta J g = 0$$

for $i = 1, \dots, 2n$. Then for any $z \in I$ we have

$$\alpha^i g_{x_i} \sin z + \gamma g \sin z + \delta J g \sin z = 0$$

and

$$\alpha^i g_{x_i} \cos z + \gamma g \cos z + \delta J g \cos z = 0.$$

Adding the above equalities to each other and taking into account that

$$f_{x_i} = J g_{x_i} \cos z + g_{x_i} \sin z, \qquad f_z = -J g \sin z + g \cos z,$$

we obtain

$$\alpha^i f_{x_i} + \gamma f - \delta f_z = 0.$$

But since f is an immersion and $C = -\lambda f$ is a transversal vector field, the above implies

$$\alpha^i = \gamma = \delta = 0.$$

Thus $\{g_{x_i}\}$, g, Jg are linearly independent. So g and, in consequence, ζ is a transversal vector field to g. From the Weingarten formula for g we have

$$\mathrm{D}_{\partial_{x_i}} \zeta = -g_*(\widetilde{S} \partial_{x_i}) + \tau_1(\partial_{x_i})\zeta + \tau_2(\partial_{x_i}) J\zeta.$$

On the other hand, we compute

$$\mathrm{D}_{\partial_{x_i}} \zeta = \partial_{x_i}(\zeta) = -|\lambda|^{\frac{2n+3}{2n+4}} g_*(\partial_{x_i}).$$

Summarizing, we obtain

$$\widetilde{S} = |\lambda|^{\frac{2n+3}{2n+4}} I, \qquad \tau_1 = 0, \qquad \tau_2 = 0. \tag{8}$$

Now, to prove that ζ is an affine normal vector field, it is enough to show that $|H_\zeta| = 1$. Since g is Kählerian, J is a complex structure on TU thus, without loss of generality, we may assume that

$$\partial_{x_{n+i}} = J \partial_{x_i}$$

for $i = 1 \dots n$. Let us denote

$$A := \Theta_\zeta(\partial_{x_1}, \dots, \partial_{x_n}, J\partial_{x_1}, \dots, J\partial_{x_n}).$$

Then the basis

$$\frac{1}{A}\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n}, J\partial_{x_1}, \dots, J\partial_{x_n}$$

is unimodular relative to Θ_ζ. Now (according to H_ζ definition) we have

$$H_\zeta = \frac{1}{A^2} \cdot \det h_1,$$

where

$$\det h_1 = \det \begin{bmatrix} h_1(\partial_{x_1}, \partial_{x_1}) & h_1(\partial_{x_1}, \partial_{x_2}) & \cdots & h_1(\partial_{x_1}, \partial_{x_{2n}}) \\ h_1(\partial_{x_2}, \partial_{x_1}) & h_1(\partial_{x_2}, \partial_{x_2}) & \cdots & h_1(\partial_{x_2}, \partial_{x_{2n}}) \\ \vdots & \vdots & \ddots & \vdots \\ h_1(\partial_{x_{2n}}, \partial_{x_1}) & h_1(\partial_{x_{2n}}, \partial_{x_2}) & \cdots & h_1(\partial_{x_{2n}}, \partial_{x_{2n}}) \end{bmatrix}.$$

From the Gauss formula for g we have

$$\begin{aligned} g_{x_i x_j} &= g_*(\widetilde{\nabla}_{\partial_{x_i}} \partial_{x_j}) + h_1(\partial_{x_i}, \partial_{x_j})\zeta + h_2(\partial_{x_i}, \partial_{x_j})J\zeta \\ &= g_*(\widetilde{\nabla}_{\partial_{x_i}} \partial_{x_j}) - |\lambda|^{\frac{2n+3}{2n+4}} h_1(\partial_{x_i}, \partial_{x_j})g - |\lambda|^{\frac{2n+3}{2n+4}} h_2(\partial_{x_i}, \partial_{x_j})Jg. \end{aligned} \tag{9}$$

From the Gauss formula for f we have

$$\begin{aligned} f_{x_i x_j} &= Jg_{x_i x_j} \cos z + g_{x_i x_j} \sin z \\ &= f_*(\nabla_{\partial_{x_i}} \partial_{x_j}) - \lambda h(\partial_{x_i}, \partial_{x_j})(Jg \cos z + g \sin z). \end{aligned} \tag{10}$$

Using (9) in (10), we obtain

$$\begin{aligned} &f_*(\nabla_{\partial_{x_i}} \partial_{x_j}) - \lambda h(\partial_{x_i}, \partial_{x_j})(Jg \cos z + g \sin z) \\ &\quad = Jg_*(\widetilde{\nabla}_{\partial_{x_i}} \partial_{x_j}) \cos z + g_*(\widetilde{\nabla}_{\partial_{x_i}} \partial_{x_j}) \sin z \\ &\qquad - |\lambda|^{\frac{2n+3}{2n+4}} (h_1(\partial_{x_i}, \partial_{x_j})Jg - h_2(\partial_{x_i}, \partial_{x_j})g) \cos z \\ &\qquad - |\lambda|^{\frac{2n+3}{2n+4}} (h_1(\partial_{x_i}, \partial_{x_j})g + h_2(\partial_{x_i}, \partial_{x_j})Jg) \sin z \\ &\quad = f_*(\widetilde{\nabla}_{\partial_{x_i}} \partial_{x_j}) - |\lambda|^{\frac{2n+3}{2n+4}} h_1(\partial_{x_i}, \partial_{x_j})(Jg \cos z + g \sin z) \\ &\qquad - |\lambda|^{\frac{2n+3}{2n+4}} h_2(\partial_{x_i}, \partial_{x_j})(-g \cos z + Jg \sin z) \\ &\quad = f_*(\widetilde{\nabla}_{\partial_{x_i}} \partial_{x_j}) - |\lambda|^{\frac{2n+3}{2n+4}} h_1(\partial_{x_i}, \partial_{x_j}) \cdot f - |\lambda|^{\frac{2n+3}{2n+4}} h_2(\partial_{x_i}, \partial_{x_j}) \cdot Jf. \end{aligned}$$

Since $f_*(\widetilde{\nabla}_{\partial_{x_i}} \partial_{x_j})$ and Jf are tangent, we immediately get that

$$-\lambda h(\partial_{x_i}, \partial_{x_j}) = -|\lambda|^{\frac{2n+3}{2n+4}} h_1(\partial_{x_i}, \partial_{x_j}).$$

By the Gauss formula for f, we also have

$$h(\partial_z, \partial_z) = \frac{1}{\lambda}$$

and

$$h(\partial_z, \partial_{x_i}) = h(\partial_{x_i}, \partial_z) = 0$$

for $i = 1 \dots 2n$. Hence

$$\begin{aligned}
\det h := &\begin{bmatrix} h(\partial_{x_1}, \partial_{x_1}) & h(\partial_{x_1}, \partial_{x_2}) & \cdots & h(\partial_{x_1}, \partial_{x_{2n}}) & 0 \\ h(\partial_{x_2}, \partial_{x_1}) & h(\partial_{x_2}, \partial_{x_2}) & \cdots & h(\partial_{x_2}, \partial_{x_{2n}}) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h(\partial_{x_{2n}}, \partial_{x_1}) & h(\partial_{x_{2n}}, \partial_{x_2}) & \cdots & h(\partial_{x_{2n}}, \partial_{x_{2n}}) & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{\lambda} \end{bmatrix} \\
&= \frac{1}{\lambda} \det[h(\partial_{x_i}, \partial_{x_j})] = \frac{1}{\lambda} \cdot (\frac{1}{\lambda} \cdot |\lambda|^{\frac{2n+3}{2n+4}})^{2n} \det h_1 \\
&= \frac{1}{\lambda} \cdot |\lambda|^{-\frac{2n}{2n+4}} \det h_1.
\end{aligned}$$

Finally, we get

$$|\det h_1| = |\lambda|^{\frac{4n+4}{2n+4}} |\det h| = |\lambda|^{\frac{2n+2}{n+2}} |\det h|. \tag{11}$$

Now, since $C = -\lambda f$ is the Blaschke field, we have

$$\begin{aligned}
\omega_h = \sqrt{|\det h|} &= \Theta(\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z) = \det[f_{x_1}, \dots, f_{x_{2n}}, f_z, C] \\
&= -\lambda \det[Jg_{x_1} \cos z + g_{x_1} \sin z, \dots, Jg_{x_{2n}} \cos z + g_{x_{2n}} \sin z, \\
&\quad - Jg \sin z + g \cos z, Jg \cos z + g \sin z].
\end{aligned}$$

Using the fact that a determinant is $(2n+2)$-linear and antisymmetric, and since

$$g_{x_{n+i}} = Jg_{x_i}$$

for $i = 1 \dots n$, we obtain

$$\begin{aligned}
\Theta(\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z) &= -\lambda \det[g_{x_1}, \dots, g_{x_n}, Jg_{x_1}, \dots, Jg_{x_n}, g, Jg] \\
&= -\lambda(|\lambda|^{\frac{2n+3}{2n+4}})^{-2} \det[g_*(\partial_{x_1}), \dots, g_*(\partial_{x_{2n}}), \zeta, J\zeta] \\
&= -\lambda \cdot (|\lambda|^{-\frac{2n+3}{n+2}}) \Theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}}).
\end{aligned}$$

Now, it easily follows that

$$|\det h| = [\Theta(\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z)]^2$$

$$
\begin{aligned}
&= |\lambda|^2 \cdot |\lambda|^{\frac{-4n-6}{n+2}} [\Theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}})]^2 \\
&= |\lambda|^{\frac{-2n-2}{n+2}} [\Theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}})]^2 \\
&= |\lambda|^{\frac{-2n-2}{n+2}} \cdot A^2.
\end{aligned}
$$

The above implies (see (11)) that

$$|\det h_1| = A^2.$$

Summarizing,

$$|H_\zeta| = \frac{1}{A^2} |\det h_1| = 1,$$

that is, ζ is an affine normal field and due to (8) g is an affine hypersphere.
("$\Leftarrow$") Let $g \colon U \to \mathbb{R}^{2n+2}$ be a proper complex affine hypersphere. In particular, g is Kählerian, and there exists $\alpha \neq 0$ such that $\zeta = -\alpha g$ is an affine normal vector field. Without loss of generality, we may assume that $\alpha > 0$. Since g is transversal, Jg is transversal too, thus $\{g_{x_1}, \dots, g_{x_{2n}}, g, Jg\}$ form the basis of $\mathbb{R}^{2n+2}$. The above implies that

$$f \colon U \times I \ni (x_1, \dots, x_{2n}, z) \mapsto f(x_1, \dots, x_{2n}, z) \in \mathbb{R}^{2n+2},$$

given by the formula:

$$f(x_1, \dots, x_{2n}, z) := Jg(x_1, \dots, x_{2n}) \cos z + g(x_1, \dots, x_{2n}) \sin z$$

is an immersion, and $C := -\alpha^{\frac{2n+4}{2n+3}} \cdot f$ is a transversal vector field. Of course, C is J-tangent because $JC = \alpha^{\frac{2n+4}{2n+3}} f_z$. Since C is equiaffine, it is enough to show that $\omega_h = \Theta$ for some positively oriented (relative to Θ) basis on $U \times I$. Let $\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z$ be a local coordinate system on $U \times I$. Since g is Kählerian, we may assume that $\partial_{x_{n+i}} = J\partial_{x_i}$ for $i = 1 \dots n$. Then, in a similar way as in the proof of the first implication, we compute

$$
\begin{aligned}
\Theta(\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z) &= \det[f_{x_1}, \dots, f_{x_{2n}}, f_z, -\alpha^{\frac{2n+4}{2n+3}} f] \\
&= -\alpha^{\frac{2n+4}{2n+3}} \det[Jg_{x_1} \cos z + g_{x_1} \sin z, \dots, Jg_{x_{2n}} \cos z + g_{x_{2n}} \sin z, \\
&\quad - Jg \sin z + g \cos z, Jg \cos z + g \sin z] \\
&= -\alpha^{\frac{2n+4}{2n+3}} \det[g_*(\partial_{x_1}), \dots, g_*(\partial_{x_{2n}}), g, Jg] \\
&= -\alpha^{\frac{2n+4}{2n+3}} \cdot \frac{1}{\alpha^2} \Theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}}) \\
&= -\alpha^{-\frac{2n+2}{2n+3}} \Theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}}).
\end{aligned}
$$

Again, in a similar way, as in the proof of the first implication, we get

$$\det h = \alpha^{-\frac{2n+4}{2n+3}} \cdot \Big(\frac{\alpha}{\alpha^{\frac{2n+4}{2n+3}}}\Big)^{2n} \det h_1 = \alpha^{-\frac{2n+4}{2n+3}} \cdot \alpha^{-\frac{2n}{2n+3}} \det h_1 = \alpha^{\frac{-4n-4}{2n+3}} \det h_1.$$

The above implies that

$$\omega_h := \sqrt{|\det h|} = \alpha^{\frac{-2n-2}{2n+3}} \sqrt{|\det h_1|}.$$

It is easy to see that

$$|\det h_1| = |H_\zeta|[\Theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}})]^2,$$

because

$$\frac{1}{\Theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}})}\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_{2n}}$$

is a unimodular basis relative to Θ_ζ. Hence, (since $|H_\zeta| = 1$)

$$\omega_h = \alpha^{\frac{-2n-2}{2n+3}} |\Theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}})|.$$

Finally, we get $\omega_h = |\Theta(\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z)|$. The proof is completed. $\square$

Immediately, from the proof of the above theorem, we get

Remark 4.1. If f is an affine hypersphere with $S = \lambda I$, and g is a complex affine hypersphere with $\widetilde{S} = \alpha I$, then we have the following relation $|\lambda| = |\alpha|^{\frac{2n+4}{2n+3}}$.

Complex affine hyperspheres of complex dimension one we call *complex affine circles in* $\mathbb{C}^2$. We have the following classification of the complex affine circles

Theorem 4.2 ([3])**.** *A complex affine curve in $\mathbb{C}^2$ is a complex affine circle if and only if it is a quadratic complex curve, respectively, of parabolic or hyperbolic type according to the circle being improper or proper.*

As a consequence of the above theorem and Theorem 4.1, one may obtain simple proof of Theorem 4.2 from [7]. That is

Theorem 4.3 ([7])**.** *Let $f\colon M \mapsto \mathbb{R}^4$ be a J-tangent affine hypersphere with an involutive distribution $\mathcal{D}$. Then f can be locally expressed in the form:*

$$f(x, y, z) = \lambda^{-\frac{5}{8}} \begin{bmatrix} \sin\sqrt{\lambda}x \sinh\sqrt{\lambda}y \\ -\cos\sqrt{\lambda}x \sinh\sqrt{\lambda}y \\ cos\sqrt{\lambda}x \cosh\sqrt{\lambda}y \\ \sin\sqrt{\lambda}x \cosh\sqrt{\lambda}y \end{bmatrix} \cos\lambda z + \lambda^{-\frac{5}{8}} \begin{bmatrix} \cos\sqrt{\lambda}x \cosh\sqrt{\lambda}y \\ \sin\sqrt{\lambda}x \cosh\sqrt{\lambda}y \\ -\sin\sqrt{\lambda}x \sinh\sqrt{\lambda}y \\ \cos\sqrt{\lambda}x \sinh\sqrt{\lambda}y \end{bmatrix} \sin\lambda z \in \mathbb{R}^4$$

for some $\lambda > 0$.

Proof. From Theorem 4.1, f can be locally expressed in the form:

$$f(x,y,z) = Jg(x,y)\cos z + g(x,y)\sin z,$$

where g is a complex affine hypersphere. Since g is a 1-dimensional (in a complex sense) affine hypersphere, thus g is a complex affine circle. Now, by Theorem 4.2, g is a quadratic complex curve. Moreover, since g is a proper hypersphere, it must be of hyperbolic type, that is

$$z_1 z_2 = \alpha,$$

where $\alpha > 0$. Equivalently, using the following complex equiaffine transformation

$$\begin{bmatrix} \frac{i}{2} & \frac{1}{2} \\ -i & 1 \end{bmatrix},$$

g can be locally expressed in a parametric form as follows:

$$g(u) = \sqrt{2\alpha}\begin{bmatrix} \cos u \\ \sin u \end{bmatrix}.$$

Now, moving to real numbers ($u = x + iy, x, y \in \mathbb{R}$), we have

$$g(x,y) = \sqrt{2\alpha}\begin{bmatrix} \operatorname{Re}\cos u \\ \operatorname{Re}\sin u \\ \operatorname{Im}\cos u \\ \operatorname{Im}\sin u \end{bmatrix} = \sqrt{2\alpha}\begin{bmatrix} \cosh x\cosh y \\ \sin x\cosh y \\ -\sin x\sinh y \\ \cos x\sinh y \end{bmatrix}$$

so

$$f(x,y,z) = \sqrt{2\alpha}\begin{bmatrix} \sin x\sinh y \\ -\cos x\sinh y \\ cos x\cosh y \\ \sin x\cosh y \end{bmatrix}\cos z + \sqrt{2\alpha}\begin{bmatrix} \cos x\cosh y \\ \sin x\cosh y \\ -\sin x\sinh y \\ \cos x\sinh y \end{bmatrix}\sin z.$$

Taking into account that $\widetilde{S} = \frac{1}{(2\alpha)^{\frac{2}{3}}}I$ for g (see Example 2 in [3]) we easily get that $\lambda = (2\alpha)^{-\frac{4}{5}}$ (see Remark 4.1). Now, replacing x with $\sqrt{\lambda}x$, y with $\sqrt{\lambda}y$, and z with λz, we obtain f in the required form. $\square$

5. Some examples

Example 1. Let $\tilde{g}\colon \mathbb{C}^n \to \mathbb{C}^{n+1}$ be a standard complex hypersphere of complex dimension n. That is

$$\tilde{g}(z_1,\dots,z_n) = \begin{bmatrix} \tilde{g}_1(z_1,\dots,z_n) \\ \tilde{g}_2(z_1,\dots,z_n) \\ \tilde{g}_3(z_1,\dots,z_n) \\ \vdots \\ \tilde{g}_{n-1}(z_1,\dots,z_n) \\ \tilde{g}_n(z_1,\dots,z_n) \\ \tilde{g}_{n+1}(z_1,\dots,z_n) \end{bmatrix} = \begin{bmatrix} \cos z_1 \\ \sin z_1 \cdot \cos z_2 \\ \sin z_1 \cdot \sin z_2 \cdot \cos z_3 \\ \vdots \\ \sin z_1 \cdot \sin z_2 \cdot \ldots \cdot \sin z_{n-2} \cdot \cos z_{n-1} \\ \sin z_1 \cdot \sin z_2 \cdot \ldots \cdot \sin z_{n-1} \cdot \cos z_n \\ \sin z_1 \cdot \sin z_2 \cdot \ldots \cdot \sin z_{n-1} \cdot \sin z_n \end{bmatrix}.$$

Let $z_k = x_k + iy_k$ for $k = 1,\dots,n$. Then

$$g\colon \mathbb{R}^{2n} \ni (x_1,y_1,x_2,y_2,\dots,x_n,y_n) \mapsto g(x_1,y_1,\dots,x_n,y_n) \in \mathbb{R}^{2n+2},$$

given by the formula

$$g(x_1,y_1,\dots,x_n,y_n) = \begin{bmatrix} \operatorname{Re}\tilde{g}_1(z_1,\dots,z_n) \\ \operatorname{Re}\tilde{g}_2(z_1,\dots,z_n) \\ \vdots \\ \operatorname{Re}\tilde{g}_n(z_1,\dots,z_n) \\ \operatorname{Re}\tilde{g}_{n+1}(z_1,\dots,z_n) \\ \operatorname{Im}\tilde{g}_1(z_1,\dots,z_n) \\ \operatorname{Im}\tilde{g}_2(z_1,\dots,z_n) \\ \vdots \\ \operatorname{Im}\tilde{g}_n(z_1,\dots,z_n) \\ \operatorname{Im}\tilde{g}_{n+1}(z_1,\dots,z_n) \end{bmatrix}.$$

is a complex affine hypersphere. Now, by Theorem 4.1

$$\begin{aligned} &f(x_1,y_1,x_2,y_2,\dots,x_n,y_n,z) \\ &\quad := Jg(x_1,y_1,\dots,x_n,y_n)\cos z + g(x_1,y_1,\dots,x_n,y_n)\sin z \end{aligned}$$

is a J-tangent affine hypersphere with an involutive contact distribution.

Example 2. Let us consider a complex affine hypersphere (see Example 1 in [3]), given by the formula

$$z_1 \cdot z_2 \cdot \ldots \cdot z_n \cdot z_{n+1} = 1 \tag{12}$$

(when $n > 1$, this hypersphere is not affinely equivalent with the hypersphere from Example 1). Rewriting (1) in a parametric form, we get

$$\tilde{g}(z_1, \dots, z_n) = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ 1/(z_1 \cdot z_2 \cdot \ldots \cdot z_n) \end{bmatrix}.$$

Now, moving to real numbers, we have

$$g(x_1, y_1, \dots, x_n, y_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \operatorname{Re} 1/(z_1 \cdot z_2 \cdot \ldots \cdot z_n) \\ y_1 \\ y_2 \\ \vdots \\ y_n \\ \operatorname{Im} 1/(z_1 \cdot z_2 \cdot \ldots \cdot z_n) \end{bmatrix},$$

and, by Theorem 4.1,

$$\begin{aligned} &f(x_1, y_1, x_2, y_2, \dots, x_n, y_n, z) \\ &\quad := Jg(x_1, y_1, \dots, x_n, y_n) \cos z + g(x_1, y_1, \dots, x_n, y_n) \sin z \end{aligned}$$

is a J-tangent affine hypersphere with an involutive contact distribution.

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