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A note on normal idempotents

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Abstract. Let S be a regular semigroup and $\overline{E}$ be the subsemigroup of S generated by the set E of all idempotent elements of S. By [1], an idempotent element u of S is called a normal idempotent if $xux = x$ for every $x \in \overline{E}$ and $u\overline{E}u$ is a semilattice. In this paper we introduce the notion of a quasi-normal idempotent of S as an idempotent element u of S which satisfies the conditions: $eue = e$ for every $e \in E$ and uEu is a semilattice. It is clear that every normal idempotent is quasi-normal. The main purpose of our paper is to show the converse statement, that is, every quasi-normal idempotent of a regular semigroup is also normal.

1. Introduction and preliminaries

In [1], Blyth and McFadden introduce the concept of *normal idempotent* of a regular semigroup. An idempotent u of a regular semigroup S is a *medial idempotent* if for any element x of the regular semigroup $\overline{E}$ generated by the set E of idempotents of S, $xux = x$. A medial idempotent u is called *normal* if $u\overline{E}u$ is a semilattice. Guo [5] generalizes the normal idempotent on *abundant semigroup.* He calls an idempotent u of *an abundant semigroup* S a *weak medial idempotent* if for any idempotent x of S, $xux = x$, and then a weak medial is called a *weak normal idempotent* if uSu is an *adequate semigroup.*

In this paper, as analogous to weak normal idempotent of an abundant semigroup, we introduce the concept of *quasi-normal idempotent* of regular semigroup. Then by exploring the relationships between (quasi-)normal idempotents and

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some inverse transversals, the main theorem of this paper is hold, that is, every idempotent element of a regular semigroup is quasi-normal if and only if it is normal.

The reader is referred to [2], [4], [6], [7] and [8] for all the notation and terminology not defined in this paper.

A regular semigroup S° is called an *inverse transversal* for a regular semigroup S, if S° is a subsemigroup of S and for any $x \in S$, $|V_{S^\circ}(x)| = 1$. Particularly, we denote the unique inverse of x by x°.

Moreover, if $S^\circ S S^\circ \subseteq S^\circ$, then S° is called a *quasi-ideal* transversal for S; if for any $e \in E$, $e^\circ \in E$, then S° is called a *weakly multiplicative* inverse transversal for S.

Let $I = \{aa^\circ : a \in S, a^\circ \in V_{S^\circ}(a)\}$ and $\Lambda = \{a^\circ a : a \in S, a^\circ \in V_{S^\circ}(a)\}$.

If $\Lambda I \subseteq E$, then S° is called a *multiplicative* transversal.

Lemma 1.1 ([3]). *Let S° be an inverse transversal for a regular semigroup S. Then S° is multiplicative if and only if S° is a weakly multiplicative transversal and a quasi-ideal for S.*

2. On normal idempotents

In this section, for any semigroup S, we denote the set of idempotents by $E(S)$ and the regular semigroup generated by E by $\overline{E(S)}$. If there are no ambiguities, we shall denote them by E, $\overline{E}$ respectively.

For convenience, in this sequel, let S be a regular semigroup with the set E of all idempotents of S without mention.

Definition 2.1. An idempotent u of S is called a quasi-normal idempotent if for any $e \in E$, $eue = e$ and uEu is a semilattice.

The above notion shows that every normal idempotent of S is clearly a quasi-normal idempotent. Throughout what follows, we devote ourselves to prove that the converse statement is also true by means of Lemma 1.1.

For this purpose, we explore the relationships between (quasi-)normal idempotents and some inverse transversals firstly.

Lemma 2.2. *Let u be a quasi-normal idempotent of S. Then*

(1) $(\forall e \in E)$ $eu \mathcal{R} e \mathcal{L} ue$;

(2) $(\forall e \in E)$ $eu, ue \in E$;

(3) $(\forall e, f \in E)\ e \mathcal{R} f \Leftrightarrow eu = fu$;

(4) $(\forall e, f \in E)\ e \mathcal{L} f \Leftrightarrow ue = uf$.

Proof. (1), (2) Follows from the Definition 2.1. (3), (4) According to (1) and (2), it is easy to check. □

We here add a basic property of regular elements that will be useful.

Lemma 2.3. *For any $e \in E$ and $x \in S$, if $V_{eSe}(x) \neq \emptyset$, then*

$$V_{eSe}(x) = V_{eSe}(ex) = V_{eSe}(xe) = V_{eSe}(exe).$$

Proof. In fact, we only need to check the first equation. The proof of the left equations could be proved similarly. Let $x' \in V_{eSe}(x)$. Then

$$exx'ex = exx'x = ex \quad \text{and} \quad x'exx' = x'xx' = x'.$$

It means $x' \in V_{eSe}(ex)$. On the other hand, let $y \in V_{eSe}(ex)$. Then

$$yxy = yexy = y \quad \text{and} \quad xyx = xyex = xx'exyex = xx'ex = xx'x = x.$$

It means $y \in V_{eSe}(x)$. Therefore, $V_{eSe}(x) = V_{eSe}(ex)$. □

Lemma 2.4. *If u is a quasi-normal idempotent of S, then*

(1) $uEu = E(uSu)$;

(2) uSu *is an inverse subsemigroup of* S;

(3) $(\forall x \in S)\ |\ V_{uSu}(x)| = 1$;

(4) $(\forall e \in E)\ ueu \in V(e) \cap E(uSu)$.

Proof. (1) As u is quasi-normal, we have $uEu \subseteq E$. From this, it follows that $uEu \subseteq E(uSu)$. For arbitrary $f = utu \in E(uSu)$, we have $f = utu = uutuu = ufu \in uEu$ and so $E(uSu) \subseteq uEu$. Hence $E(uSu) = uEu$.

(2) Let u be a quasi-normal idempotent element of a regular semigroup S. It is clear that uSu is a subsemigroup of S. Let $x \in S$ be an arbitrary element. As S is regular, there is an element $x' \in S$ such that $xx'x = x$ and $x'xx' = x'$. We show $ux'u \in V(x) \cap uSu$ by showing $ux'u \in V(x)$. As $xx', x'x \in E$, we have

$$\begin{aligned}(ux'u)x(ux'u) &= ux'(xx')u(xx')xux'xx'u \\ &= ux'xx'xux'xx'u = u(x'x)u(x'x)x'u \\ &= ux'xx'u = ux'u\end{aligned}$$

and

$$\begin{aligned} x(ux'u)x &= x(x'x)u(x'x)x'uxx'x \\ &= xx'(xx')u(xx')x \\ &= xx'xx'x = xx'x = x \end{aligned}$$

and so $ux'u \in V(x)$, indeed. From the above result it follows that, for every $x \in uSu$, $ux'u \in V_{uSu}(x)$ and so uSu is a regular semigroup. By (1), $E(uSu) = uEu$ is a semilattice and so the idempotents of uSu commute with each other. Thus uSu is an inverse semigroup.

(3) By the proof of (2), we have $ux'u \in V_{uSu}(x)$ for every $x \in S$ and every $x' \in V(x)$. Hence $V(x) \cap uSu \neq \emptyset$ for every element x of S. Suppose that $x'', x^\circ \in V_{uSu}(x)$. Then by Lemma 2.3, $x'', x^\circ \in V_{uSu}(uxu)$. Since uSu is an inverse semigroup, $x'' = x^\circ$.

(4) Let $e \in E$ be arbitrary. Then $e \in V(e)$ and, by the above, $ueu \in V(e) \cap uSu$. As $ueu \in uEu$, we have $ueu \in V(e) \cap E(uSu)$. □

Guided by Lemma 2.4, it is seen that for every idempotent element u of S, if u is a quasi-normal idempotent, then uSu is a weakly multiplicative transversal for S. Naturally, we wonder whether the the converse statement is true. To answer the question, we need some particular results.

Lemma 2.5. *Let $u \in E$ and uSu be an inverse subsemigroup for S. If for any $e \in E$, $V(e) \cap E(uSu) \neq \emptyset$, then*

(1) *Eu is a left normal band;*

(2) *uE is a right normal band;*

(3) *uEu is a semilattice;*

(4) *u is a quasi-normal idempotent.*

Proof. (1) Let $e^\circ \in V(e) \cap E(uSu)$. Then $e^\circ \in V(eu) \cap E(uSu)$. It follows that $(e^\circ eu)^2 = e^\circ eue^\circ eu = e^\circ eu$. It means $e^\circ eu \in E(uSu)$. Since $e^\circ \in E(uSu)$ and $E(uSu)$ is a semilattice, $eu = ee^\circ eu = ee^\circ e^\circ eu = ee^\circ eue^\circ = ee^\circ$ and so $eu \in E$. As a dual, $ue \in E$. Hence for any $f \in E$, $ufu \in E(uSu)$ and so $(eufu)^2 = eufueufu = eueufufu = eufu$. It shows that Eu is a band. Let $g \in E$. Then $eufugu = eugufu$. Therefore, Eu is a left normal band.

(2) Similar to the proof of (1).

(3) By the above results, $ufu \in uEu$ for every $f \in E$. Then we have $uEu \subseteq E(uSu)$. Assume $x \in E(uSu)$. Then $x \in E$ and $x = utu$ for some $t \in S$. Hence $x = utu = uutuu = uxu \in uEu$ and so $E(uSu) \subseteq uEu$. Consequently,

$uEu = E(uSu)$. As uSu is an inverse semigroup, their idempotents commute with each other and so $E(uSu)$ is a semilattice. Consequently, uEu is a semilattice.

(4) By (3), it is sufficient to show that $eue = e$ for every $e \in E$. Let $e^\circ \in V(e) \cap E(uSu)$. Then $e = ee^\circ e = eue^\circ e$ and so $(eu)e = (eu)^2 e^\circ e = eue^\circ e = e$, because $eu \in E$. □

The following characterizations of a quasi-normal idempotent will be helpful for us to investigate the relationships between quasi-normal idempotents and weakly multiplicative inverse transversals.

Theorem 2.6. *Let $u \in E$. For any $e \in E$ and $x \in S$, the following statements are equivalent:*

(1) *u is quasi-normal;*

(2) *uEu is a semilattice and $V(x) \cap uSu \neq \emptyset$;*

(3) *uE is a right normal band and $V(x) \cap Su \neq \emptyset$;*

(4) *Eu is a left normal band and $V(x) \cap uS \neq \emptyset$;*

(5) *uEu is a semilattice and $V(e) \cap uSu \neq \emptyset$;*

(6) *uE is a right normal band and $V(e) \cap Su \neq \emptyset$;*

(7) *Eu is a left normal band and $V(e) \cap uS \neq \emptyset$.*

Proof. We only prove that (1) ⇒ (2) ⇒ (3) ⇒ (5) ⇒ (6) ⇒ (1).

(1) ⇒ (2) As u is quasi-normal, uEu is a semilattice by Definition 2.1. $V(x) \cap uSu \neq \emptyset$ follows from Lemma 2.4(3).

(2) ⇒ (3) Let $e \in E$. Then there exists $e' \in V(e) \cap uSu$ and so $e = eue'e$. It follows that $e \mathcal{R} eu$. As $\mathcal{R}$ is a left congruence, $ue \mathcal{R} ueu$, which together with $uEu \subseteq E$ implies $(ueu)ue = ue$. It means $ue \in E$. Let $f, g \in E$. Since uEu is a semilattice, $(ueuf)^2 = ueufueuf = ueufuf = uefu$ and $ueufug = ufueug$. Hence uE is a right normal band, while $V(x) \cap Su \neq \emptyset$ is obvious.

(3) ⇒ (5) $V(e) \cap uSu \neq \emptyset$ is evident. For any $e \in E$, $ue \in E$ implies that $ueu \in E$. Let $f \in E$. Since uE is a right normal idempotent, $(ueufu)^2 = ueufueufu = ueufu$ and $ueufu = ufueu$. Hence uEu is a semilattice.

(5) ⇒ (6) Similar to the proof of (2) ⇒ (3).

(6) ⇒ (1) In view of the proof of (3) ⇒ (5), uEu is a semilattice. Suppose that $e' \in V(e) \cap Su$. Then $e = ee'ue$ and so $e \mathcal{L} ue$. Since uE is a band, $eue = e$. Hence u is a quasi-normal idempotent. □

By reviewing Lemma 2.4 and 2.5 and the above theorem, we have the following theorem which will be useful for our main purpose.

Theorem 2.7. *For any $u \in E$, u is a quasi-normal idempotent of S if and only if uSu is a weakly multiplicative inverse transversal for S.*

As we know, every normal idempotent is also quasi-normal. Now we pass to consider the situations in which a quasi-normal idempotent is normal.

Proposition 2.8. *If u is a quasi-normal idempotent of S, then*

(1) *$I = Eu = E(Su)$ is a left normal band;*

(2) *$\Lambda = uE = E(uS)$ is a right normal band;*

(3) *$I \cap \Lambda = uEu$.*

Proof. (1) By Lemma 2.5 and Theorem 2.7, Eu is a left normal band, and so it is easy to check that $Eu = E(Su)$. For the first equation, let $e \in E$, then $ueu \in V(eu)$. Since $eu = eu(ueu)$, $eu \in I$. While $i \in Eu$ for any $i \in I$ is obvious. Hence $I = Eu$.

(2) As a dual of (1).

(3) Easily. □

Proposition 2.9. *Let u be a quasi-normal idempotent of S. The following statements are equivalent:*

(1) *u is normal;*

(2) *$(\forall e, f \in E)$ $uefu \in E$;*

(3) *$(\forall e, f \in E)$ $uef \in E$;*

(4) *$(\forall e, f \in E)$ $efu \in E$;*

(5) *$(\forall i \in I, \forall \lambda \in \Lambda)$ $V_{uSu}(\lambda i) \in uEu$;*

(6) *$(\forall i \in I, \forall \lambda \in \Lambda)$ $\lambda i \in uEu$;*

(7) *$I\Lambda$ is a subsemigroup of S;*

(8) *$I\Lambda = \overline{E}$.*

Proof. We just prove that (1) ⇒ (2) ⇒ (3) ⇒ (1) and that (1) ⇒ (5) ⇒ (6) ⇒ (7) ⇒ (8) ⇒ (1).

(1) ⇒ (2) Follows from the definition of normal idempotent.

(2) ⇒ (3) By Lemma 2.2, $f \mathcal{R} fu$. Then $uef \mathcal{R} uefu$ and so $uefu(uef) = uef$. It means $uef \in E$.

(3) ⇒ (1) Since $ue \mathcal{L} e$, $uef \mathcal{L} ef$ and then $ef = ef(uef)$ as $uef \in E$. According to this, it is easy to check that $uefu \in E$. In view of the above proof, we also have $efu \in E$. By the induction of mathematics, for any $x \in \overline{E}$, $xu \mathcal{R} x$ and $xu \in E$. Hence $xux = x$ and so $uxu \in E$. It follows that $uEu = u\overline{E}$. Therefore, u is a normal idempotent.

(1) $\Rightarrow$ (5) As $I = Eu$ and $\Lambda = uE$, for any $i \in I$ and $\lambda \in \Lambda$, $\lambda i \in u\overline{E}u \subseteq E$ and then $\lambda i \in V_{uSu}(\lambda i)$.

(5) $\Rightarrow$ (6) As uSu is an inverse semigroup.

(6) $\Rightarrow$ (7) For any $i, j \in I$ and $\lambda, k \in \Lambda$, $i\lambda jk \in I\Lambda$ since $jk \in uEu$ and $I = Eu$ is a band. Hence $I\Lambda$ is a subsemigroup of S.

(7) $\Rightarrow$ (8) It follows from $E \subseteq I\Lambda$.

(8) $\Rightarrow$ (1) Since $uI = uEu = \Lambda u$, $u\overline{E}u = uEu$ is a semilattice. □

In view of Proposition 2.9, a normal idempotent can be characterized via some multiplicative inverse transversal.

Theorem 2.10. *For any $u \in E$, u is a normal idempotent of S if and only if uSu is a multiplicative inverse transversal for S.*

Proof. If u is a normal idempotent, then uSu is a weakly multiplicative inverse transversal for S. In this case, $I = Eu$ and $\Lambda = uE$. Obviously, $\Lambda I \subseteq u\overline{E}u = uEu = E(uSu)$. Hence uSu is multiplicative. Conversely, by Theorem 2.7, u is a quasi-normal idempotent of S. So it follows from Proposition 2.8 and 2.9 that u is a normal idempotent. □

The main result is as follows, which is induced by Lemma 1.1 and Theorem 2.7 and 2.10.

Theorem 2.11. *For any $u \in E$, u is a normal idempotent of S if and only if u is a quasi-normal idempotent of S.*

The normal idempotent can be described in the following way, which should be viewed in comparison with the notion of a normal idempotent.

Corollary 2.12. *An idempotent u of a regular semigroup S is a normal idempotent if for any $e \in E$, $eue = e$ and uEu is a semilattice.*

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