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Erdős–Surányi sequences and trigonometric integrals

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Abstract. We study representations of integers as sums of the form $\pm a_1 \pm a_2 \pm \cdots \pm a_n$, where $a_1, a_2, \ldots$ is a prescribed sequence of integers. Such a sequence is called an Erdős–Surányi sequence if every integer can be written in this form for some $n \in \mathbb{N}$ and choices of signs, in infinitely many ways. We study the number of representations of a fixed integer, which can be written as a trigonometric integral, and obtain an asymptotic formula under a rather general scheme due to Roth and Szekeres. Our approach, which is based on Laplace's method for approximating integrals, can also be easily extended to find higher-order expansions. As a corollary, we settle a conjecture of Andrica and Ionaşcu on the number of solutions to the signum equation $\pm 1^k \pm 2^k \pm \cdots \pm n^k = 0$.

1. Introduction

1.1. Erdős–Surányi sequences and solutions to signum equations. A sequence of positive integers $(a_n)_{n=1}^{\infty}$ is called an Erdős–Surányi sequence if every integer can be written in the form $\pm a_1 \pm a_2 \pm \cdots \pm a_n$ for some $n \in \mathbb{N}$ and choices of signs + and −, in infinitely many ways. Representations of this kind were first studied systematically by Erdős and Surányi [13], who provided sufficient conditions for a sequence of integers to have this property (that cover e.g. the sequence of primes).

The sequence of k-th powers, which will be of particular interest to us, was shown to be an Erdős–Surányi sequence by Mitek [16], and later, independently, by Bleicher [6], who also discusses the behaviour of the minimal choice of n.

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Drimbe [11] showed that, generally, any sequence $(p(n))_{n=1}^{\infty}$ where $p(n) \in \mathbb{Q}[n]$ takes an integer value whenever $n \in \mathbb{Z}$, and $\gcd\{p(n) \mid n \in \mathbb{Z}\} = 1$, is an Erdős–Surányi sequence, and this result was rediscovered more recently by Yu [21] and also generalised further by Boulanger and Chabert [7] (to the ring of algebraic integers over a cyclotomic field), by Chen and Chen [8] (to weights other than ± 1), and again by Chen and Chen [9] (who provided a necessary and sufficient condition for arbitrary sequences of integers).

For an Erdős–Surányi sequence $\mathbf{a} = (a_n)_{n=1}^{\infty}$, the signum equation of $\mathbf{a}$ is $\pm a_1 \pm a_2 \pm \cdots \pm a_n = 0$, and for a fixed $n \in \mathbb{N}$, a solution to the signum equation is a choice of $+$ and $-$ such that the equation holds. We denote the number of solutions to the signum equation of $\mathbf{a}$ by $S_{\mathbf{a}}(n)$, and more generally the number of representations of an integer k as $\pm a_1 \pm a_2 \pm \cdots \pm a_n$ by $S_{\mathbf{a}}(n, k)$. In [3], it is shown that the number of solutions to the signum equation can be given by the following integral formula:

$$S_{\mathbf{a}}(n) = \frac{2^n}{2\pi} \int_0^{2\pi} \prod_{i=1}^{n} \cos(a_i t)\, \mathrm{d}t, \tag{1}$$

which follows from expanding each cosine into a sum of exponentials, multiplying out and using the fact that for $m \in \mathbb{Z}$, $\int_0^{2\pi} \exp(imt)\, \mathrm{d}t$ equals 2π if $m = 0$, and 0 if $m \neq 0$. From this, it can be easily seen that the number of representations of k as $\pm a_1 \pm a_2 \pm \cdots \pm a_n$ is given by $S_{\mathbf{a}}(n, k) = S_{\mathbf{a}'}(n+1)/2$ where $\mathbf{a}' = (k, a_1, a_2, \dots)$, as was shown in [4].

Andrica and Tomescu [3] conjectured that the number of solutions to the signum equation in the case $a_i = i$ is asymptotically equal to $\sqrt{6/\pi} \cdot n^{-3/2} 2^n$, which was recently proved by Sullivan [19]. The related question of representing numbers as sums of the form $\sum_{k=-n}^{n} \epsilon_k k$ with $\epsilon_k \in \{0, 1\}$ (and determining the asymptotic number of representations) was also studied in several papers, see van Lint [20], Entringer [12], Clark [10], and Louchard and Prodinger [15]. Prodinger [17] determined an asymptotic formula for the number of ways to partition the set $\{1, 2, \dots, n\}$ into two subsets of equal cardinality and sum (note that representations of zero of the form $0 = \pm 1 \pm 2 \pm \cdots \pm n$ correspond exactly to partitions of this type, where, however, the cardinalities are not necessarily equal). The asymptotic behaviour of an integral similar to the one in (1) (but with sines rather than cosines) was studied recently in [14].

A more general conjecture in the case $a_n = n^k$ was recently formulated by Andrica and Ionaşcu [1], [2]: namely, that (for $n \equiv 0, 3 \bmod 4$)

$$S_{\mathbf{a}}(n) \sim \sqrt{\frac{2(2k+1)}{\pi}} \cdot \frac{2^n}{n^{k+1/2}}. \tag{2}$$

The main theorem of this paper establishes an asymptotic formula for sequences $(a_n)_{n=0}^{\infty}$ that belong to an analytic scheme due to Roth and Szekeres (see the following section). The conjecture of Andrica and Ionaşcu will be included as a special case. It will also follow that all these sequences are Erdős–Surányi sequences.

1.2. Roth–Szekeres sequences. In [18], Roth and Szekeres investigated partitions into elements of a sequence $(a_n)_{n=0}^{\infty}$ satisfying the following conditions:

C1. $a_{n+1} \geq a_n$ for sufficiently large n;

C2. $s = \lim\limits_{n\to\infty} \dfrac{\log a_n}{\log n}$ exists and is positive;

C3. $J_n = \inf\limits_{(2a_n)^{-1}<t\leq 1/2} \dfrac{\sum_{i=1}^{n} \|a_i t\|^2}{\log n} \to \infty$ as $n \to \infty$, where $\|\cdot\|$ denotes the distance from the nearest integer.

For brevity, we will call such sequences Roth–Szekeres sequences. Roth and Szekeres themselves showed that the following classes of sequences are Roth–Szekeres sequences:

(1) $a_n = p_n$, the n-th prime number;

(2) $a_n = f(n)$, where f is a polynomial with rational coefficients taking integer values at integer places, such that $\gcd f(\mathbb{Z}) = \gcd\{f(n) \mid n \in \mathbb{Z}\} = 1$ (for brevity, we will call such polynomials *primitive*);

(3) $a_n = f(p_n)$, p_n is the n-th prime number and f is a polynomial with rational coefficients taking integer values at integer places, such that $\gcd\{nf(n) \mid n \in \mathbb{Z}\} = 1$.

In particular, we see that if f is a primitive polynomial, then $(f(n))_{n=0}^{\infty}$ is both a Roth–Szekeres sequence and has been proved to be an Erdős–Surányi sequence. In fact, a corollary to the main theorem of this paper shows that all Roth–Szekeres sequences are indeed Erdős–Surányi sequences.

2. Main theorem and applications

2.1. Main theorem and sequences of applicability. For sequences that satisfy conditions C1–C3, we are able to provide an asymptotic formula for the integral in (1):

Theorem 1. *Let $(a_n)_{n=1}^{\infty}$ be a Roth–Szekeres sequence. Then*

$$\int_0^{\pi/2} \prod_{i=1}^{n} \cos(a_i t)\,\mathrm{d}t = \frac{1}{2}\sqrt{\frac{2\pi}{\sum_{i=1}^n a_i^2}} - \frac{\sqrt{2\pi}\sum_{i=1}^n a_i^4}{8\left(\sum_{i=1}^n a_i^2\right)^{5/2}} + O\left(n^{-s-5/2+\epsilon}\right) \tag{3}$$

for any $\epsilon > 0$.

As will become clear from the proof, it would be possible to derive further terms of an asymptotic expansion.

Corollary 1. *If $\mathbf{a} = (a_n)_{n=0}^{\infty}$ is a Roth–Szekeres sequence, then $\mathbf{a}$ is also an Erdős–Surányi sequence.*

Proof. Let $k \in \mathbb{Z}$, and let $\mathbf{a}' = (k, a_1, a_2, \dots)$. Then for $n \in \mathbb{N}$, the number of representations of k as $\pm a_1 \pm a_2 \pm \cdots \pm a_n$ is

$$\begin{aligned}
&S_{\mathbf{a}}(n,k)\\
&= \frac{2^n}{2\pi}\int_0^{2\pi} \cos(kt)\prod_{i=1}^n \cos(a_i t)\,\mathrm{d}t = \frac{2^{n+1}}{2\pi}\int_0^{\pi} \cos(kt)\prod_{i=1}^n \cos(a_i t)\,\mathrm{d}t\\
&= \frac{2^{n+1}}{2\pi}\int_0^{\pi/2} \cos(kt)\prod_{i=1}^n \cos(a_i t) + \cos(k(\pi - t))\prod_{i=1}^n \cos(a_i(\pi-t))\,\mathrm{d}t\\
&= \frac{2^{n+1}}{2\pi}\left(1 + (-1)^{k+\sum_{i=1}^n a_i}\right)\int_0^{\pi/2} \cos(kt)\prod_{i=1}^n \cos(a_i t)\,\mathrm{d}t\\
&= \frac{2^n}{\sqrt{2\pi}}\left(1+(-1)^{k+\sum_{i=1}^n a_i}\right)\left[\frac{1}{\sqrt{k^2+\sum_{i=1}^n a_i^2}} - \frac{k^4+\sum_{i=1}^n a_i^4}{4\left(k^2+\sum_{i=1}^n a_i^2\right)^{5/2}} + O\left(n^{-s-5/2+\epsilon}\right)\right] \qquad (4)
\end{aligned}$$

for any $\epsilon > 0$ by Theorem 1. Now, since $\mathbf{a}$ is a Roth–Szekeres sequence, we know that $\sum_{i=1}^n \|a_i/2\|^2/\log n \to \infty$ as $n \to \infty$ by Condition C3, and so in particular there are infinitely many $i \in \mathbb{N}$ such that a_i is odd. Hence there are infinitely many $n \in \mathbb{N}$ such that $k + \sum_{i=1}^n a_i$ is even. For these n, $S_{\mathbf{a}}(n,k) \to \infty$ as $n \to \infty$, and hence $\mathbf{a}$ is an Erdős–Surányi sequence. □

The following proposition further expands the applicability of the main theorem:

Proposition 1.

(1) *If $S \subset \mathbb{N}$ has the property that $\#\{k \in S \mid k \le n\} = O(\log n)$ as $n \to \infty$ and $\mathbf{a} = (a_n)_{n=1}^{\infty}$ is a Roth–Szekeres sequence, then the subsequence of $(a_n)_{n=1}^{\infty}$ consisting of all elements with indices* not *in S is also a Roth–Szekeres sequence for the same value of s.*

(2) *If* $\mathbf{a} = (a_n)_{n=1}^{\infty}$ *is a Roth–Szekeres sequence and is also a subsequence of a sequence* $\mathbf{b} = (b_m)_{m=1}^{\infty}$ *which satisfies conditions C1 and C2 (with a possibly different value of* s*), then* $\mathbf{b} = (b_m)_{m=1}^{\infty}$ *is also a Roth–Szekeres sequence (i.e. also satisfies condition C3).*

Proof. (1) Let $\mathbf{a}' = (a_{n_m})_{m=1}^{\infty}$ denote the subsequence of $\mathbf{a}$ with indices not in S. It is obvious that $\mathbf{a}'$ also satisfies condition C1. Moreover, for large m we have $m = \#\{k \notin S \mid k \le n_m\} = n_m + O(\log n_m)$, so $\lim_{m\to\infty} \log n_m / \log m = \lim_{m\to\infty} n_m/m = 1$; thus

$$s = \lim_{n\to\infty} \log a_n / \log n = \lim_{m\to\infty} \log a_{n_m} / \log n_m = \lim_{m\to\infty} \log a_{n_m} / \log m,$$

and so $\mathbf{a}'$ also satisfies condition C2 with the same value of s. Finally,

$$\begin{aligned}\inf_{(2a_{n_m})^{-1}<t\le 1/2} \frac{\sum_{i=1}^{m} \|a_{n_i} t\|^2}{\log m} &\ge \inf_{(2a_{n_m})^{-1}<t\le 1/2} \frac{\sum_{i=1}^{n_m} \|a_i t\|^2 \ - (n_m - m)}{\log m} \\ &= \inf_{(2a_{n_m})^{-1}<t\le 1/2} \frac{\sum_{i=1}^{n_m} \|a_i t\|^2 + O(\log n_m)}{\log m} \\ &= \inf_{(2a_{n_m})^{-1}<t\le 1/2} \frac{\sum_{i=1}^{n_m} \|a_i t\|^2}{\log n_m} + O(1) \to \infty\end{aligned}$$

as $m \to \infty$, and so $\mathbf{a}'$ also satisfies condition C3.

(2) Suppose that

$$\lim_{n\to\infty} \frac{\log a_n}{\log n} = s_1 \quad \text{and} \quad \lim_{m\to\infty} \frac{\log b_m}{\log m} = s_2,$$

and that $\mathbf{a} = (a_n)_{n=1}^{\infty}$ is the subsequence $(b_{m_n})_{n=1}^{\infty}$ of $\mathbf{b}$. It remains to show that $\mathbf{b}$ also satisfies Condition C3. Now, for all $M \in \mathbb{N}$ let $n(M)$ be the smallest $n \in \mathbb{N}$ such that $m_n \ge M$ (and thus $a_n = b_{m_n} \ge b_M > b_{m_{n-1}}$). Then

$$\frac{s_2}{s_1} = \lim_{m\to\infty} \frac{\log b_m}{\log m} \left(\lim_{n\to\infty} \frac{\log a_n}{\log n} \right)^{-1} = \lim_{n\to\infty} \frac{\log b_{m_n}}{\log m_n} \left(\lim_{n\to\infty} \frac{\log a_n}{\log n} \right)^{-1} = \lim_{n\to\infty} \frac{\log n}{\log m_n},$$

and so $\lim_{M\to\infty} \frac{\log n(M)}{\log M} = \frac{s_2}{s_1}$. Finally,

$$\begin{aligned}\inf_{(2b_M)^{-1}<t\le 1/2} \frac{\sum_{m=1}^{M} \|b_m t\|^2}{\log M} &\ge \inf_{(2a_{n(M)})^{-1}<t\le 1/2} \frac{\sum_{m=1}^{M} \|b_m t\|^2}{\log M} \\ &\ge \inf_{(2a_{n(M)})^{-1}<t\le 1/2} \frac{\sum_{k=1}^{n(M)-1} \|b_{m_k} t\|^2}{\log M}\end{aligned}$$

$$\geq \inf_{(2a_{n(M)})^{-1}<t\leq 1/2} \frac{\left(\sum_{k=1}^{n(M)}\|a_k t\|^2\right)-1}{\log n(M)} \cdot \frac{\log n(M)}{\log M}$$

$$\sim \frac{s_2}{s_1} \cdot \inf_{(2a_{n(M)})^{-1}<t\leq 1/2} \frac{\left(\sum_{k=1}^{n(M)}\|a_k t\|^2\right)-1}{\log n(M)} \to \infty$$

as $M \to \infty$, and so **b** also satisfies Condition C3. □

The first part of the proposition shows in particular that removing finitely many elements from a Roth–Szekeres sequence still yields a Roth–Szekeres sequence (by similar arguments, this is also true if finitely many elements are added). The second part shows, for instance, that sequences of the form $a_n = \lfloor n^s \rfloor$ for arbitrary rational numbers s are also Roth–Szekeres sequences, since the sequence of numbers of the form $\lfloor n^{p/q} \rfloor$ contains the sequence of all p-th powers as a subsequence.

2.2. Applications of the main theorem to more specific sequences.

2.2.1. *Polynomial-like sequences.* As an application of Theorem 1, consider the case when $\mathbf{a} = (a_n)_{n=0}^{\infty}$ has the asymptotic expansion $a_n = \alpha n^s + \beta n^{s-1} + O(n^{s-2})$ for some real numbers α, β, s, where $\alpha > 0$ and $s > 0$. Then

$$a_n^2 = \alpha^2 n^{2s} + 2\alpha\beta n^{2s-1} + O(n^{2s-2}) \quad \text{and} \quad a_n^4 = \alpha^4 n^{4s} + O\left(n^{4s-1}\right),$$

so we have that

$$\sum_{i=1}^{n} a_i^2 = \frac{\alpha^2}{2s+1} n^{2s+1} + \left(\frac{\alpha^2}{2} + \frac{\alpha\beta}{s}\right) n^{2s} + O\left(n^{2s-1}\right)$$

and

$$\sum_{i=1}^{n} a_i^4 = \frac{\alpha^4}{4s+1} n^{4s+1} + O\left(n^{4s}\right).$$

If the sequence **a** also satisfies C3, then it follows by Theorem 1 that

$$\int_0^{\pi/2} \prod_{i=1}^{n} \cos(a_i t)\,\mathrm{d}t$$
$$= \frac{\sqrt{2\pi(2s+1)}}{2\alpha n^{s+1/2}} \left[1 - \frac{(2s+1)(s(3s+1)\alpha + (4s+1)\beta)}{2s(4s+1)\alpha n}\right] + O\left(n^{-s-5/2+\epsilon}\right)$$

for any $\epsilon > 0$. In fact, in following the proof of Theorem 1 in Section 3 for this sequence, we can see that ϵ may be set to zero.

In the special case that $\mathbf{a}$ is the polynomial sequence $a_n = n^s$, which is an Erdős–Surányi sequence as remarked earlier, we obtain the following result:

$$S_{\mathbf{a}}(n) = \left(1 + (-1)^{\sum_{i=1}^{n} i^s}\right) \sqrt{\frac{2s+1}{2\pi}} \frac{2^n}{n^{s+1/2}} \left[1 - \frac{(2s+1)(3s+1)}{2(4s+1)n} + O\left(\frac{1}{n^2}\right)\right],$$

which in particular proves the asymptotic formula (2) conjectured by Andrica and Ionaşcu.

If we only have the weaker property that $a_n \sim \alpha n^s$, it still follows that

$$\int_0^{\pi/2} \prod_{i=1}^{n} \cos(a_i t)\,\mathrm{d}t \sim \frac{\sqrt{2\pi(2s+1)}}{2\alpha n^{s+1/2}}.$$

For example, if $\mathbf{a}$ is the sequence of square-free numbers, for which it is well known that $a_n \sim \pi^2 n/6$ (this is a Roth–Szekeres sequence, e.g. by part (2) of Proposition 1 applied to the sequence of primes, which are all square-free), we get

$$\int_0^{\pi/2} \prod_{i=1}^{n} \cos(a_i t)\,\mathrm{d}t \sim \frac{3\sqrt{6}}{(\pi n)^{3/2}},$$

and a corresponding asymptotic formula for the number of solutions to the signum equation.

2.2.2. *Polynomials in primes.* Let us also consider the case when f is a polynomial satisfying the properties mentioned in Section 1.2, $f(n) = \alpha n^s + O\left(n^{s-1}\right)$, and $a_n = f(p_n)$, where p_n is the n-th prime number. We will use the following standard lemma, whose proof is given for completeness:

Lemma 1. *If $q(x) = \sum_{i=0}^{s} c_i x^i$ is a polynomial with $c_s > 0$, then*

$$\sum_{i=1}^{n} q(p_i) \sim \frac{c_s}{s+1} n^{s+1} (\log n)^s.$$

Proof. We use the ideas described in Section 2.7 of [5]. Let $\pi(x)$ be the prime counting function, $\mathrm{li}(x) = \int_2^x \mathrm{d}t/\log t$ be the logarithmic integral, and define $\epsilon(x) = \pi(x) - \mathrm{li}(x)$ which is $o(t/\log t)$ by the prime number theorem. Then we write the above sum as a Stieltjes integral, where $b < 2$:

$$\sum_{i=1}^{n} q(p_i) = \sum_{p \le p_n} q(p) = \int_b^{p_n} q(t)\,\mathrm{d}\pi(t) = \int_b^{p_n} q(t)\,\mathrm{d}(\mathrm{li}(t) + \epsilon(t)).$$

Now, we note that $\mathrm{d}(\mathrm{li}(t)) = \mathrm{d}t/\log t$ and perform integration by parts on $\int_b^{p_n} q(t)\,\mathrm{d}\epsilon(t)$, giving

$$\sum_{i=1}^n q(p_i) = \int_b^{p_n} \frac{q(t)\,\mathrm{d}t}{\log t} + [q(t)\epsilon(t)]_b^{p_n} - \int_b^{p_n} \epsilon(t)q'(t)\,\mathrm{d}t.$$

Letting $b \to 2^-$ and noting that $\epsilon(b) \to 0$, this becomes

$$\begin{aligned}\sum_{i=1}^n q(p_i) &= \int_2^{p_n} \frac{q(t)\,\mathrm{d}t}{\log t} + q(p_n)\epsilon(p_n) - \int_2^{p_n} \epsilon(t)q'(t)\,\mathrm{d}t \\ &= \sum_{i=0}^s c_i \left[\int_2^{p_n} \frac{t^i\,\mathrm{d}t}{\log t} + p_n^i\epsilon(p_n) - \int_2^{p_n} \epsilon(t)it^{i-1}\,\mathrm{d}t\right].\end{aligned}$$

In the i-th summand, the first integral is an example of an exponential integral and has asymptotic expansion $\frac{p_n^{i+1}}{(i+1)\log p_n}(1+O(1/\log p_n))$, whereas using the asymptotic bound on $\epsilon(t)$ it is easily seen that the other two terms are $o(p_n^{i+1}/\log p_n)$. Hence, we have the desired asymptotic expansion, using the asymptotic formula $p_n \sim n\log n$ that follows from the prime number theorem:

$$\sum_{i=1}^n q(p_i) \sim \frac{c_s}{s+1}\frac{p_n^{s+1}}{\log p_n} \sim \frac{c_s}{s+1}n^{s+1}(\log n)^s. \qquad \square$$

Applying this lemma, we get that

$$\sum_{i=1}^n a_i^2 = \sum_{i=1}^n f(p_n)^2 \sim \frac{\alpha^2}{2s+1}n^{2s+1}(\log n)^{2s}.$$

Thus Theorem 1 gives us

$$\int_0^{\pi/2} \prod_{i=1}^n \cos(a_i t)\,\mathrm{d}t \sim \frac{1}{2}\sqrt{\frac{2\pi}{\frac{\alpha^2}{2s+1}n^{2s+1}(\log n)^{2s}}} = \frac{\sqrt{2\pi(2s+1)}}{2\alpha}\frac{1}{n^{s+1/2}(\log n)^s},$$

and so

$$S_{\mathbf{a}}(n) \sim \frac{\sqrt{2s+1}}{\sqrt{2\pi}\alpha}\frac{2^{n+1}}{n^{s+1/2}(\log n)^s} \quad \text{as } n \to \infty \text{ and } \sum_{i=1}^n f(p_n) \text{ is even.}$$

Specifically, if $\mathbf{a}$ is the sequence of primes,

$$S_{\mathbf{a}}(n) \sim \sqrt{\frac{6}{\pi}}\frac{2^n}{n^{3/2}\log n} \quad \text{as } n \to \infty \text{ for odd } n,$$

which was conjectured in [2].

3. Proof of the main theorem

To prove Theorem 1, we will require the following lemmas; in both of them, we assume that the sequence **a** satisfies Condition C2.

Lemma 2. *If $b > 0$, then $\sum_{i=1}^n a_i^b = O\left(n^{bs+1+\epsilon}\right)$ and $\left(\sum_{i=1}^n a_i^b\right)^{-1} = O\left(n^{-bs-1+\epsilon}\right)$ for any $\epsilon > 0$. As corollaries, we have the following:*

If $b > d > 0$ and $c \in \mathbb{R}$, then

$$\frac{\sqrt[d]{\sum_{i=1}^n a_i^d}}{(\log n)^c \sqrt[b]{\sum_{i=1}^n a_i^b}} \to \infty \text{ as } n \to \infty, \quad \text{and} \quad \frac{\sqrt[d]{\sum_{i=1}^n a_i^d}}{(\log n)^c a_n} \to \infty \text{ as } n \to \infty. \tag{5}$$

Proof. This follows immediately from the fact that for any $\delta > 0$, $i^{s-\delta} < a_i < i^{s+\delta}$ for sufficiently large i according to Condition C2. □

Lemma 3. *If $b_n > 0$ and $b_n^2 \sum_{i=1}^n a_i^2 / \log n \to \infty$ as $n \to \infty$, then for any $m \geq 0$ and any fixed $\ell > 0$, we have*

$$\int_0^{b_n} \pi^m t^m \exp\left(-\frac{\pi^2 t^2}{2} \sum_{i=1}^n a_i^2\right) \mathrm{d}t = 2^{(m-1)/2} \pi^{-1} \Gamma\left(\frac{m+1}{2}\right) \left(\sum_{i=1}^n a_i^2\right)^{-(m+1)/2} + O\left(n^{-\ell}\right).$$

Proof.

$$\int_0^{b_n} t^m \exp\left(-\frac{\pi^2 t^2}{2} \sum_{i=1}^n a_i^2\right) \mathrm{d}t = \int_0^{\infty} t^m \exp\left(-\frac{\pi^2 t^2}{2} \sum_{i=1}^n a_i^2\right) \mathrm{d}t - \int_{b_n}^{\infty} t^m \exp\left(-\frac{\pi^2 t^2}{2} \sum_{i=1}^n a_i^2\right) \mathrm{d}t.$$

The first integral is (substituting $u = (\pi^2 t^2/2) \sum_{i=1}^n a_i^2$)

$$\int_0^{\infty} t^m \exp\left(-\frac{\pi^2 t^2}{2} \sum_{i=1}^n a_i^2\right) \mathrm{d}t = 2^{(m-1)/2} \left(\pi^2 \sum_{i=1}^n a_i^2\right)^{-(m+1)/2} \int_0^{\infty} u^{(m-1)/2} e^{-u} \, \mathrm{d}u = 2^{(m-1)/2} \pi^{-m-1} \Gamma\left(\frac{m+1}{2}\right) \left(\sum_{i=1}^n a_i^2\right)^{-(m+1)/2},$$

and by a similar procedure the second integral can be written as follows:

$$\int_{b_n}^{\infty} t^m \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=1}^n a_i^2\right)\,\mathrm{d}t \\ = 2^{(m-1)/2}\pi^{-m-1}\left(\sum_{i=1}^n a_i^2\right)^{-(m+1)/2}\int_{x_n}^{\infty} u^{(m-1)/2}e^{-u}\,\mathrm{d}u,$$

where $x_n = \pi^2 b_n^2 \sum_{i=1}^n a_i^2/2 \to \infty$ as $n \to \infty$. Now, for u sufficiently large, $u^{(m-1)/2} \le e^{u/2}$, so for n sufficiently large,

$$0 \le \int_{x_n}^{\infty} u^{(m-1)/2}e^{-u}\,\mathrm{d}u \le \int_{x_n}^{\infty} e^{-u/2}\,\mathrm{d}u = 2e^{-x_n/2} = O\left(n^{-\ell}\right).$$

Here, the last estimate follows from the assumption made on b_n, which implies that $x_n/\log n \to \infty$ as $n \to \infty$. This completes the proof. □

Now, we are ready to prove Theorem 1:

Proof of Theorem 1. We assume throughout that n is large and thus that a_n is large positive, and is not less than a_i for $i < n$. We rewrite the integral in (3) as follows:

$$\int_0^{\pi/2}\prod_{i=1}^n \cos(a_i t)\,\mathrm{d}t = \pi\int_0^{1/(2a_n)}\prod_{i=1}^n \cos(a_i\pi t)\,\mathrm{d}t + \pi\int_{1/(2a_n)}^{1/2}\prod_{i=1}^n \cos(a_i\pi t)\,\mathrm{d}t = I_1 + I_2.$$

The second integral, I_2, can be estimated as follows, making use of the simple inequality $|\cos(\pi x)| \le \exp(-\pi^2\|x\|^2/2)$ that is valid for all real x:

$$\left|\int_{1/(2a_n)}^{1/2}\prod_{i=1}^n \cos(a_i\pi t)\,\mathrm{d}t\right| \le \int_{1/(2a_n)}^{1/2}\prod_{i=1}^n |\cos(a_i\pi t)|\,\mathrm{d}t \\ \le \int_{1/(2a_n)}^{1/2}\prod_{i=1}^n \exp\left(-\frac{\pi^2}{2}\|a_i t\|^2\right)\,\mathrm{d}t \le \int_{1/(2a_n)}^{1/2}\exp\left(-\frac{\pi^2}{2}\sum_{i=1}^n \|a_i t\|^2\right)\,\mathrm{d}t \\ \le \int_{1/(2a_n)}^{1/2}\exp\left(-\frac{\pi^2}{2}J_n\log n\right)\,\mathrm{d}t = \left[\frac{1}{2}-\frac{1}{2a_n}\right]\exp\left(-\frac{\pi^2}{2}J_n\log n\right) < \frac{1}{2}n^{-\pi^2 J_n/2}.$$

Since $J_n \to \infty$ as $n \to \infty$ by condition C3, it follows that

$$I_2 = \pi\int_{1/2a_n}^{1/2}\prod_{i=1}^n \cos(a_i\pi t)\,\mathrm{d}t = O(n^{-\ell}) \quad \text{for any } \ell > 0. \tag{6}$$

We now split up the first integral, I_1, again:

$$\pi\int_0^{1/(2a_n)}\prod_{i=1}^n\cos(a_i\pi t)\,\mathrm{d}t=\pi\int_0^{b_n}\prod_{i=1}^n\cos(a_i\pi t)\,\mathrm{d}t+\pi\int_{b_n}^{1/(2a_n)}\prod_{i=1}^n\cos(a_i\pi t)\,\mathrm{d}t=I_3+I_4$$

where $b_n\in(0,1/(2a_n))$ will be chosen later. I_4 can be estimated as before (note that $\|a_it\|=a_it$ for $0\le t\le 1/(2a_n)\le 1/(2a_i)$):

$$\begin{aligned}&\left|\int_{b_n}^{1/2a_n}\prod_{i=1}^n\cos(a_i\pi t)\,\mathrm{d}t\right|\\&\quad\le\int_{b_n}^{1/2a_n}\exp\left(-\frac{\pi^2t^2}{2}\sum_{i=1}^n a_i^2\right)\,\mathrm{d}t<\int_{b_n}^{1/2a_n}\exp\left(-\frac{\pi^2b_n^2}{2}\sum_{i=1}^n a_i^2\right)\,\mathrm{d}t\\&\quad<\int_0^{1/2a_n}\exp\left(-\frac{\pi^2b_n^2}{2}\sum_{i=1}^n a_i^2\right)\,\mathrm{d}t=\frac{1}{2a_n}\exp\left(-\frac{\pi^2b_n^2}{2}\sum_{i=1}^n a_i^2\right).\end{aligned}$$

We then have the following estimate:

$$I_4=O(n^{-\ell})\text{ for any }\ell>0,\ \text{provided that }b_n^2\sum_{i=1}^n a_i^2/\log n\to\infty\text{ as }n\to\infty.\quad(7)$$

Now, for I_3 we use the Taylor expansion

$$\log\cos x=-\frac{x^2}{2}-\frac{x^4}{12}+O(x^6),$$

which gives

$$\begin{aligned}\prod_{i=1}^n\cos(a_i\pi t)&=\exp\left(-\frac{\pi^2t^2}{2}\sum_{i=1}^n a_i^2-\frac{\pi^4t^4}{12}\sum_{i=1}^n a_i^4+O\left(t^6\sum_{i=1}^n a_i^6\right)\right)\\&=\exp\left(-\frac{\pi^2t^2}{2}\sum_{i=1}^n a_i^2\right)\left[1-\frac{\pi^4t^4}{12}\sum_{i=1}^n a_i^4+O\left(t^6\sum_{i=1}^n a_i^6+t^8\left(\sum_{i=1}^n a_i^4\right)^2\right)\right]\quad(8)\end{aligned}$$

for $|t|\le b_n$, provided that $b_n^4\sum_{i=1}^n a_i^4\to 0$ and $b_n^6\sum_{i=1}^n a_i^6\to 0$ as $n\to\infty$ (in fact, the former implies the latter by Lemma 2). Thus by Lemma 3,

$$\int_0^{b_n} \prod_{i=1}^n \cos(a_i \pi t)\,\mathrm{d}t$$

$$= \int_0^{b_n} \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=1}^n a_i^2\right)\mathrm{d}t - \sum_{i=1}^n a_i^4 \int_0^{b_n} \frac{\pi^4 t^4}{12}\exp\left(-\frac{-\pi^2 t^2}{2}\sum_{i=1}^n a_i^2\right)\mathrm{d}t$$

$$+ O\left(\sum_{i=1}^n a_i^6 \int_0^{b_n} t^6 \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=1}^n a_i^2\right)\mathrm{d}t\right)$$

$$+ O\left(\left(\sum_{i=1}^n a_i^4\right)^2 \int_0^{b_n} t^8 \exp\left(-\frac{\pi^2 t^2}{2}\sum_{i=1}^n a_i^2\right)\mathrm{d}t\right)$$

$$= \frac{1}{\sqrt{2}\pi}\Gamma(1/2)\left(\sum_{i=1}^n a_i^2\right)^{-1/2} - \frac{2^{3/2}}{12\pi}\Gamma(5/2)\sum_{i=1}^n a_i^4\left(\sum_{i=1}^n a_i^2\right)^{-5/2}$$

$$+ O\left(\sum_{i=1}^n a_i^6\left(\sum_{i=1}^n a_i^2\right)^{-7/2} + \left(\sum_{i=1}^n a_i^4\right)^2\left(\sum_{i=1}^n a_i^2\right)^{-9/2}\right) + O\left(n^{-\ell}\right)$$

for any $\ell > 0$. Now, by Lemma 2, we have

$$\sum_{i=1}^n a_i^6\left(\sum_{i=1}^n a_i^2\right)^{-7/2} = O\left(n^{-s-5/2+\epsilon}\right)$$

and

$$\left(\sum_{i=1}^n a_i^4\right)^2\left(\sum_{i=1}^n a_i^2\right)^{-9/2} = O\left(n^{-s-5/2+\epsilon}\right)$$

for any $\epsilon > 0$. Thus it follows that

$$I_3 = \frac{1}{2}\sqrt{\frac{2\pi}{\sum_{i=1}^n a_i^2}} - \frac{\sqrt{2\pi}}{8}\frac{\sum_{i=1}^n a_i^4}{\left(\sum_{i=1}^n a_i^2\right)^{5/2}} + O\left(n^{-s-5/2+\epsilon}\right) \quad \text{for any } \epsilon > 0. \tag{9}$$

Finally, combining (6), (7) and (9), we arrive at the following second-order approximation for our initial integral:

$$\int_0^{\pi/2}\prod_{i=1}^n \cos(a_i t)\,\mathrm{d}t = I_3 + I_4 + I_2 = \frac{1}{2}\sqrt{\frac{2\pi}{\sum_{i=1}^n a_i^2}} - \frac{\sqrt{2\pi}\sum_{i=1}^n a_i^4}{8\left(\sum_{i=1}^n a_i^2\right)^{5/2}} + O\left(n^{-s-5/2+\epsilon}\right)$$

for any $\epsilon > 0$. The only issue which yet remains is the existence of a sequence $(b_n)_{n=1}^\infty$ satisfying the conditions imposed on it in Lemma 3, (7) and (8). Using Lemma 2, it is easy to see that $b_n = n^{-s-1/3}$ satisfies these conditions, and hence our proof is complete. $\square$

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