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On a class of Finsler metrics with relatively isotropic mean Landsberg curvature

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Abstract. In this paper, we find an equation which characterizes a class of Finsler metrics with relatively isotropic mean Landsberg curvature. Furthermore, we determine the local structure of a class of Douglas metrics with relatively isotropic Landsberg curvature.

1. Introduction

In Finsler geometry, there are several very important non-Riemannian quantities, the simplest of them is the Cartan torsion $\mathbf{C}$. There is another, more sophisticated quantity which is determined by the Busemann–Hausdorff volume form: the so-called distortion τ. The vertical differential of τ on each tangent space gives rise to the mean Cartan torsion $\mathbf{I} = \tau_{y^k} dx^k$. $\mathbf{C}$, τ and $\mathbf{I}$ are the basic geometric data which characterize Riemannian metrics among Finsler metrics. Differentiating $\mathbf{C}$ along geodesics leads to the Landsberg curvature $\mathbf{L}$. The horizontal derivative of τ along geodesics is the so-called S-curvature $S := \tau_{|k} y^k$. The horizontal derivative of $\mathbf{I}$ along geodesics is the mean Landsberg curvature $\mathbf{J} := \mathbf{I}_k y^k$. The Riemann curvature measures the shape of the space, while the non-Riemannian quantities describe the change of the "color" on the space. Thus, figuratively saying, Finsler spaces are "colorful" geometric spaces. It also turned

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out that the flag curvature is closely related to these non-Riemannian quantities [3], [13], [14].

Recall that a Finsler metric is a *Landsberg metric* if $\mathbf{L} = 0$. Landsberg metrics can be generalized as follows. Let F be a Finsler metric on an n-dimensional manifold M. We say that F has *relatively isotropic Landsberg curvature* if $\mathbf{L} + cF\mathbf{C} = 0$, where c is a scalar function on M. We say that F has *relatively isotropic mean curvature* if $\mathbf{J} + cF\mathbf{I} = 0$. By the definitions, if F has relatively isotropic Landsberg curvature, it must have relatively isotropic mean Landsberg curvature. The converse may not be true. Many known Finsler metrics satisfy $\mathbf{J} + cF\mathbf{I} = 0$ (see [3], [5], [13]). X. Cheng and Z. Shen classify Randers metrics of isotropic flag curvature satisfying $\mathbf{J} + cF\mathbf{I} = 0$ for some c [5]. Further, Cheng–Mo–Shen characterize flag curvature of Finsler metrics of scalar flag curvature with relatively isotropic mean Landsberg curvature [3]. In [4], Cheng–Wang–Wang obtain a sufficient and necessary condition for an (α, β)-metric to be of relatively isotropic mean Landsberg curvature. Recently, Cheng–Li–Zou have studied conformally flat (α, β)-metrics with relatively isotropic mean Landsberg curvature [2].

The following Randers metric F given by

$$F(x, y) := \frac{\sqrt{(1 - \|x\|^2)\|y\|^2 + \langle x, y\rangle^2}}{1 - \|x\|^2} + \frac{\langle x, y\rangle}{1 - \|x\|^2} \tag{1}$$

is the Funk metric [7]. It is a projectively flat Finsler metric on $\mathbf{B}^n(1)$ with flag curvature $K = -\frac{1}{4}$. The Randers metric (1) satisfies $\mathbf{J} \pm \frac{1}{2}F\mathbf{I} = 0$. In [13], for Randers metrics, Shen showed that $\mathbf{J} + cF\mathbf{I} = 0$ if and only if $\mathbf{L} + cF\mathbf{C} = 0$. Moreover, the Finsler metric (1) satisfies

$$F(Ax, Ay) = F(x, y), \tag{2}$$

for all $A \in O(n)$. A Finsler metric with this property is called *spherically symmetric*. Such metrics were first studied by Rutz in [12]. They can be locally expressed on a ball $\mathbf{B}^n(\delta) \subset \mathbf{R}^n$ in the form

$$F(x, y) = \|y\|\phi\left(\|x\|, \frac{\langle x, y\rangle}{\|y\|}\right),$$

where $\|\cdot\|$ stands for the Euclidean norm in $\mathbf{R}^n$. Many known examples such as Bryant metric and Chern–Shen metric [11], [16] belong to this class. Moreover, spherically symmetric Finsler metrics form an important class of generalized (α, β)-metrics [15]. Hence, the researches on them offer references to study generalized (α, β)-metrics. Recently, some works have been carried out on spherically symmetric Finsler metrics [9], [10], [11], [16], [18]. In this paper, we mainly study spherically symmetric Finsler metrics with relatively isotropic mean Landsberg curvature and prove the following

Theorem 1.1. *Let F be a spherically symmetric Finsler metric on $\mathbf{B}^n(\delta) \subset \mathbf{R}^n$, given by $F(x,y) := \|y\|\phi(r,s) := \|y\|\phi\left(\|x\|, \frac{\langle x,y\rangle}{\|y\|}\right)$. Then F is of relatively isotropic mean Landsberg curvature if and only if*

$$[(n-2)\tilde{\rho}_0 + 3\Xi](L_2 - c\rho_1) + (r^2 - s^2)\Xi(L_1 - cT) = 0, \tag{3}$$

where c is a scalar function,

$$\tilde{\rho}_0 := \frac{1}{\phi(\phi - s\phi_s)}, \quad \rho_1 := (\phi - s\phi_s)\phi_s - s\phi\phi_{ss}, \tag{4}$$

$$T := 3\phi_s\phi_{ss} + \phi\phi_{sss}, \quad \Xi := \frac{1}{\phi[\phi - s\phi_s + (r^2 - s^2)\phi_{ss}]}, \tag{5}$$

$$L_1 := \phi P_{sss} + sQ_{sss}(\phi - s\phi_s) + 3\phi_s P_{ss} + r^2\phi_s Q_{sss}, \tag{6}$$

$$L_2 := -s\phi P_{ss} + \phi_s(P - sP_s) + [s\phi + (r^2 - s^2)\phi_s](Q_s - sQ_{ss}). \tag{7}$$

Recall that a Finsler metric is called a *Douglas metric* if its Douglas curvature vanishes. Douglas metrics form a rich class of Finsler metrics including locally projectively flat Finsler metrics. In this paper, we obtain the following classification theorem

Theorem 1.2. *Let $(\mathbf{B}^n(\delta), F)$ be a non-Riemannian spherically symmetric Douglas manifold. If F has relatively isotropic Landsberg curvature, then one of the following holds:*

(1) *F is a Berwald metric;*

(2) *F is a Randers metric which is of the following form*

$$F(x,y) := \sqrt{f(r)\|y\|^2 + g(r)\langle x,y\rangle^2} + h(r)\langle x,y\rangle, \quad y \in T_x\mathbf{B}^n(\delta) \cong \mathbf{R}^n, \tag{8}$$

where the smooth functions f, g and h satisfy

$$f[2(f + gr^2)h' - (2f' + g'r^2)h] = rh(2f + rf')(g - h^2), \tag{9}$$

with $r := \|x\|$.

Note that by (9), we can construct a lot of spherically symmetric Randers metrics with isotropic S-curvature (see [8]). Taking

$$f(r) := \frac{\varepsilon}{1 + \varepsilon r^2}, \quad h(r) := \frac{\sqrt{1 - \varepsilon^2}}{1 + \varepsilon r^2} \quad \text{and} \quad g = h^2,$$

we have the following

Example 1.3. Consider the Randers metric $F = \alpha + \beta$ on $\mathbf{R}^n$ defined by

$$\alpha(x,y) := \frac{\sqrt{\varepsilon\|y\|^2(1+\varepsilon\|x\|^2) + (1-\varepsilon^2)\langle x,y\rangle^2}}{1+\varepsilon\|x\|^2}, \quad \beta(x,y) := \frac{\sqrt{1-\varepsilon^2}\langle x,y\rangle}{1+\varepsilon\|x\|^2},$$

where ε is an arbitrary constant with $0 < \varepsilon \le 1$. Then F has relatively isotropic Landsberg curvature and isotropic S-curvature.

2. Preliminaries

Let F be a Finsler metric on an n-dimensional manifold M. The components of the fundamental tensor of (M, F) are

$$g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}. \tag{10}$$

Given a non-zero vector $y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$, F induces an inner product on T_xM given by

$$g_y(u,v) = g_{ij}u^i v^j,$$

where

$$u = u^i \frac{\partial}{\partial x^i}, \quad v = v^j \frac{\partial}{\partial x^j} \in T_xM.$$

Lemma 2.1 ([10]). *If F is a spherically symmetric Finsler metric given by $F := \|y\|\phi(r,s)$, then*

$$g_{ij} = \rho\delta_{ij} + \rho_0 x^i x^j + \rho_1\left(x^i \frac{y^j}{u} + x^j \frac{y^i}{u}\right) + \rho_2 \frac{y^i}{u}\frac{y^j}{u}, \tag{11}$$

where

$$\rho = \phi(\phi - s\phi_s), \quad \rho_0 = \phi_s^2 + \phi\phi_{ss}, \quad \rho_1 = (\phi - s\phi_s)\phi_s - s\phi\phi_{ss}, \quad \rho_2 = -s\rho_1. \tag{12}$$

The components of the inverse of (g_{ij}) are

$$g^{jk} = \tilde{\rho}_0\delta^{jk} + \frac{\tilde{\rho}_1}{u^2}y^j y^k + \frac{\tilde{\rho}_2}{u}(x^j y^k + x^k y^j) + \tilde{\rho}_3 x^j x^k, \tag{13}$$

where

$$\tilde{\rho}_0 = \frac{1}{\phi(\phi - s\phi_s)}, \quad \tilde{\rho}_1 = -\frac{s\phi + (r^2 - s^2)\phi_s}{\phi}\tilde{\rho}_2, \tag{14}$$

$$\tilde{\rho}_2 = \frac{s\phi\phi_{ss} - (\phi - s\phi_s)\phi_s}{\phi^2(\phi - s\phi_s)[\phi - s\phi_s + (r^2 - s^2)\phi_{ss}]}, \tag{15}$$

$$\tilde{\rho}_3 = -\frac{\phi_{ss}}{\phi(\phi - s\phi_s)[\phi - s\phi_s + (r^2 - s^2)\phi_{ss}]}. \tag{16}$$

Let

$$C_{ijk} := \frac{1}{4}[F^2]_{y^i y^j y^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}. \tag{17}$$

The symmetric trilinear form $\mathbf{C} := C_{ijk}dx^i \otimes dx^j \otimes dx^k$ on TM is the *Cartan torsion* of (M, F). The *mean Cartan torsion* $\mathbf{I} = I_i dx^i$ is defined by

$$I_i := g^{jk}C_{ijk}. \tag{18}$$

For a Finsler metric F, the geodesics are locally characterized by

$$\ddot{x}^i(t) + 2G^i(x(t), \dot{x}(t)) = 0,$$

where the functions

$$G^i = \frac{1}{4}g^{il}\left\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\right\} \tag{19}$$

are called the geodesic coefficients of F. A straightforward computation gives the following result:

Lemma 2.2 ([9], [10], [18])**.** *Let F be a spherically symmetric Finsler metric on $\mathbf{B}^n(\delta) \subset \mathbf{R}^n$ given by $F := \|y\|\phi(r, s)$. Let $x^1, \cdots, x^n$ be the standard coordinates on $\mathbf{R}^n$ and let $y = \sum y^i \frac{\partial}{\partial x^i}$. Then the geodesic coefficients of F are of the form:*

$$G^i = u(Py^i + uQx^i), \tag{20}$$

where $u := \|y\|$,

$$Q := \frac{1}{2r}\frac{r\phi_{ss} - \phi_r + s\phi_{rs}}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}}, \quad r := \|x\|, \quad s := \frac{\langle x, y\rangle}{\|y\|}, \tag{21}$$

and

$$P := \frac{r\phi_s + s\phi_r}{2r\phi} - \frac{Q}{\phi}[s\phi + (r^2 - s^2)\phi_s]. \tag{22}$$

The Landsberg curvature $\mathbf{L} = L_{ijk}dx^i \otimes dx^j \otimes dx^k$ is a type $(0, 3)$ tensor on $TM\backslash\{0\}$ with components

$$L_{ijk} := -\frac{1}{2}FF_{y^m}[G^m]_{y^i y^j y^k} \tag{23}$$

If $\mathbf{L} = 0$, then F is called a *Landsberg metric*. The mean Landsberg curvature $\mathbf{J} = J_i dx^i$ is defined by (cf. [13], [14])

$$J_i := g^{jk}L_{ijk}. \tag{24}$$

If $\mathbf{J} = 0$, then F is a *weakly Landsberg metric*. We say that F has *relatively isotropic Landsberg curvature* if $\mathbf{L} + cF\mathbf{C} = 0$, where c is a scalar function on M. Finally, F has *relatively isotropic mean Landsberg curvature* if $\mathbf{J} + cF\mathbf{I} = 0$.

Recall that a Finsler metric F on a manifold M is called a *Berwald metric* if $G^i(x,y) = \frac{1}{2}\Gamma^i{}_{jk}(x)y^k y^j$, i.e., G^i is quadratic in $y = y^i \frac{\partial}{\partial x^i}|_x$. A Finsler metric is called a *Douglas metric* if its geodesic coefficients satisfy

$$G^i = \frac{1}{2}\Gamma^i{}_{jk}(x)y^j y^k + P(x,y)y^i,$$

which means that it is pointwise projectively related to a Berwald metric (for details, see [1]).

For a spherically symmetric Finsler metric, Mo–Solórzano–Tenenblat proved the following

Lemma 2.3 ([9])**.** *A spherically symmetric Finsler metric on* $\mathbf{B}^n(\delta) \subset \mathbf{R}^n$ *is a Douglas metric if and only if* $Q = f(r) + g(r)s^2$*, where* Q *is given by* (21).

Let γ be a geodesic of F with $\gamma(0) = x$ and $\dot\gamma(0) = y$. Define

$$S(x,y) = \frac{d}{dt}[\tau(\gamma(t)), \dot\gamma(t)]|_{t=0},$$

where τ is the distortion of F. The function $S(x,y)$ is called the *S-curvature* of (M,F) [5], [13], [17]. A Finsler metric F is said to have *isotropic S-curvature* if there is a scalar function κ on M such that

$$S = (n+1)\kappa F. \tag{25}$$

3. Proof of Theorem 1.1

Note first that

$$u_{y^i} = \frac{y^i}{u}, \quad s_{y^i} = \frac{1}{u}\left(x^i - \frac{s}{u}y^i\right). \tag{26}$$

From (11) and (26) we get

$$\frac{\partial g_{ij}}{\partial y^k} = \rho_{y^k}\delta_{ij} + (\rho_0)_{y^k}x^i x^j + (\rho_1)_{y^k}\left(x^i\frac{y^j}{u} + x^j\frac{y^i}{u}\right) - \frac{\rho_1}{u^3}(x^i y^j + x^j y^i)y^k$$
$$+ \frac{\rho_1}{u}(x^i\delta^j{}_k + x^j\delta^i{}_k) + (\rho_2)_{y^k}\frac{y^i y^j}{u^2} + \frac{\rho_2}{u^2}(y^i\delta^j{}_k + y^j\delta^i{}_k) - \frac{2\rho_2}{u^4}y^i y^j y^k. \tag{27}$$

By (12) we obtain

$$\rho_s = \rho_1, \quad (\rho_0)_s = T, \quad (\rho_1)_s = -sT, \quad (\rho_2)_s = -\rho_1 + s^2 T, \tag{28}$$

where

$$T := 3\phi_s \phi_{ss} + \phi\phi_{sss}. \tag{29}$$

From (26)–(28) we get

$$\begin{aligned}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{u}&\left[\rho_1 \left(x^k - \frac{s}{u}y^k\right)\delta^i{}_j - sTx^i x^j \frac{y^k}{u} - (\rho_1 - s^2 T)x^i \frac{y^k}{u}\frac{y^j}{u}\right] \\ &(i \to j \to k \to i) + \frac{T}{u}x^i x^j x^k + \frac{s(3\rho_1 - s^2 T)}{u^4} y^i y^j y^k,\end{aligned} \tag{30}$$

where $i \to j \to k \to i$ denotes cyclic permutation. By (30), the Cartan torsion of a spherically symmetric Finsler metric is

$$\begin{aligned}C_{ijk} &= \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} \\ &= \frac{1}{2u}\left\{\left[\rho_1 \left(x^k - \frac{s}{u}y^k\right)\delta^i{}_j - sTx^i x^j \frac{y^k}{u} - (\rho_1 - s^2 T)x^i \frac{y^k}{u}\frac{y^j}{u}\right]\right. \\ &\quad \left.(i \to j \to k \to i) + Tx^i x^j x^k + s(3\rho_1 - s^2 T)\frac{y^i}{u}\frac{y^j}{u}\frac{y^k}{u}\right\}.\end{aligned} \tag{31}$$

It follows from (13), (18) and (31) that

$$\begin{aligned}I_i &= g^{jk} C_{ijk} \\ &= \frac{1}{2u}\left\{[(n+1)\tilde\rho_0 + 3(r^2 - s^2)\tilde\rho_3]\rho_1 + (r^2 - s^2)[\tilde\rho_0 + (r^2 - s^2)\tilde\rho_3]T\right\}\left(x^i - \frac{s}{u}y^i\right) \\ &= \frac{1}{2u}\left\{(n-2)\tilde\rho_0\rho_1 + [3\rho_1 + (r^2 - s^2)T]\Xi\right\}\left(x^i - \frac{s}{u}y^i\right),\end{aligned}$$

where we used (16) and the first equality of (4). Moreover, Ξ is given by the second equality of (5). Therefore, we have the following

Proposition 3.1. *For a spherically symmetric Finsler metric given by* $F(x, y) := \|y\|\phi(r, s)$, *where* $r := \|x\|$ *and* $s := \frac{\langle x,y\rangle}{\|y\|}$, *the mean Cartan torsion can be expressed as*

$$I_i = \frac{1}{2u}\left\{(n-2)\tilde\rho_0\rho_1 + [3\rho_1 + (r^2 - s^2)T]\Xi\right\}\left(x^i - \frac{s}{u}y^i\right), \tag{32}$$

where $\tilde\rho_0$, ρ_1, T *and* Ξ *are given by* (14), (12), (29) *and* (5), *respectively.*

Corollary 3.2. *A spherically symmetric Finsler metric is a Riemannian metric if and only if*

$$(n-2)\tilde{\rho}_0\rho_1 + [3\rho_1 + (r^2 - s^2)T]\Xi = 0.$$

In [10], the Landsberg curvature of a spherically symmetric Finsler metric has already been calculated:

$$L_{ijk} = -\frac{\phi}{2}\left\{ \left[L_2\left(x^j - \frac{s}{u}y^j\right)\delta_{ki} - sL_1x^ix^k\frac{y^j}{u} + (s^2L_1 - L_2)x^j\frac{y^k}{u}\frac{y^i}{u}\right] \right.$$
$$\left. (i \to j \to k \to i) + L_1x^ix^jx^k + s(3L_2 - s^2L_1)\frac{y^i}{u}\frac{y^j}{u}\frac{y^k}{u} \right\}, \tag{33}$$

where L_1 and L_2 are given by (6) and (7), respectively. From (13), (24) and (33) we obtain the mean Landsberg curvature of spherically symmetric Finsler metrics as

$$J_i = -\frac{\phi}{2}\{(n-2)\tilde{\rho}_0L_2 + [3L_2 + (r^2 - s^2)L_1]\Xi\}\left(x^i - \frac{s}{u}y^i\right). \tag{34}$$

Proof of Theorem 1.1. By definition, we know that F is of relatively isotropic mean Landsberg curvature if and only if $J + cFI = 0$. Thus, applying (32) and (34), we conclude the proof . □

In the two-dimensional case, Theorem 1.1 becomes simpler and we have the following corollary.

Corollary 3.3. *A spherically symmetric Finsler metric on* $\mathbf{B}^2(\delta) \subset \mathbf{R}^2$ *is of relatively isotropic Landsberg curvature if and only if*

$$3(L_2 - c\rho_1) + (r^2 - s^2)(L_1 - cT) = 0, \tag{35}$$

where L_1 *and* L_2 *are given by* (6) *and* (7)*, respectively.*

Proof. Theorem 1.1 tells us that a two-dimensional Finsler metric has relatively isotropic Landsberg curvature if and only if

$$\Xi\big[3(L_2 - c\rho_1) + (r^2 - s^2)(L_1 - cT)\big] = 0,$$

where Ξ is given by the second equality of (5). Since Ξ nowhere vanishes, (35) follows. □

4. Spherically symmetric Finsler metrics of relatively isotropic Landsberg curvature

In this section, we classify a class of spherically symmetric Finsler metrics with relatively isotropic Landsberg curvature.

From (31) and (33) we obtain

$$\begin{aligned} L_{ijk} + cFC_{ijk} = -\frac{\phi}{2}\Bigg\{ & \Big[(L_2 - c\rho_1)\left(x^j - \frac{s}{u}y^j\right)\delta_{ki} - s(L_1 - cT)x^i x^j \frac{y^k}{u} \\ & + \big[s^2(L_1 - cT) - (L_2 - c\rho_1)\big]x^j \frac{y^k}{u}\frac{y^i}{u}\Big](i \to j \to k \to i) \\ & + (L_1 - cT)x^i x^j x^k + s[3(L_2 - c\rho_1) - s^2(L_1 - cT)]\frac{y^i}{u}\frac{y^j}{u}\frac{y^k}{u}\Bigg\}, \quad (36) \end{aligned}$$

where ρ_1, T, L_1 and L_2 are given by the second equality of (4), the first equality of (5), (6) and (7), respectively.

By (36), F has relatively isotropic Landsberg curvature, i.e., there exists a scalar function c on M such that $L + cFC = 0$, if and only if ϕ satisfies

$$L_1 = cT, \quad L_2 = c\rho_1. \tag{37}$$

Plugging T, L_1 and L_2 into (37), we conclude the following result:

Proposition 4.1. *A spherically symmetric Finsler metric on $\mathbf{B}^n(\delta) \subset \mathbf{R}^n$ is of relatively isotropic Landsberg curvature if and only if the following system of equations holds:*

$$-s\phi(P_{ss} - c\phi_{ss}) + \phi_s[P - sP_s - c(\phi - s\phi_s)] + [s\phi + (r^2 - s^2)\phi_s](Q_s - sQ_{ss}) = 0, \tag{38}$$

$$\phi(P_{sss} - c\phi_{sss}) + 3\phi_s(P_{ss} - c\phi_{ss}) + [s\phi + (r^2 - s^2)\phi_s]Q_{sss} = 0. \tag{39}$$

By solving (38) and (39), we will get:

Lemma 4.2. *A spherically symmetric Finsler metric on $\mathbf{B}^n(\delta) \subset \mathbf{R}^n$ has relatively isotropic Landsberg curvature if and only if there exist functions $t_i(r)$, $i \in \{0, 1, 2, 3\}$, such that the geodesic coefficients of F are of the form $G^i = u(Py^i + uQx^i)$, where*

$$P = c(x)\phi + t_1(r)s + \frac{t_2(r)\sqrt{r^2 - s^2}}{r^2} \tag{40}$$

and

$$Q = t_0(r)s^2 - \frac{t_2(r)s\sqrt{r^2 - s^2}}{r^4} + t_3(r). \tag{41}$$

Proof. First we prove the necessity. Let

$$\Phi := P - sP_s - c(\phi - s\phi_s), \quad \Pi := s\phi + (r^2 - s^2)\phi_s. \tag{42}$$

Differentiating (42) with respect to s, yields

$$\Phi_s = -s(P_{ss} - c\phi_{ss}), \quad \Pi_s = \phi - s\phi_s + (r^2 - s^2)\phi_{ss}. \tag{43}$$

Moreover,

$$\Phi_{ss} = -(P_{ss} - c\phi_{ss}) - s(P_{sss} - c\phi_{sss}). \tag{44}$$

By (42) and (43), Eq. (38) is changed to the following:

$$(\phi\Phi)_s + \Pi(Q_s - sQ_{ss}) = 0. \tag{45}$$

Differentiating (45) with respect to s, yields

$$(\phi\Phi)_{ss} = -\Pi_s(Q_s - sQ_{ss}) + s\Pi Q_{sss}. \tag{46}$$

By (44) and the first equality of (42) and (43), we find that

$$\begin{aligned}(\phi\Phi)_{ss} &= (\phi_s\Phi + \phi\Phi_s)_s \\ &= \phi_{ss}\Phi + 2\phi_s\Phi_s + \phi\Phi_{ss} \\ &= -\phi[P_{ss} - c\phi_{ss} + s(P_{sss} - c\phi_{sss})] - 2s\phi_s(P_{ss} - c\phi_{ss}) + \phi_{ss}\Phi. \end{aligned} \tag{47}$$

It follows from (46) and (47) that

$$\begin{aligned}\Pi_s(Q_s - sQ_{ss}) &= s\Pi Q_{sss} - (\phi\Phi)_{ss} \\ &= s\Pi Q_{sss} + \phi[P_{ss} - c\phi_{ss} + s(P_{sss} - c\phi_{sss})] \\ &\quad + 2s\phi_s(P_{ss} - c\phi_{ss}) - \phi_{ss}\Phi. \end{aligned} \tag{48}$$

Plugging (39) into (48), yields

$$\begin{aligned}\Pi_s(Q_s - sQ_{ss}) &= (\phi - s\phi_s)(P_{ss} - c\phi_{ss}) - \phi_{ss}\Phi \\ &= -\frac{\phi - s\phi_s}{s}\Phi_s - \phi_{ss}\Phi. \end{aligned} \tag{49}$$

Note that Π is nowhere zero. Plugging (45) into (49), we obtain

$$\Pi\left(\phi_{ss}\Phi + \frac{\phi - s\phi_s}{s}\Phi_s\right) - \Pi_s(\phi_s\Phi + \phi\Phi_s) = 0. \tag{50}$$

I) $\Phi = 0$

In this case, $P - sP_s - c(\phi - s\phi_s) = 0$. It is easy to obtain that $P = c\phi + t_1(r)s$. Moreover, from Eq. (38), we have $Q_s - sQ_{ss} = 0$, where we have used $\Pi \neq 0$. It is easy to see that $Q = t_0(r)s^2 + t_3(r)$.

II) $\Phi \neq 0$ (nowhere 0)

Multiplying both sides of (50) by $\frac{1}{\Phi}$, (50) changes to the following:

$$\big[(\phi - s\phi_s)\Pi - s\Pi_s\phi\big]\frac{\Phi_s}{\Phi} = s(\Pi_s\phi_s - \Pi\phi_{ss}). \tag{51}$$

Inserting the second equality in (42) and (43) into (51), we obtain

$$\big[(\phi - s\phi_s)\phi_s - s\phi\phi_{ss}\big]\left[(r^2 - s^2)\frac{\Phi_s}{\Phi} - s\right] = 0. \tag{52}$$

1) $(\phi - s\phi_s)\phi_s - s\phi\phi_{ss} = 0$

Substituting $\eta = \phi^2$, the above equation reduces to $\eta_s - s\eta_{ss} = 0$. It is easy to see that $\eta = a(r) + b(r)s^2$, i.e., $\phi = \sqrt{a(r) + b(r)s^2}$. The corresponding spherically symmetric Finsler metric is a Riemannian metric. At the same time, plugging ϕ into (21) and (22), it is easy to verify that

$$P = t_1(r)s, \quad Q = t_0(r)s^2 + t_3(r),$$

where

$$t_0(r) := -\frac{2a'b - ab'}{4ar(a + br^2)}, \quad t_1(r) := \frac{a'}{2ar}, \quad t_3(r) := -\frac{a' - 2br}{4r(a + br^2)}.$$

2) $(\phi - s\phi_s)\phi_s - s\phi\phi_{ss} \neq 0$

In this case $\dfrac{\Phi_s}{\Phi} = \dfrac{s}{r^2 - s^2}$. From this, we obtain $\Phi = \dfrac{t_2(r)}{\sqrt{r^2 - s^2}}$. Hence, by the first equality of (42), we have

$$P - c\phi - s(P_s - c\phi_s) = \frac{t_2(r)}{\sqrt{r^2 - s^2}}. \tag{53}$$

By solving (53), we find that

$$P = c\phi + t_1(r)s + \frac{t_2(r)\sqrt{r^2 - s^2}}{r^2}. \tag{54}$$

Plugging (54) into (38) yields

$$\Pi\left[Q_s - sQ_{ss} + \frac{t_2(r)}{(r^2-s^2)^{\frac{3}{2}}}\right] = 0. \tag{55}$$

Since $\Pi \neq 0$, it follows that

$$Q_s - sQ_{ss} = -\frac{t_2(r)}{(r^2-s^2)^{\frac{3}{2}}}. \tag{56}$$

By solving (56), we obtain its solution

$$Q = t_0(r)s^2 - \frac{t_2(r)s\sqrt{r^2-s^2}}{r^4} + t_3(r). \tag{57}$$

Conversely, it is easy to verify that (40) and (41) satisfy (38) and (39), so, the sufficiency also holds. □

Remark. It is difficult to classify all of the spherically symmetric Finsler metrics with relatively isotropic Landsberg curvature by (54) and (57). However, we believe that all the *regular* spherically symmetric Finsler metrics with relatively isotropic Landsberg curvature are either Berwald metrics or Randers metrics.

Let us consider a spherically symmetric Douglas metric F. By Lemma 2.3 and Lemma 4.2, F has relatively isotropic Landsberg curvature if and only if there exist functions $t_i(r)$, $i = \{0, 1, 3\}$, such that the geodesic coefficients of F are of the form $G^i = u(Py^i + uQx^i)$, where

$$P = c(x)\phi + t_1(r)s \tag{58}$$

and

$$Q = t_0(r)s^2 + t_3(r). \tag{59}$$

By (22) and (21), we have

$$\frac{s\phi_r + r\phi_s}{2r\phi} - (t_0s^2 + t_3)\frac{s\phi + (r^2-s^2)\phi_s}{\phi} = c\phi + t_1s, \tag{60}$$

$$\frac{r\phi_{ss} - (\phi_r - s\phi_{rs})}{2r\left[\phi - s\phi_s + (r^2-s^2)\phi_{ss}\right]} = t_0s^2 + t_3. \tag{61}$$

Equations (60) and (61) are equivalent to

$$r\left[1 - 2(r^2-s^2)(t_0s^2+t_3)\right]\phi_s + s\phi_r - 2rs(t_0s^2 + t_1 + t_3)\phi - 2cr\phi^2 = 0 \tag{62}$$

and

$$r\big[1-2(r^2-s^2)(t_0s^2+t_3)\big]\phi_{ss}-\phi_r+s\phi_{rs}-2r(t_0s^2+t_3)(\phi-s\phi_s)=0. \quad (63)$$

Differentiating (62) with respect to s yields

$$r\big[1-2(r^2-s^2)(t_0s^2+t_3)\big]\phi_{ss}+\phi_r+s\phi_{rs}+2rs\big(3t_0s^2+t_3-t_1-2t_0r^2\big)\phi_s \\ -2r\big(3t_0s^2+t_3+t_1\big)\phi-4cr\phi\phi_s=0. \quad (64)$$

From (64)–(63) we get

$$\phi_r-rs\big[2t_0(r^2-s^2)+t_1\big]\phi_s-r(2t_0s^2+t_1)\phi-2cr\phi\phi_s=0. \quad (65)$$

From (62)–(65)$\times s$, we obtain

$$\big[1-2r^2t_3+(t_1+2t_3)s^2\big]\phi_s-s(t_1+2t_3)\phi-2c\phi^2+2cs\phi\phi_s=0. \quad (66)$$

Note that (66) is equivalent to

$$\left(\frac{1-2r^2t_3+(t_1+2t_3)s^2}{\phi^2}\right)_s+\left(\frac{4cs}{\phi}\right)_s=0. \quad (67)$$

Case 1. $c\neq 0$

1) $1-2r^2t_3+(t_1+2t_3)s^2\neq 0$

Integrating (67) yields

$$\phi=\frac{2cs+\sqrt{(1-2r^2t_3)\sigma+\big[4c^2+\sigma(t_1+2t_3)\big]s^2}}{\sigma}, \quad (68)$$

where σ is any non-zero smooth function. Then the corresponding spherically symmetric Finsler metric is a Randers metric.

2) $1-2r^2t_3+(t_1+2t_3)s^2=0$

In this case (67) reduces to $(\frac{s}{\phi})_2=0$, and it is easy to obtain that $\phi=\frac{s}{a(r)}$. Hence, the corresponding spherically symmetric Finsler metric is a Kropina metric, which is singular. Here we omit this case since the Finsler metric is assumed to be regular.

Case 2. $c = 0$

Then the geodesic coefficients of F are

$$\begin{aligned} G^i &= uPy^i + u^2Qx^i \\ &= t_1(r)usy^i + u^2\big[t_0(r)s^2 + t_3(r)\big]x^i \\ &= t_1(r)\langle x,y\rangle y^i + t_0(r)\langle x,y\rangle^2 x^i + t_3(r)|y|^2 x^i. \end{aligned} \tag{69}$$

From (69), it is easy to see that the functions G^i are quadratic in $y = y^i\dfrac{\partial}{\partial x^i}|_x$. Hence F is a Berwald metric. Next, we find explicitly the function ϕ.

It is easy to see that (66) reduces to

$$\big[1 - 2r^2t_3 + (t_1 + 2t_3)s^2\big]\phi_s - s(t_1 + 2t_3)\phi = 0. \tag{70}$$

i) $1 - 2r^2t_3 + (t_1 + 2t_3)s^2 \neq 0$

By (70) we obtain

$$\phi = \sigma(r)\sqrt{1 - 2r^2t_3 + (t_1 + 2t_3)s^2}, \tag{71}$$

where σ is any positive smooth function. Hence, in this case, the corresponding spherically symmetric Finsler metric is a Riemannian metric.

ii) $1 - 2r^2t_3 + (t_1 + 2t_3)s^2 = 0$

Note that $\phi > 0$ and $s \neq 0$. In this case, (70) is equivalent to

$$t_1 + 2t_3 = 0, \quad 1 - 2r^2t_3 + (t_1 + 2t_3)s^2 = 0. \tag{72}$$

It follows from (72) that

$$t_1 = -\frac{1}{r^2}, \quad t_3 = \frac{1}{2r^2}. \tag{73}$$

In this case, (65) implies that (62) holds. By the above caculations, it is easy to see that (62) and (65) imply (63). Therefore, we only need to solve (65). Plugging (73) into (65) yields

$$\phi_r + rs[\frac{1}{r^2} - 2(r^2 - s^2)t_0]\phi_s = r\left(-\frac{1}{r^2} + 2t_0s^2\right)\phi. \tag{74}$$

The characteristic equation of the PDE (74) is

$$\frac{dr}{1} = \frac{ds}{rs[\frac{1}{r^2} - 2(r^2 - s^2)t_0]} = \frac{d\phi}{r(-\frac{1}{r^2} + 2t_0s^2)\phi}. \tag{75}$$

It follows that

$$\frac{s^2}{e^{\int \frac{2}{r}(1-2r^4t_0)dr}+4s^2\int t_0e^{\int \frac{2}{r}(1-2r^4t_0)dr}dr}=a_1 \text{ and } \ln\frac{s}{\phi}-2\int\frac{1-r^4t_0}{r}dr=a_2$$

are independent integrals of (75). Hence the solution of (74) is

$$\phi=f\left(\frac{s^2}{e^{\int \frac{2}{r}(1-2r^4t_0)dr}+4s^2\int t_0e^{\int \frac{2}{r}(1-2r^4t_0)dr}dr}\right)e^{-2\int\frac{1-r^4t_0}{r}dr}s, \tag{76}$$

where $f(\cdot)$ is any continuously differentiable positive function. So, we have the following theorem:

Theorem 4.3. *If a non-Riemannian spherically symmetric Douglas metric F on $\mathbf{B}^n(\delta)\subset\mathbf{R}^n$ has relatively isotropic Landsberg curvature, then one of the following holds:*

(1) *F is a Berwald metric given by* (76)*;*

(2) *F is a Randers metric.*

Proof of Theorem 1.2. Under the conditions, from Theorem 4.3, we obtain that F is either a Berwald metric or a Randers metric given by (8). If F is a Randers metric, then from Lemma 5.1 in [8], we know that F is a Douglas metric. Since F has relatively isotropic Landsberg curvature, then it follows from Theorem 1.1 in [6] that F has isotropic mean Berwald curvature. Combining this with Theorem 1.1 in [5], F is of isotropic S-curvature. By Theorem 5.2 in [8], we obtain that f, g and h satisfy (9). □

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