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H-projectively Euclidean Kähler tangent bundles of natural diagonal type

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Dedicated to Professor Lajos Tamássy

Abstract. We obtain the characterization of the natural diagonal Kähler manifolds (TM, G, J) which have constant holomorphic sectional curvature, or equivalently, which are H-projectively Euclidean. Moreover, we classify the natural diagonal Kähler manifolds (TM, G, J) which are horizontally H-projectively flat (resp. vertically H-projectively flat).

1. Introduction

The holomorphically planar curves were introduced in 1954 by Otsuki and Tashiro [19] to generalize in some extent, in the Kählerian context, the notion of geodesics from the Riemannian case. In this sense, the projective transformations, i.e. the transformations preserving the geodesics (see [2], [8], [9], [26]), have as a Kählerian correspondent the holomorphically projective transformations, i.e. the transformations preserving the holomorphically planar (H-planar) curves (see [19]). A well-known result is that a Kählerian space holomorphically projective to an Euclidean space (called also H-projectively Euclidean space) has constant holomorphic sectional curvature (see [27]).

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The (para-)holomorphically-projective (i.e. (para-)H-projective) curvature tensor fields, which are invariant with respect to (para-)hollomorphically projective transformations, were studied in the context of the Kähler manifolds (e.g. in [30], [22]) and resp. para-Kähler manifolds (e.g. in [20], [21]). The holomorphically-projective transformations (i.e. preserving H-planar curves) were generalized by the holomorphically–projective mappings (see e.g. [14], [23], [24] and the references therein).

On the other hand, the theory of the natural metrics on the total space TM of the tangent bundle of a (pseudo-)Riemannian manifold (M, g), initiated by Kowalski and Sekizawa in [13], was developed by Abbassi, Sarih, Oproiu, Calvaruso, Perrone and others, including the present authors, in papers such as [1], [3]–[7], [10]–[12], [15]–[18], [25].

A natural diagonal metric on TM was obtained in [17], by lifting the metric g from the base manifold M, using four smooth functions depending on the energy density t on TM.

In [6], it was shown that the constant holomorphic sectional curvature of TM, endowed with a Kähler structure (G, J) of natural diagonal lift type, is proportional to the constant sectional curvature of the base manifold. We go further, and show that (TM, G, J) has constant holomorphic sectional curvature if and only if the base manifold in flat, and a coefficient involved in the definition of the metric G is a real constant. Moreover, the natural diagonal Kähler manifold (TM, G, J) cannot have nonzero constant holomorphic sectional curvature.

Then, we classify the natural diagonal Kähler manifolds (TM, G, J) for which the H-projective curvature tensor, restricted to the horizontal (resp. vertical) distribution, vanishes, and we call them horizontally (resp. vertically) H-projectively Euclidean Kähler manifolds.

Note that we will use throughout this paper the well-known Einstein summation convention.

2. The holomorphic sectional curvature of the tangent bundle endowed with a natural diagonal Kähler structure

Consider a Riemannian manifold (M, g), and denote by $\dot{\nabla}$ the Levi–Civita connection of g. Let $\pi : TM \to M$ be the tangent bundle of M, and let $(x^1, \dots, x^n)$ (resp. $(x^1, \dots, x^n, y^1, \dots, y^n)$) be the local coordinates on an open subset U of M (resp. on $\pi^{-1}(U) \subset TM$).

There are many ways of lifting a vector field from the base manifold M to the total space of the tangent bundle, TM. We shall use here the horizontal lift X^H and the vertical lift X^V of a vector field X to TM. More precisely, if X is locally expressed on U as $X = X^i \frac{\partial}{\partial x^i}$, then, on $\pi^{-1}(U)$, we have

$$X^H = X^i \delta_i, \quad X^V = X^i \partial_i,$$

where $\{\delta_i, \partial_j\}_{i,j=\overline{1,n}}$ is the adapted local frame on $\pi^{-1}(U)$, given by:

$$\delta_i = \frac{\partial}{\partial x^i} - \Gamma^h_{ki} y^k \frac{\partial}{\partial y^h}, \quad \partial_i = \frac{\partial}{\partial y^i}, \ \forall i = \overline{1,n},$$

$\Gamma^h_{ki}(x)$ being the Christoffel symbols of $\dot{\nabla}$.

An almost complex structure on TM, obtained as a natural diagonal lift of the Riemannian metric g, was characterized in [17] by:

$$\begin{aligned} JX^H_y &= a_1(t) X^V_y + b_1(t) g_{\pi(y)}(X, y) y^V_y, \\ JX^V_y &= -\frac{1}{a_1(t)} X^H_y + \frac{b_1(t)}{a_1(t)(a_1(t) + 2tb_1(t))} g_{\pi(y)}(X, y) y^H_y, \end{aligned}$$

for every tangent vector $y \in TM$ and every vector field X on M, where a_1, b_1 are smooth functions on $\mathbb{R}^+$, and t is the energy density of y, i.e.,

$$t = \frac{1}{2} g_{\pi(y)}(y, y). \tag{1}$$

With respect to the adapted local frame $\{\delta_i, \partial_j\}_{i,j=\overline{1,n}}$, the almost complex structure J has the expression:

$$J\delta_i = (J_1)^j_i \partial_j, \quad J\partial_i = -(J_2)^j_i \delta_j, \tag{2}$$

where the M-tensor fields $(J_\alpha)^j_i$, $\alpha = \overline{1,2}$ are defined by:

$$\begin{aligned} (J_1)^j_i &= a_1(t) \delta^j_i + b_1(t) g_{0i} y^j, \\ (J_2)^j_i &= \frac{1}{a_1(t)} \delta^j_i - \frac{b_1(t)}{a_1(t)(a_1(t) + 2tb_1(t))} g_{0i} y^j, \quad \forall i, j = \overline{1,n}. \end{aligned} \tag{3}$$

It was proved in [17] that the almost complex structure J on TM is integrable (i.e. a complex structure) if and only if the base manifold (M, g) has constant sectional curvature c and

$$b_1(t) = \frac{a_1(t) a_1'(t) - c}{a_1(t) - 2ta_1'(t)}. \tag{4}$$

A natural diagonal metric G on TM, has the expression:

$$\begin{cases} G(X_y^H, Y_y^H) &= c_1(t)g_{\pi(y)}(X,Y) + d_1(t)g_{\pi(y)}(X,y)g_{\pi(y)}(Y,y), \\ G(X_y^V, Y_y^V) &= c_2(t)g_{\pi(y)}(X,Y) + d_2(t)g_{\pi(y)}(X,y)g_{\pi(y)}(Y,y), \\ G(X_y^V, Y_y^H) &= 0, \end{cases} \tag{5}$$

for all $X, Y \in \Gamma(TM)$, $y \in TM$, where $c_1,\ c_2,\ d_1,\ d_2$ are smooth functions on $\mathbb{R}^+$.

The metric G is positive definite if and only if the functions $c_1,\ c_2,\ x \mapsto c_1(x) + 2xd_1(x),\ x \mapsto c_2(x) + 2xd_2(x)$ on $\mathbb{R}^+$ are strictly positive.

Remark 2.1. Hereafter, unless otherwise stated, all the functions $a_1,\ b_1,\ c_1,\ c_2,\ d_1,\ d_2$ are evaluated at the energy density t, given by (1).

In the adapted local frame $\{\delta_i, \partial_j\}_{i,j=\overline{1,n}}$, the matrix of the metric G is

$$\begin{pmatrix} G_{ij}^{(1)} & 0 \\ 0 & G_{ij}^{(2)} \end{pmatrix} = \begin{pmatrix} c_1 g_{ij} + d_1 g_{0i} g_{0j} & 0 \\ 0 & c_2 g_{ij} + d_2 g_{0i} g_{0j} \end{pmatrix}, \tag{6}$$

where $g_{0i}(y) = g_{ij}y^j$, and its inverse has the form

$$\begin{pmatrix} H_{(1)}^{ij} & 0 \\ 0 & H_{(2)}^{ij} \end{pmatrix} = \begin{pmatrix} \frac{1}{c_1}(g^{ij} - \frac{d_1}{c_1+2td_1} y^i y^j) & 0 \\ 0 & \frac{1}{c_2}(g^{ij} - \frac{d_2}{c_2+2d_2t} y^i y^j) \end{pmatrix}. \tag{7}$$

Adapting a result from [16] to the diagonal case (i.e. the case when the coefficients with the index 3 vanish), we have that (TM, G, J) is a Hermitian manifold of natural diagonal type if and only if the integrability conditions for J and the following relations hold good:

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} = \lambda, \quad \frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \lambda + 2t\mu, \tag{8}$$

where the proportionality coefficients $\lambda > 0$ and $\lambda + 2t\mu > 0$ are some functions on $\mathbb{R}^+$, depending on the energy density t.

Moreover, the Hermitian manifold (TM, G, J) is a Kähler manifold if and only if

$$\mu = \lambda'. \tag{9}$$

Remark 2.2. The natural diagonal Kähler structures on TM depend on two essential coefficients, a_1 and λ, which must satisfy the conditions $a_1 > 0,\ a_1 + 2tb_1 > 0$, $\lambda > 0$, $\lambda + 2t\lambda' > 0$, where b_1 is given by (4).

In [5], we obtained the following:

Proposition 2.1. *The Levi–Civita connection* ∇ *of* G *has the following expression in the adapted local frame* $\{\partial_i, \delta_j\}_{i,j=\overline{1,n}}$*:*

$$\begin{cases} \nabla_{\partial_i}\partial_j &= Q^h_{ij}\partial_h, \ \nabla_{\delta_i}\partial_j = \Gamma^h_{ij}\partial_h + P^h_{ji}\delta_h, \\ \nabla_{\partial_i}\delta_j &= P^h_{ij}\delta_h, \ \nabla_{\delta_i}\delta_j = \Gamma^h_{ij}\delta_h + S^h_{ij}\partial_h, \end{cases}$$

where Γ^h_{ij} *are the Christoffel symbols of* $\dot{\nabla}$ *and the coefficients involved in the above expressions are given as*

$$\begin{cases} Q^h_{ij} &= \frac{1}{2}(\partial_i G^{(2)}_{jk} + \partial_j G^{(2)}_{ik} - \partial_k G^{(2)}_{ij})H^{kh}_{(2)}, \\ P^h_{ij} &= \frac{1}{2}(\partial_i G^{(1)}_{jk} + R^l_{0jk} G^{(2)}_{li})H^{kh}_{(1)}, \\ S^h_{ij} &= -\frac{1}{2}(\partial_k G^{(2)}_{ij} + R^l_{0ij} G^{(2)}_{lk})H^{kh}_{(2)}, \end{cases}$$

where R^h_{kij} *are the components of the curvature tensor field of the base manifold* (M, g)*, and* ∂_i *denotes the derivative with respect to the tangential coordinates* y^i*.*

The curvature tensor field K of the connection ∇, defined by the well-known formula

$$K(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad X,Y,Z \in \Gamma(TM),$$

has the following components with respect to the adapted local frame $\{\delta_i, \partial_j\}_{i,j=\overline{1,n}}$:

$$\begin{aligned} K(\delta_i,\delta_j)\delta_k &= (P^h_{li}S^l_{jk} - P^h_{lj}S^l_{ik} + R^l_{0ij}P^h_{lk} + R^h_{kij})\delta_h, \\ K(\delta_i,\delta_j)\partial_k &= (P^l_{kj}S^h_{il} - P^l_{ki}S^h_{jl} + R^l_{0ij}Q^h_{lk} + R^h_{kij})\partial_h, \\ K(\partial_i,\partial_j)\delta_k &= (\partial_i P^h_{jk} - \partial_j P^h_{ik} + P^l_{jk}P^h_{il} - P^l_{ik}P^h_{jl})\delta_h, \\ K(\partial_i,\partial_j)\partial_k &= (\partial_i Q^h_{jk} - \partial_j Q^h_{ik} + Q^l_{jk}Q^h_{il} - Q^l_{ik}Q^h_{jl})\partial_h, \\ K(\partial_i,\delta_j)\delta_k &= (\partial_i S^h_{jk} + S^l_{jk}Q^h_{il} - P^l_{ik}S^h_{jl} - \dot{\nabla}_j R^r_{0ik}G^{(2)}_{rl}H^{(1)}_{hl})\partial_h, \\ K(\partial_i,\delta_j)\partial_k &= (\partial_i P^h_{kj} + P^l_{kj}P^h_{il} - Q^l_{ik}P^h_{lj})\delta_h. \end{aligned} \tag{10}$$

The curvature tensor field K_0 of a Kähler manifold (TM, G, J) of constant holomorphic sectional curvature k is given by:

$$\begin{aligned} K_0(X,Y)Z = \frac{k}{4}[&G(Y,Z)X - G(X,Z)Y + G(JY,Z)JX \\ &- G(JX,Z)JY + 2G(X,JY)JZ], \ \forall X,Y,Z \in \Gamma(TM), \end{aligned}$$

and it has the following components with respect to $\{\delta_i, \partial_j\}_{i,j=\overline{1,n}}$:

$$\begin{aligned}
K_0(\delta_i,\delta_j)\delta_k &= \frac{k}{4}\Big[G^{(1)}_{jk}\delta^h_i - G^{(1)}_{ik}\delta^h_j\Big]\delta_h\\
K_0(\delta_i,\delta_j)\partial_k &= \frac{k}{4}\Big[(J_1)^h_i(J_1)^l_jG^{(2)}_{lk} - (J_1)^h_j(J_1)^l_iG^{(2)}_{lk}\Big]\partial_h,\\
K_0(\partial_i,\delta_j)\delta_k &= \frac{k}{4}\Big[G^{(1)}_{ik}\delta^h_j + (J_1)^h_j(J_2)^l_iG^{(1)}_{lk} + 2(J_1)^h_k(J_1)^l_jG^{(2)}_{il}\Big]\partial_h,\\
K_0(\partial_i,\delta_j)\partial_k &= -\frac{k}{4}\Big[G^{(2)}_{ik}\delta^h_j + (J_2)^h_i(J_1)^l_jG^{(2)}_{lk} + 2(J_2)^h_k(J_1)^l_jG^{(2)}_{il}\Big]\delta_h,\\
K_0(\partial_i,\partial_j)\delta_k &= \frac{k}{4}\Big[(J_2)^h_i(J_2)^l_jG^{(1)}_{lk} - (J_2)^h_j(J_2)^l_iG^{(1)}_{lk}\Big]\delta_h,\\
K_0(\partial_i,\partial_j)\partial_k &= \frac{k}{4}\Big[G^{(2)}_{jk}\delta^h_i - G^{(2)}_{ik}\delta^h_j\Big]\partial_h.
\end{aligned}$$

Now we prove the following result.

Proposition 2.2. *Let (M, g) be a Riemannian manifold and TM the total space of its tangent bundle. The natural diagonal Kähler manifold (TM, G, J) is a complex space form (or equivalently, it is H-projectively flat) if and only if the base manifold is flat and the coefficient c_1 of G is an arbitrary real constant. Moreover, the natural diagonal Kähler manifold (TM, G, J) cannot have nonzero constant holomorphic sectional curvature.*

Proof. We have to study the conditions of vanishing of the difference between the curvature tensor fields K and K_0.

After some straightforward computations, the components of $K{-}K_0$ with respect to the adapted local frame have some expressions of the form

$$\begin{aligned}
&\alpha_1\delta^h_i g_{jk} + \alpha_2 g_{ik}\delta^h_j + \alpha_3 g_{ij}\delta^h_k + \alpha_4 g_{0i}g_{0j}\delta^h_k + \alpha_5 g_{0i}g_{0k}\delta^h_j + \alpha_6 g_{0j}g_{0k}\delta^h_i\\
&\quad + \alpha_7 g_{jk}g_{0i}y^h + \alpha_8 g_{ik}g_{0j}y^h + \alpha_9 g_{ij}g_{0k}y^h + \alpha_{10}g_{0i}g_{0j}g_{0k}y^h, \qquad (11)
\end{aligned}$$

and according to [7, Lemma 3.2], they vanish if and only if $\alpha_i = 0,\ \forall i = \overline{1,10}$, where the coefficients α_i are some smooth functions on $\mathbb{R}^+$, depending on $a_1,\ \lambda$, their three first order derivatives, the energy density t, the constant sectional curvature c of the base manifold, and the constant holomorphic sectional curvature k of TM.

In the expression of $(K - K_0)(\partial_i, \delta_j)\partial_k$, the coefficient of $\delta^h_k g_{ij}$ is

$$\alpha_3 = \frac{a_1^2a_1'\lambda - 2a_1c\lambda - a_1^2k\lambda^2 + a_1^3\lambda' + 2a_1'c\lambda t + 2a_1a_1'k\lambda^2t - 2a_1c\lambda't}{2a_1^2\lambda(a_1 - 2a_1't)},$$

hence it vanishes if and only if

$$\lambda' = -\lambda \frac{a_1^2 a_1' - 2a_1 c - a_1^2 k\lambda + 2a_1' ct + 2a_1 a_1' k\lambda t}{a_1(a_1^2 - 2ct)}. \tag{12}$$

By replacing this value of λ' into the expression of $(K - K_0)(\partial_i, \partial_j)\partial_k$, we obtain that in this component, the coefficient of $g_{ik}\delta_j^h$ is

$$\alpha_2 = -\alpha_1 = \frac{4a_1 c + a_1^2 k\lambda + 2ck\lambda t}{4a_1(a_1^2 + 2ct + 2a_1 k\lambda t)},$$

and it is zero if and only if

$$k = -\frac{4ca_1}{\lambda(a_1^2 + 2ct)}. \tag{13}$$

Now we study two cases:

Case I) When $c \neq 0$, or equivalently $k \neq 0$, we have

$$\lambda = -\frac{4ca_1}{k(a_1^2 + 2ct)}. \tag{14}$$

Then, replacing (14) into (12), the expressions of the coefficients α_1 and α_2 from $(K - K_1)(\delta_i, \delta_j)\delta_k$ take the form:

$$\alpha_1 = -\alpha_2 = \frac{2a_1^2 c}{a_1^2 + 2ct}, \tag{15}$$

which cannot vanish for $c \neq 0$, hence Case I) is not a valid one.

Case II) When $c = 0$ it follows that $k = 0$, and (12) becomes:

$$\lambda' = -\lambda \frac{a_1'}{a_1}, \tag{16}$$

which has the solution:

$$\lambda = \frac{c_0}{a_1}, \tag{17}$$

where c_0 is an arbitrary real constant.

Replacing (17) into (8), we obtain:

$$c_1 = c_0 \in \mathbb{R}.$$

Now, it is easy to verify that the flatness of the base manifold and the constancy of the coefficient c_1 are the necessary and sufficient conditions for (TM, G, J) to be a complex space form. Moreover, from (13), it follows that TM cannot have nonzero constant holomorphic sectional curvature. □

3. The H-projective curvature of (TM, G, J)

The projective curvature tensor field associated to a linear connection on a manifold, introduced in [29] and studied e.g. in [5], [28] and the references therein, is invariant under any projective transformation of the connection. Similarly, the H-projective curvature tensor field associated to a J-connection ∇ on a Kähler manifold is invariant under a H-projective transformation of ∇, i.e. a transformation which preserves the H-planar paths (see [19] and [22]). It was shown that a connected Kähler manifold is H-projectively flat if and only if it has constant holomorphic sectional curvature (see e.g. [22]).

Definition 3.1. On an n-dimensional Kähler manifold (M, g, J), the H-projective curvature tensor field associated to a J-connection ∇, is the $(1,3)$-tensor field HP, defined by:

$$\begin{aligned} HP(X,Y)Z &= R(X,Y)Z - L(Y,Z)X + L(X,Z)Y + [L(X,Y) - L(Y,X)]Z \\ &+ L(Y,JZ)JX - L(X,JZ)JY - [L(X,JY) - L(Y,JX)]JZ,\ \forall X,Y,Z \in \Gamma(TM), \end{aligned}$$

where R and Ric are respectively the curvature tensor field and the Ricci tensor field of ∇, and L is the Brinkman tensor field, given by:

$$L(X,Y) = \frac{1}{2(n+1)}\{\mathrm{Ric}(X,Y) + \frac{1}{n-1}[\widetilde{\mathrm{Ric}}(X,Y) + \widetilde{\mathrm{Ric}}(Y,X)]\}, \quad \forall X,Y \in \Gamma(TM),$$

where

$$2\widetilde{\mathrm{Ric}}(X,Y) = \mathrm{Ric}(X,Y) - \mathrm{Ric}(JX,JY), \quad \forall X,Y \in \Gamma(TM).$$

A Kähler manifold (M, g, J) is called H-projectively flat (or H-projectively Euclidean) if the H-projective curvature tensor field associated to the Levi–Civita connection of g vanishes identically.

Since the Ricci tensor associated to the Levi–Civita connection is symmetric, and the Ricci tensor on a Kähler manifold is hybrid, it follows that the H-projective curvature tensor field associated to the Levi–Civita connection on a Kähler manifold has the expression:

$$\begin{aligned} HP(X,Y)Z = R(X,Y)Z &- \frac{1}{2(n+1)}[\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y \\ &- \mathrm{Ric}(Y,JZ)JX + \mathrm{Ric}(X,JZ)JY + 2\,\mathrm{Ric}(X,JY)JZ], \end{aligned} \tag{18}$$

for every $X, Y, Z \in \Gamma(TM)$.

Now we introduce the following definition, for further use.

Definition 3.2. The Kähler manifold (TM, G, J) is called horizontally (resp. vertically) H-projectively flat if the H-projective curvature tensor field associated to the Levi–Civita connection of G vanishes on the horizontal (resp. vertical) distribution of TTM.

In the sequel, we shall characterize the H-projectively Euclidean Kähler tangent bundles of natural diagonal type.

Theorem 3.1. *Let (M, g) be a Riemannian manifold. The Kähler manifold (TM, G, J) of natural diagonal type is horizontally H-projectively flat if and only if the base manifold is flat and the coefficient c_1 of G is an arbitrary real constant, i.e. if and only if (TM, G, J) is H-projectively Euclidean, or equivalently, (TM, G, J) has constant holomorphic sectional curvature.*

Proof. We consider the H-projective curvature tensor field HP associated to the Levi–Civita connection ∇ on the natural diagonal Kähler manifold (TM, G, J). On the horizontal distribution we have:

$$HP\big(\delta_i, \delta_j\big)\delta_k = K\big(\delta_i, \delta_j\big)\delta_k + \frac{1}{2(n+1)}\Big[\operatorname{Ric}\big(\delta_i, \delta_k\big)\delta_j - \operatorname{Ric}\big(\delta_j, \delta_k\big)\delta_i\Big], \quad (19)$$

where K and Ric are, respectively, the curvature tensor field of ∇ and the corresponding Ricci tensor field, given by:

$$\operatorname{Ric}(\delta_i, \delta_k) = K^h_{ihk} + K^{\bar{h}}_{i\bar{h}k}, \quad \forall i, j, k, h, \bar{h} = \overline{1, n}, \quad (20)$$

where the indices i, j, k, h correspond to the horizontal arguments and $\bar{h}$ to the vertical argument.

Now we study the conditions under which (TM, G, J) is horizontally H-projectively flat, i.e. $HP(\delta_i, \delta_j)\delta_k$ vanishes identically.

From (10), (19) and (20), we obtain:

$$\begin{aligned} HP(\delta_i, \delta_j)\delta_k = \Bigg[&\frac{A_1 + B_1 n}{N_1} g_{jk}\delta^h_i + \frac{A_2 + B_2 n}{N_2} g_{ik}\delta^h_j + \frac{A_3}{N_3} g_{ij}\delta^h_k + \frac{A_4}{N_4} g_{0i}g_{0j}\delta^h_k \\ &+ \frac{A_5 + B_5 n}{N_5} g_{0i}g_{0k}\delta^h_j + \frac{A_6 + B_6 n}{N_6} g_{0j}g_{0k}\delta^h_i + \frac{A_7}{N_7}(g_{ik}g_{0j} - g_{kj}g_{0i})y^h\Bigg]\delta_h, \quad (21) \end{aligned}$$

where $A_\alpha,\ B_\alpha,\ N_\alpha,\ \alpha = \overline{1, 7}$ have some quite long expressions, depending on $a_1,\ \lambda$, their first three order derivatives, the constant sectional curvature c of the base manifold, and the energy density t of $y \in TM$.

According to [7, Lemma 3.2], the above expression vanishes if and only if $A_\alpha + B_\alpha n = 0,\ \alpha = \overline{1,7}$. Moreover, since we study the conditions of vanishing of the expression of $HP(\delta_i, \delta_j)\delta_k$ for the tangent bundle of a Riemannian manifold of arbitrary dimension n, it follows that $A_\alpha + B_\alpha n = 0,\ \alpha = \overline{1,7}$ for every $n > 1$, i.e. if and only if $A_\alpha = B_\alpha = 0,\ \alpha = \overline{1,7}$.

After the computations, the coefficient B_1 has the form

$$B_1 = (a_1 - 2a_1't)(\lambda + 2\lambda' t)(a_1^2 a_1' \lambda - 2a_1 c\lambda + a_1^3 \lambda' + 2a_1' c\lambda t - 2a_1 c\lambda' t).$$

Since the integrability conditions (4) must be satisfied, and taking into account Remark 1, it follows that $B_1 = 0$ if and only if

$$\lambda' = \frac{2a_1 c - a_1'(a_1^2 + 2ct)}{a_1(a_1^2 - 2ct)}\lambda. \tag{22}$$

After replacing this value of λ', the coefficients B_1 and A_2 vanish, and the expressions of other coefficients become very simple:

$$A_1 = B_2 = -A_3 = a_1^2 c, \quad A_4 = c^2, \quad A_7 = -2a_1^2 c^2,$$

hence they vanish if and only if the base manifold is flat.

Then, replacing $c = 0$ into (22), it follows that λ has the same expression as in the case of the natural diagonal tangent bundle of constant holomorphic sectional curvature, which leads to the constancy of the coefficient c_1, and thus the proof is complete. □

Theorem 3.2. *The natural diagonal Kähler manifold (TM, G, J) is vertically H-projectively flat if and only if the base manifold is flat, and the coefficient c_1 of G is a real constant, i.e. (TM, G, J) is H-projectively flat, or equivalently a complex space form.*

Proof. On the vertical distribution, the component of the H-projective curvature tensor corresponding to the Levi–Civita connection of G is:

$$HP(\partial_i, \partial_j)\partial_k = K(\partial_i, \partial_j)\partial_k + \frac{1}{2(n+1)}\Big[\mathrm{Ric}(\partial_i, \partial_k)\partial_j - \mathrm{Ric}(\partial_j, \partial_k)\partial_i\Big], \tag{23}$$

where K is the curvature tensor field of ∇ and Ric is the corresponding Ricci tensor, whose component on the vertical distribution is given as:

$$\mathrm{Ric}\left(\partial_i, \partial_k\right) = K^h_{\bar{i}h\bar{k}} + K^{\bar{h}}_{\bar{i}\bar{h}\bar{k}}, \quad \forall i, k, h, \bar{i}, \bar{k}, \bar{h} = \overline{1,n}, \tag{24}$$

where the indices i, k, h correspond to the horizontal arguments and $\bar{i}, \bar{k}, \bar{h}$ to the vertical arguments.

Using (10), (23) and (24), we obtain:

$$HP(\partial_i, \partial_j)\partial_k = \Bigg[\frac{\bar{A}_1 + \bar{B}_1 n}{\bar{N}_1}(g_{jk}\delta_i^h - g_{ik}\delta_j^h) + \frac{\bar{A}_2}{\bar{N}_2}g_{ij}\delta_k^h + \frac{\bar{A}_3}{\bar{N}_3}g_{0i}g_{0j}\delta_k^h$$
$$+ \frac{\bar{A}_4 + \bar{B}_4 n}{\bar{N}_4}g_{0k}g_{0j}\delta_i^h + \frac{\bar{A}_5 + \bar{B}_5 n}{\bar{N}_5}g_{0i}g_{0k}\delta_j^h + \frac{\bar{A}_6}{\bar{N}_6}(g_{ik}g_{0j} - g_{kj}g_{0i})y^h\Bigg]\partial_h,$$

where $\bar{A}_\alpha,\ \bar{B}_\alpha,\ \alpha = \overline{1,6}$, have some quite long expressions, depending on a_1, their first two order derivatives, the constant sectional curvature c of the base manifold, and the energy density t of $y \in TM$.

It follows that $HP(\partial_i, \partial_j)\partial_k = 0$, $\forall i, j, k = \overline{1,n}$, if and only if $\bar{A}_\alpha = \bar{B}_\alpha = 0$, $\alpha = \overline{1,6}$.

Then, after some computations and a reasoning similar to that in the previous proof, it follows that the natural diagonal Kähler manifold (TM, G, J) is vertically H-projectively flat if and only if (TM, G, J) is H-projectively flat, or equivalently, is a complex space form. □

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