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An elementary proof for the time-monotonicity of the solutions of linear parabolic equations

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In Banach spaces, POLÁČIK [12] has recently investigated the monotonicity properties with respect to the time variable of solutions of semilinear parabolic problems of form

$$\begin{cases} u' + Au = f(u) \\ u(0) = u_0, \end{cases}$$

where A is a sectorial operator, f is smooth enough and the domain of the fractional power A^α is strongly ordered for some α. Later MIERCZYŃSKI [8] generalized Poláčik's result for C^1 strongly monotone semiflows. They proved that under certain conditions the set of points near the equilibrium point having not eventually strongly monotone trajectories lie on a manifold of co-dimension one.

Both the above mentioned papers include the case of the present paper as certain linear parabolic equations are treated here using the technique of [11] to obtain a new elementary proof.

Let $n \in \mathbb{N}^+$, $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\partial\Omega$ belonging to the Hölder class $C^{2+\alpha}$ for some positive α, and L the following symmetric second order linear differential operator

$$Lu := \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u) + du,$$

where $a_{ij} \in C^{1+\alpha}(\bar{\Omega})$, $a_{ij} = a_{ji}$, $i, j = 1, \dots, n$; $d \in C^\alpha(\bar{\Omega})$, $d \leq 0$ and suppose that L is uniformly elliptic in Ω, i.e. there exists a positive number

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κ such that for every $\zeta \in \mathbb{R}^n$

$$\kappa|\zeta|^2 \leq \sum_{i,j=1}^{n} a_{ij}\zeta_i\zeta_j.$$

Let $Q := (0,+\infty) \times \Omega$, $\Gamma := [0,+\infty) \times \partial\Omega$ and $\Omega_0 := \{0\} \times \Omega$.

It is well-known [7] that there exists a sequence of solutions of the classical eigenvalue problem

$$\begin{cases} Lw + \lambda w = 0 \quad \text{in } \Omega \\ w = 0 \text{ on } \partial\Omega \\ w \in C^{2+\alpha}(\bar{\Omega}) \end{cases} . \tag{1}$$

Denote the sequence of eigenvalues by λ_k, $k \in \mathbb{N}^+$ (let them form a monotone nondecreasing sequence) and the corresponding eigenfunctions normed in $L^2(\Omega)$ by w_k.

Let $\varphi \in L^2(\Omega)$ be a given function. We examine the generalized solution of the initial-boundary value problem

$$\begin{cases} \partial_0 u - Lu = 0 \quad \text{in } Q \\ u \mid_\Gamma = 0 \\ u \mid_{\Omega_0} = \tilde{\varphi} \\ u \in H^{0,1}(Q) \end{cases} \tag{2}$$

where $\tilde{\varphi}(0,x) := \varphi(x)$ for $x \in \Omega$. For the definition of $H^{0,1}(Q)$ see e.g. [14]. Let

$$\xi_k := \int_\Omega \varphi w_k, \quad k \in \mathbb{N}^+.$$

We recall that there exists a unique weak solution of (2),

$$u(t,x) = \sum_{k=1}^{\infty} \xi_k e^{-\lambda_k t} w_k(x), \quad (t,x) \in Q$$

(convergence is understood in the norm of $H^{0,1}(Q)$) and it is smooth in $\bar{Q} \setminus \bar{\Omega}_0$ (see e.g. [14]). If $\varphi \in C^{2+\alpha}(\bar{\Omega})$ then $u \in C^{1+\alpha/2,2+\alpha}(\bar{Q})$ [3].

Results of Narasimhan [10] and Friedman [3] claim a solution of the classical initial-boundary value problem corresponding to (2) with $\varphi \in C(\Omega)$ tends to zero uniformly in Ω as t tends to infinity.

Under weaker conditions on the coefficients of L and $\partial\Omega$ we have proved [11] for any $\varphi \in L^2(\Omega)$ and fixed $x \in \Omega$ the monotonicity of the function $t \mapsto u(t,x)$ for t large enough. Moreover, we have shown that for any compact subset K of Ω there exists a positive number T such that for every $x \in K$ the function $t \mapsto u(t,x)$ is monotone in $[T,+\infty)$ provided the first Fourier coefficient of φ is not equal to zero.

Now under the given stronger conditions which ensure the existence of eigenfunctions in the classical sense we prove the same result instead of a compact subset for the whole Ω.

Due to the theorem of Krein and Rutman ([2], [6]) the principal eigenvalue λ_1 of L is simple and the corresponding eigenfunction w_1 does not vanish in Ω, thus it can be chosen a positive function in Ω.

Theorem 1. *Let u be the (unique) weak solution of the initial-boundary value problem* (2) *with the conditions given previously. Suppose that the first Fourier coefficient ξ_1 of φ is not equal to zero. Then there exists a positive number T such that for every $x \in \Omega$ the function $t \mapsto u(t,x)$, $t > T$ is strictly decreasing if $\xi_1 > 0$, and strictly increasing if $\xi_1 < 0$.*

Proof. Theorem 3 in [11] gives the following estimate for the maximum of the absolute value of w_k:

$$\max_{\bar{\Omega}}|w_k| \le M^* \lambda_k^{s^*}, \quad k \in \mathbb{N}^+, \tag{3}$$

where M^* and s^* are appropriate positive constants independent of k.

Therefore we have

$$\begin{gathered}\partial_0 u(t,x) = \\ = -e^{-\lambda_1 t} w_1(x) \left(\xi_1 \lambda_1 + e^{-(\lambda_2-\lambda_1)t} \sum_{k=2}^{\infty} \xi_k \lambda_k e^{-(\lambda_k-\lambda_2)t} \cdot \frac{w_k(x)}{w_1(x)} \right), \\ \text{where } (t,x) \in Q.\end{gathered} \tag{4}$$

First, we examine term

$$\frac{w_k(x)}{w_1(x)}\,. \tag{5}$$

Under our assumptions the outward normal derivative of w_1 does not vanish on $\partial\Omega$ (see e.g. [4] or [13]), thus there exists a positive ε such that

$$\partial_\nu w_1 \le -\varepsilon \text{ on } \partial\Omega. \tag{6}$$

For every $y \in \partial\Omega$ let us take an open, convex neighbourhood $U_y \subset \mathbb{R}^n$ such that in a system of coordinates chosen appropriately $U_y \cap \partial\Omega$ is the graph of a function belonging to the $C^{2+\alpha}$ class. We can take U_y such that $\partial_\nu w_1 \le -\varepsilon/2$ is valid in $U_y \cap \Omega$ since $w_1 \in C^1(\bar{\Omega})$. In addition we may assume for every $x \in U_y \cap \Omega$ the existence of a point $\beta_x \in \partial\Omega$ such that the direction $\beta_x - x$ coincides with the outward normal direction at β_x (e.g. let $\min\{|\beta - x| : \beta \in \partial\Omega\}$ be attained at β_x). From the open cover

$$\partial\Omega \subset \bigcup_{y\in\partial\Omega} U_y$$

we can select a finite cover $\{U_{y_1}, \dots, U_{y_N}\}$. Let K be the following compact set:

$$K := \Omega \setminus \bigcup_{i=1}^{N} U_{y_i}.$$

With $\delta := \min\{w_1(x) : x \in K\}$ we have

$$\left|\frac{w_k(x)}{w_1(x)}\right| \le \frac{M^*}{\delta}\lambda_k^{s^*} \quad \text{for } x \in K. \tag{7}$$

Now we will examine term (5) near the boundary. Due to the homogeneous Dirichlet boundary condition we can write

$$\left|\frac{w_k(x)}{w_1(x)}\right| = \left|\frac{w_k(x) - w_k(\beta_x)}{w_1(x) - w_1(\beta_x)}\right| = \left|\frac{\partial_\nu w_k(\eta_x)}{\partial_\nu w_1(\eta_x)}\right| \quad \text{for } x \in \Omega \cap \left(\bigcup_{i=1}^{N} U_{y_i}\right),$$

where $\beta_x \in \partial\Omega$, the direction $\beta_x - x$ coincides with the outward normal direction ν, and η_x is an appropriate point in the segment (β_x, x). Therefore, by using (6) we have

$$\left|\frac{w_k(x)}{w_1(x)}\right| \le \frac{2}{\varepsilon}\max_{\bar\Omega}|\operatorname{grad} w_k| \le \frac{2}{\varepsilon}\|w_k\|_{C^1(\bar\Omega)} \quad \text{in } \Omega \cap \left(\bigcup_{i=1}^{N} U_{y_i}\right). \tag{8}$$

Ladyženskaja and Ural'ceva [7] proved boundedness in $C^1(\bar\Omega)$-norm for the solution of the generalized elliptic boundary value problem under certain conditions. By using their proof we have obtained a bound in $C^1(\bar\Omega)$-norm for the solution w_k of the eigenvalue problem (1) depending on the eigenvalue λ_k. In the Appendix we have shown the existence of positive numbers N^* and r^* such that

$$\|w_k\|_{C^1(\bar\Omega)} \le N^*\lambda_k^{r^*}, \quad k \in \mathbb{N}^+. \tag{9}$$

(For the details see Theorem 2.)

By using estimates (7), (8) and (9) we obtain

$$\left|\frac{w_k(x)}{w_1(x)}\right| \le C\lambda_k^{\sigma}, \quad k \in \mathbb{N}^+, \quad x \in \Omega$$

where $C := \max\left\{\frac{2N^*}{\varepsilon}, \frac{M^*}{\delta}\right\}$ and $\sigma := \max\{r^*, s^*\}$.

Finally we examine the series in (4) as it was done in [11].

$$\left|\sum_{k=2}^{\infty} \xi_k \lambda_k e^{-(\lambda_k-\lambda_2)t} \cdot \frac{w_k(x)}{w_1(x)}\right| \le C \sum_{k=2}^{\infty} |\xi_k|\ |\lambda_k|^{\sigma+1} e^{-(\lambda_k-\lambda_2)t} \tag{10}$$

for $(t,x) \in Q$. The series on the right-hand side of (10) admits a finite sum for every $t \in \mathbb{R}^+$ due to the following estimate for the eigenvalues λ_k:

$$C_1 k^{2/n} \le \lambda_k \le C_2 k^{2/n}, \quad k \in \mathbb{N}^+ \tag{11}$$

(C_1 and C_2 are appropriate positive constants, see e.g. [9], [14]). Moreover, it is easy to see that both series in (10) have an upper bound independent of t (see [11]), thus the function

$$t \mapsto e^{-(\lambda_2-\lambda_1)t} \sum_{k=2}^{\infty} \xi_k \lambda_k e^{-(\lambda_k-\lambda_2)t} \cdot \frac{w_k(x)}{w_1(x)}$$

tends to zero uniformly in Ω as $t \to +\infty$. For this reason there exists a positive number T such that

$$\operatorname{sign}\{\partial_0 u(t,x)\} = \operatorname{sign}\{-\xi_1\lambda_1\} \quad \text{for } (t,x) \in (T,+\infty) \times \Omega.$$

Theorem 1 is proved.

Appendix

Here we prove formula (9), i.e. we give an upper bound for the $C^1(\bar{\Omega})$-norm of the eigenfunctions w_k of (1) depending on the eigenvalue λ_k. The proof was obtained by complementing the proof of Theorem 15.1 in [7].

Theorem 2. *There exist positive numbers $N^*, r^* \in \mathbb{R}^+$ such that for the eigenfunctions w_k of* (1) *normed in $L^2(\Omega)$*

$$\|w_k\|_{C^1(\bar{\Omega})} \le N^* \lambda_k^{r^*}, \quad k \in \mathbb{N}^+ \tag{12}$$

holds (or, equivalently $\|w_k\|_{C^1(\bar{\Omega})} \le N k^r$ for some $N, r \in \mathbb{R}^+$).

Proof. Let $p > n$. According to Ladyženskaja and Ural'ceva there exists a positive constant K_1 such that

$$\|v\|_{W^{2,p}(\Omega)} \le K_1 \left(\|Lv\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}\right) \tag{13}$$

for arbitrary $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [7], formula (11.8) in part III). Applying this a priori estimate to w_k we obtain

$$\|w_k\|_{W^{2,p}(\Omega)} \le K_1(\lambda_k + 1)\|w_k\|_{L^p(\Omega)} \le K_2 \lambda_k \|w_k\|_{L^p(\Omega)}, \quad k \in \mathbb{N}^+ \tag{14}$$

for an appropriate positive number K_2.

According to the Sobolev imbedding theorem (see e.g. [1]) $W^{2,p}(\Omega) \subset C^1(\bar{\Omega})$ for $p > n$, and there exists a positive number K_3 such that for every $k \in \mathbb{N}^+$

$$\|w_k\|_{C^1(\bar{\Omega})} \le K_3 \|w_k\|_{W^{2,p}(\Omega)}. \tag{15}$$

From (14) and (15) we obtain the following inequality with some positive constant K_4:

$$\|w_k\|_{C^1(\bar{\Omega})} \le K_4 \lambda_k \|w_k\|_{L^p(\Omega)}, \quad k \in \mathbb{N}^+.$$

The $L^p(\Omega)$-norm of w_k can trivially be estimated by using the maximum norm of w_k:

$$\|w_k\|_{L^p(\Omega)} \le \operatorname{mes}(\Omega)^{1/p} \max_{\bar{\Omega}} |w_k|, \quad k \in \mathbb{N}^+.$$

Finally we use (3), i.e. the estimate for the maximum norm of w_k to get

$$\|w_k\|_{L^p(\Omega)} \le K_5 \lambda_k^{s^*}, \quad k \in \mathbb{N}^+$$

with appropriate positive constants K_5 and s^*, which leads to

$$\|w_k\|_{C^1(\bar{\Omega})} \le K_4 K_5 \lambda_k^{s^*+1}, \quad k \in \mathbb{N}^+.$$

Applying estimate (11) we find a bound depending on k for some N, $r \in \mathbb{R}^+$:

$$\|w_k\|_{C^1(\bar{\Omega})} \le N k^r, \quad k \in \mathbb{N}^+.$$

Theorem 2 is proved.

Remark 1. Supposing some more smoothness on $\partial\Omega$, results of Koshelev [5] could have been used instead of (13). As a special case, his paper gives conditions for the existence in $W^{2,p}(\Omega)$ of the solution of (1), and gives a bound for the $W^{2,p}(\Omega)$-norm of the solution depending on its $L^p(\Omega)$-norm.

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