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Consistent invertibility and perturbations for property (ω)

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Abstract. An operator $T \in B(H)$ is said to be "consistent in invertibility" provided that for each $S \in B(H)$, TS and ST are either both or neither invertible. Using the induced spectrum, the paper investigates the permanence of property (ω) under some commuting perturbations, which extends the corresponding results in P. Aiena *et al.* (2007) [3]. In addition, the stability of property (ω) of the operators which are the products of finitely normal operators is considered.

1. Introduction

Throughout this paper, H means an infinite-dimensional complex Hilbert space, and $B(H)$ the algebra of all bounded linear operators on H. For $T \in B(H)$, let T^*, $N(T)$, $R(T)$, $\sigma(T)$ and $\sigma_p(T)$ denote the adjoint, the null space, the range, the spectrum and the point spectrum of T, respectively. Set $n(T) = \dim N(T)$ and $d(T) = \dim H/R(T) = \text{codim}R(T)$. An operator $T \in B(H)$ is called upper semi-Fredholm if $n(T) < \infty$ and $R(T)$ is closed, while T is called lower semi-Fredholm if $d(T) < \infty$. If both $n(T)$ and $d(T)$ are finite, T is a Fredholm operator. If T is upper (lower) semi-Fredholm, the index of T is denoted by

$$\text{ind}(T) = n(T) - d(T).$$

The operator T is Weyl if it is Fredholm of index zero. Recall that T is said to be bounded below if it is injective and has closed range. Define $W_+(H) =$

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$\{T \in B(H) : T$ is upper semi-Fredholm and $\operatorname{ind}(T) \leq 0\}$. The Weyl essential approximate point spectrum $\sigma_{aw}(T)$ and the approximate point spectrum $\sigma_a(T)$ are defined by: $\sigma_{aw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin W_+(H)\}$ and $\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not bounded below$\}$, respectively.

The ascent of T is defined as

$$p = p(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}.$$

If $\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\} = \emptyset$, we think p as ∞. Furthermore, the descent of T is defined as

$$q = q(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\},$$

and if $\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\} = \emptyset$, then we think q as ∞.

As we have known, if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$. If T is Fredholm with $p(T) = q(T) < \infty$, we call it a Browder operator. And then the Browder spectrum $\sigma_b(T)$ of T is defined as

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}.$$

We recall that an operator $R \in B(H)$ is said to be Riesz if $R - \lambda I$ is Fredholm for all $\lambda \in \mathbb{C} \setminus \{0\}$. Browder spectra and Weyl essential approximate point spectra are invariant under commuting Riesz perturbations (see [11]), i.e., if R is a Riesz operator such that $TR = RT$, then $\sigma_b(T) = \sigma_b(T+R)$ and $\sigma_{aw}(T) = \sigma_{aw}(T+R)$.

Definition 1.1. An operator $T \in B(H)$ is said to satisfy property (ω) if

$$\sigma_a(T) \setminus \sigma_{aw}(T) = \pi_{00}(T),$$

where $\pi_{00}(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < n(T - \lambda I) < \infty\}$, and $\operatorname{iso}\sigma(T)$ denotes the set of isolated points of $\sigma(T)$.

Definition 1.2. We call an operator $T \in B(H)$ consistent in invertibility (abbrev. CI) provided there is implication, for arbitrary $S \in B(H)$, $ST \in G$ if and only if $TS \in G$, where $G = \{T \in B(H) : T^{-1} \in B(H)\}$.

Curiously, this notion is already in Weyl's paper [12]; it was discussed comprehensively on Hilbert spaces by Gong and Han [9], and on Banach spaces and in Calkin algebras by Djordjević [8]. Also by $\sigma_{CI}(T)$ we denote

$$\sigma_{CI}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not CI}\},$$

and call it CI spectrum of T. Note that the CI spectrum need be neither closed nor nonempty (see [7]).

Recall that an "isoloid" operator is one of which the isolated points of the spectrum are all eigenvalues, while an "a-isoloid" operator is one of which the isolated points of its approximate point spectrum are all eigenvalues.

Property (ω), a variant of WEYL's theorem [12] introduced by V. RAKOČEVIČ in [10], has been studied in more recent papers, [1], [2] and [5]. This paper is a continuation of a previous paper of CAO and the author [6], where the stability of generalized property (ω) under certain classes of perturbations is studied. This paper is also inspired by [1], [3], [4]. One of their main results in [3] is as follows:

Theorem 1.1. *Suppose that $T \in B(H)$ is a-isoloid, and K is a finite rank operator commuting with T such that $\sigma_a(T) = \sigma_a(T+K)$. If T satisfies property (ω), then $T+K$ satisfies property (ω).*

However, the condition $\sigma_a(T) = \sigma_a(T+K)$ is not necessary.

In this paper, using the CI spectrum, we give the necessary and sufficient conditions for the stability of property (ω) under perturbations by finite rank operators, nilpotent operators and Riesz operators, which generalize Theorem 1.1.

2. Property (ω) and perturbations

In order to study the stability of property (ω), let us begin with a lemma, the proof of which can be found in [6].

Lemma 2.1. *Let $T \in B(H)$. If $K \in B(H)$ is a finite rank operator which commutes with T, then*

$$\operatorname{iso}\sigma(T+K) \subseteq \operatorname{iso}\sigma(T) \cup \rho(T).$$

We turn to a variant of the essential approximate point spectrum. Let

$$\rho_1(T) = \Big\{\lambda \in \mathbb{C} : n(T-\lambda I) < \infty \text{ and there exists } \epsilon > 0 \text{ such that}$$
$$T - \mu I \in W_+(H),\ N(T-\mu I) \subseteq \bigcap_{n=1}^{\infty} R[(T-\mu I)^n] \text{ if } 0 < |\mu - \lambda| < \epsilon\Big\}$$

and let $\sigma_1(T) = \mathbb{C} \setminus \rho_1(T)$. Then $\sigma_1(T) \subseteq \sigma_{aw}(T) \subseteq \sigma_b(T) \subseteq \sigma(T)$.

Theorem 2.1. *Suppose that $T \in B(H)$ satisfies property (ω). If K is a finite rank operator commuting with T, then $T+K$ is isoloid and satisfies property (ω) if and only if $\sigma_b(T) \cap \sigma_a(T+K) = \sigma_1(T) \cup [\overline{\sigma_{CI}(T)} \cap \sigma_a(T)]$.*

Proof. First, we will prove the necessary. Suppose that $T+K$ is isoloid and satisfies property (ω). The inclusion

$$\sigma_b(T)\cap\sigma_a(T+K)\supseteq\sigma_1(T)\cup[\overline{\sigma_{CI}(T)}\cap\sigma_a(T)]$$

is easily to prove.

For the reverse inclusion, let $\lambda_0\notin\sigma_1(T)\cup[\overline{\sigma_{CI}(T)}\cap\sigma_a(T)]$, then $n(T-\lambda_0 I)<\infty$, and there exists $\epsilon>0$ such that $T-\lambda I\in W_+(H)$ and

$$N(T-\lambda I)\subseteq\bigcap_{n=1}^{\infty}R[(T-\lambda I)^n]$$

if $0<|\lambda-\lambda_0|<\epsilon$. Since property (ω) holds for T, $T-\lambda I$ is bounded below if $0<|\lambda-\lambda_0|<\epsilon$. There are two cases to be considered.

Case 1. Suppose $\lambda_0\notin\overline{\sigma_{CI}(T)}$, then $T-\lambda I$ is invertible if $0<|\lambda-\lambda_0|$ is small enough. This means that $\lambda_0\in\operatorname{iso}\sigma(T)\cup\rho(T)$. Thus $\lambda_0\in\operatorname{iso}\sigma(T+K)\cup\rho(T+K)$. But since $T+K$ is isoloid and $n(T-\lambda_0 I)<\infty$ implies $n(T+K-\lambda_0 I)<\infty$, $\lambda_0\in\pi_{00}(T+K)$. The fact that property (ω) holds for $T+K$ tells us that $T+K-\lambda_0 I$ is Browder, then $T-\lambda_0 I$ is Browder, which means that $\lambda_0\notin\sigma_b(T)\cap\sigma_a(T+K)$.

Case 2. If $\lambda_0\notin\sigma_a(T)$, then $\lambda_0\notin\sigma_a(T+K)$ or $\lambda_0\in\sigma_a(T+K)\setminus\sigma_{aw}(T+K)$. We may suppose that $\lambda_0\in\sigma_a(T+K)\setminus\sigma_{aw}(T+K)$. Since property (ω) holds for $T+K$, $T+K-\lambda_0 I$ is Browder. Then $T-\lambda_0 I$ is Browder. Again, we prove that $\lambda_0\notin\sigma_b(T)\cap\sigma_a(T+K)$.

Conversely, assume that

$$\sigma_b(T)\cap\sigma_a(T+K)=\sigma_1(T)\cup[\overline{\sigma_{CI}(T)}\cap\sigma_a(T)].$$

Let $\lambda_0\in\sigma_a(T+K)\setminus\sigma_{aw}(T+K)$. Then $T-\lambda_0 I\in W_+(H)$. If $\lambda_0\in\sigma_a(T)$, using the fact that property (ω) holds for T, we know that $T-\lambda_0 I$ is Browder. This induces that $T+K-\lambda_0 I$ is Browder. In the case that $\lambda\notin\sigma_a(T)$, we get that $\lambda_0\notin\sigma_1(T)\cup[\overline{\sigma_{CI}(T)}\cap\sigma_a(T)]$. Thus $\lambda_0\notin\sigma_b(T)\cap\sigma_a(T+K)$. But since $\lambda_0\in\sigma_a(T+K)$, we know $\lambda_0\notin\sigma_b(T)$. This induces that $T-\lambda_0 I$ is Browder. Again, we have that $T+K-\lambda_0 I$ is Browder.

Now, we prove that $\lambda_0\in\pi_{00}(T+K)$. For the converse, let $\lambda_0\in\pi_{00}(T+K)$. Using Lemma 2.1, $\lambda_0\in\operatorname{iso}\sigma(T)\cup\rho(T)$. Without loss of generality, one can suppose that $\lambda_0\in\operatorname{iso}\sigma(T)$. The fact that K is finite rank tells that $n(T-\lambda_0 I)<\infty$. Now, we can see that $\lambda_0\notin\sigma_1(T)\cup[\overline{\sigma_{CI}(T)}\cap\sigma_a(T)]$. Then $T-\lambda_0 I$ is Browder, and hence $T+K-\lambda_0 I$ is also Browder, which means that $\lambda_0\in\sigma_a(T+K)\setminus\sigma_{aw}(T+K)$.

In the following, we will prove that $T+K$ is isoloid. Indeed, set $\lambda_0 \in \operatorname{iso}\sigma(T+K)$ but $N(T+K-\lambda_0 I)=\{0\}$, then $\lambda_0 \in \operatorname{iso}\sigma(T)\cup\rho(T)$ and $n(T-\lambda_0 I)<\infty$. This shows that $\lambda_0 \notin \sigma_1(T)\cup[\overline{\sigma_{CI}(T)}\cap\sigma_a(T)]$. Thus $T+K-\lambda_0 I$ is bounded below or $T-\lambda_0 I$ is Browder. In each case, we may get that $T+K-\lambda_0 I$ is invertible. It is in contradiction to the fact that $\lambda_0 \in \operatorname{iso}\sigma(T+K)$. This says that $T+K$ is isoloid. □

Example 1. Let $T \in B(\ell^2)$ and $K=0\in B(\ell^2)$ be defined by:

$$T(x_1,x_2,x_3,\cdots)=(0,x_1,x_2,x_3,\cdots).$$

Clearly, $KT=TK$, and

$$\sigma_b(T)\cap\sigma_a(T+K)=\sigma_1(T)\cup[\overline{\sigma_{CI}(T)}\cap\sigma_a(T)]=\{\lambda\in\mathbb{C}:|\lambda|=1\}.$$

According to Theorem 2.1, one can get that $T+K$ satisfies property (ω) and is isoloid.

Using Theorem 2.1, we will give another proof of Theorem 1.1.

Corollary 2.1. *Suppose that $T\in B(H)$ is a-isoloid, and K is a finite rank operator commuting with T such that $\sigma_a(T)=\sigma_a(T+K)$. If T satisfies property (ω), then $T+K$ satisfies property (ω).*

Proof. It suffices to prove that

$$\sigma_b(T)\cap\sigma_a(T+K)=\sigma_1(T)\cup[\overline{\sigma_{CI}(T)}\cap\sigma_a(T)].$$

Since T is a-isoloid and property (ω) holds for T, we can get that $\sigma_b(T)=\sigma_1(T)\cup\overline{\sigma_{CI}(T)}$. Then

$$\begin{aligned}\sigma_b(T)\cap\sigma_a(T+K)&=\sigma_b(T)\cap\sigma_a(T)=[\sigma_1(T)\cup\overline{\sigma_{CI}(T)}]\cap\sigma_a(T)\\&=[\sigma_1(T)\cup\overline{\sigma_{CI}(T)}]\cap[\sigma_1(T)\cup\sigma_a(T)]\\&=\sigma_1(T)\cup[\overline{\sigma_{CI}(T)}\cap\sigma_a(T)].\end{aligned}$$ □

Recall that T is finite-isoloid if $\operatorname{iso}\sigma(T)\subseteq\{\lambda\in\mathbb{C}:0<n(T-\lambda I)<\infty\}$.

Corollary 2.2. *Suppose that T is finite-isoloid and satisfies property (ω). If K is a compact operator commuting with T, then $T+K$ is isoloid and satisfies property (ω) if and only if*

$$\sigma_b(T)\cap\sigma_a(T+K)=\sigma_1(T)\cup[\overline{\sigma_{CI}(T)}\cap\sigma_a(T)].$$

Corollary 2.3. *Suppose that T satisfies property (ω). If K is a finite rank operator commuting with T, then $T+K$ is finite-isoloid and satisfies property (ω) if and only if*

$$\sigma_b(T) \cap \sigma_a(T+K) = [\sigma_1(T) \cap acc\,\sigma(T)] \cup [\overline{\sigma_{CI}(T)} \cap \sigma_a(T)].$$

Let $H(T)$ be the class of all complex-valued functions which are analytic on a neighborhood of $\sigma(T)$, and are not constant on any component of $\sigma(T)$. Let $\sigma_{SF_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Fredholm}\}$.

Corollary 2.4. *Suppose that T is isoloid and satisfies property (ω). If K is a finite rank operator commuting with T, then the following statements are equivalent:*

(1) *for any $f \in H(T)$, $f(T) + K$ satisfies property (ω) and $\sigma_a(f(T) + K) = \sigma(f(T) + K)$;*
(2) *for any $f \in H(T)$, $f(\sigma_1(T)) = \sigma_1(f(T))$ and $\sigma_b(T) = \sigma_1(T)$;*
(3) *for any $\lambda \in \mathbb{C} \setminus \sigma_{SF_+}(T)$, $\mathrm{ind}(T - \lambda I) \geq 0$;*
(4) *$\sigma_{CI}(T) \cap \rho_a(T) = \emptyset$.*

Proof. (1) $\Longrightarrow$ (2). First, we shall prove that $\sigma_b(T) = \sigma_1(T)$. We only need to prove that $\sigma_b(T) \subseteq \sigma_1(T)$. Let $\lambda_0 \notin \sigma_1(T)$, then $n(T - \lambda_0 I) < \infty$, $T - \lambda I \in W_+(H)$ and

$$N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n],$$

if $0 < |\lambda - \lambda_0|$ is sufficiently small. Since property (ω) holds for T, $T - \lambda I$ is bounded below. Then $T + K - \lambda I$ is bounded below or $T + K - \lambda I$ is Browder. From the fact that $\sigma(T+K) = \sigma_a(T+K)$, we know that $T+K-\lambda I$ is invertible or Browder. Therefore, $T - \lambda I$ is invertible. This shows that $\lambda_0 \in \mathrm{iso}\,\sigma(T) \cup \rho(T)$. Then $T - \lambda_0 I$ is Browder, since T is isoloid and satisfies property (ω), which means that $\lambda_0 \notin \sigma_b(T)$. Now, we have that

$$f(\sigma_1(T)) = f(\sigma_b(T)) = \sigma_b(f(T)) \supseteq \sigma_1(f(T)).$$

For the converse inclusion, let $\mu_0 \notin \sigma_1(f(T))$. Then $\dim N(f(T) - \mu_0 I) < \infty$, and there exists $\epsilon > 0$ such that $f(T) - \mu I \in W_+(H)$ and

$$N(f(T) - \mu I) \subseteq \bigcap_{n=1}^{\infty} R[(f(T) - \mu I)^n],$$

if $0 < |\mu - \mu_0| < \epsilon$. Thus $f(T) - \mu I + K \in W_+(H)$. This shows that $f(T) - \mu I + K$ is bounded below or $\mu \in \sigma_a(f(T) + K) \setminus \sigma_{aw}(f(T) + K)$. We may get that $f(T) - \mu I + K$ is invertible or Browder, hence $f(T) - \mu I$ is Browder. But since

$$N(f(T) - \mu I) \subseteq \bigcap_{n=1}^{\infty} R[(f(T) - \mu I)^n],$$

the operator $f(T) - \mu I$ is invertible. This proves that $\mu_0 \in \operatorname{iso} \sigma(f(T))$.

Set

$$f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1} (T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T),$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Then $\lambda_i \in \operatorname{iso} \sigma(T) \cup \rho(T)$. Since

$$N[f(T) - \mu_0 I] \supseteq N[(T - \lambda_i I)^{n_i}]$$

and

$$n(f(T) - \mu_0 I) < \infty,$$

it follows that $n(T - \lambda_i I) < \infty$ for every $\lambda_i, 1 \leq i \leq k$. Therefore, $\lambda_i \notin \sigma_1(T)$, and we prove that $f(\sigma_1(T)) \subseteq \sigma_1(f(T))$.

(2) $\Longrightarrow$ (3). If $\lambda_0 \in \mathbb{C} \setminus \sigma_{SF_+}(T)$ such that $\operatorname{ind}(T - \lambda_0 I) < 0$, then $\lambda_0 \notin \sigma_a(T)$ or $\lambda_0 \in \sigma_a(T) \setminus \sigma_{aw}(T)$. Thus $T - \lambda_0 I$ is Browder. It is a contradiction.

(3) $\Longrightarrow$ (4). If $\lambda_0 \in \sigma_{CI}(T) \cap \rho_a(T)$, then $\operatorname{ind}(T - \lambda_0 I) < 0$. It is in contradiction to the fact that $\operatorname{ind}(T - \lambda I) \geq 0$ for any $\lambda \in \mathbb{C} \setminus \sigma_{SF_+}(T)$.

(4) $\Longrightarrow$ (1). Suppose that $\sigma_{CI}(T) \cap \rho_a(T) = \emptyset$. Then $\sigma(T) = \sigma_a(T)$, and using the fact that property (ω) holds for T, $\operatorname{ind}(T - \lambda I) \geq 0$ for any $\lambda \in \mathbb{C} \setminus \sigma_{SF_+}(T)$. We claim that for any $f \in H(T)$, $f(T)$ is isoloid and property (ω) holds for $f(T)$.

Indeed, let $\mu_0 \in \sigma_a(f(T)) \setminus \sigma_{aw}(f(T))$, suppose that

$$f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1} (T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T),$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Then $T - \lambda_i I$ is Weyl. Since property (ω) holds for T, $T - \lambda_i I$ is Browder. Then $f(T) - \mu_0 I$ is Browder. This proves that $\mu_0 \in \pi_{00}(f(T))$.

If $\mu_0 \in \pi_{00}(f(T))$, let

$$f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1} (T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T),$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Without loss of generality, we may suppose that $\lambda_i \in \sigma(T)$. Then $\lambda_i \in \pi_{00}(T)$, and hence $T - \lambda_i I$ is Browder, since T

satisfies property (ω). We now get that $\mu_0 \in \sigma_a(f(T)) \setminus \sigma_{aw}(f(T))$. This proves that property (ω) holds for $f(T)$.

Since $\operatorname{iso}\sigma(f(T)) \subseteq f(\operatorname{iso}\sigma(T))$ and T is isoloid, we get that $f(T)$ is isoloid. By $\sigma(T) = \sigma_a(T)$, we know that $\sigma_a(f(T)) = \sigma(f(T))$. We can prove that

$$\sigma_a(f(T)+K) \supseteq \sigma_a(f(T)) \cap \sigma_b(f(T)) = \sigma(f(T)) \cap \sigma_b(f(T)) = \sigma_b(f(T)),$$

and

$$\begin{aligned}\sigma_b(f(T)) &= \sigma_1(f(T)) \cup \overline{\sigma_{CI}(f(T))} = \sigma_1(f(T)) \cup [\overline{\sigma_{CI}(f(T))} \cap \sigma(f(T))] \\ &= \sigma_1(f(T)) \cup [\overline{\sigma_{CI}(f(T))} \cap \sigma_a(f(T))].\end{aligned}$$

Thus

$$\sigma_b(f(T)) \cap \sigma_a(f(T)+K) = \sigma_b(f(T)) = \sigma_1(f(T)) \cup [\overline{\sigma_{CI}(f(T))} \cap \sigma_a(f(T))].$$

By Theorem 2.1, $f(T)+K$ is isoloid, and property(ω) holds for $f(T)+K$. Since

$$\sigma_a(f(T)+K) \supseteq \sigma_b(f(T))(= \sigma_b(f(T)+K)),$$

we know that

$$\sigma(f(T)+K) = \sigma_a(f(T)+K).$$

This completes the proof. □

In the sequel, we shall consider nilpotent perturbations of operators satisfying property (ω). It is easy to check that if N is a nilpotent operator commuting with T, then $\sigma(T) = \sigma(T+N)$ and $\sigma_a(T) = \sigma_a(T+N)$. One can see that:

Theorem 2.2. *Suppose that $N \in B(H)$ is a nilpotent operator that commutes with $T \in B(H)$. Then $T+N$ is isoloid and satisfies property (ω) if and only if T is isoloid and satisfies property (ω).*

If N is a nilpotent operator commuting with T, we get that

$$\sigma_{aw}(T) = \sigma_{aw}(T+N).$$

Similar to the proof of Corollary 2.4, we get:

Corollary 2.5. *Suppose that T is isoloid and satisfies property (ω). If N is a nilpotent operator commuting with T, then the following statements are equivalent:*

(1) *for any $f \in H(T)$, $f(T)+N$ is isoloid and satisfies property (ω);*

(2) *for any pair* $\lambda, \mu \in \mathbb{C} \setminus \sigma_{SF_+}(T)$, $\mathrm{ind}(T-\lambda I)\,\mathrm{ind}(T-\mu I) \geq 0$, *and* $\sigma_a(T) = \sigma_{aw}(T)$ *or* $\sigma(T) = \sigma_a(T)$*;*

(3) *for any* $f \in H(T)$, $f(T)$ *is isoloid and satisfies property* (ω)*;*

(4) *for any* $f \in H(T)$, $f(\sigma_1(T)) \subseteq \sigma_1(f(T))$, $\sigma_a(T) = \sigma_{aw}(T)$ *or* $\sigma(T) = \sigma_a(T)$.

For the stability of property (ω) for quasi-nilpotent operators or Riesz operators, one can obtain:

Theorem 2.3. *Suppose that* $T \in B(H)$ *satisfies property* (ω). *If* $K \in B(H)$ *is a Riesz operator commuting with* T*, then* $T+K$ *is isoloid and satisfies property* (ω) *if and only if* $\sigma_b(T) \cap \sigma_a(T+K) = [\sigma_1(T) \cap acc\,\sigma(T+K)] \cup [\overline{\sigma_{CI}(T)} \cap \sigma_a(T)] \cup \{\lambda \in \mathbb{C} : n(T+K-\lambda I) = \infty\}$.

As an application, we shall consider the stability of property (ω) for the operators which are the products of finitely many normal operators. In what follows, H will be a fixed separable complex Hilbert space. It is well known that $T \in B(H)$ is the product of finitely many normal operators if and only if $\dim N(T) = \dim N(T^*)$ or $R(T)$ is not closed [13]. If $T \in B(H)$ is the product of finitely many normal operators and T is a semi-Fredholm operator, T must be a Weyl operator.

Theorem 2.4. *Suppose that* $T \in B(H)$ *is isoloid and satisfies property* (ω). *If* $T - \lambda I$ *is the product of finitely many normal operators for any* $\lambda \in \mathbb{C}$*, then for any finite rank operator* $K \in B(H)$ *commuting with* T, $T+K$ *is isoloid and satisfies property* (ω).

Proof. Since $T-\lambda I$ is the product of finitely many normal operators for any $\lambda \in \mathbb{C}$, it follows that $\sigma(T) = \sigma_a(T)$. Then

$$\sigma_1(T) \cup [\overline{\sigma_{CI}(T)} \cap \sigma_a(T)] = \sigma_1(T) \cup \overline{\sigma_{CI}(T)}.$$

Using the fact that T is isoloid and satisfies property (ω), we can get that $\sigma_a(T+K) \supseteq \sigma_a(T) \cap \sigma_b(T) = \sigma_b(T)$ and $\sigma_b(T) = \sigma_1(T) \cup \overline{\sigma_{CI}(T)}$. Thus

$$\sigma_b(T) \cap \sigma_a(T+K) = \sigma_b(T) = \sigma_1(T) \cup \overline{\sigma_{CI}(T)} = \sigma_1(T) \cup [\overline{\sigma_{CI}(T)} \cap \sigma_a(T)].$$

By Theorem 2.1, $T+K$ is isoloid and satisfies property (ω). □

Remark 2.1. (1) In Theorem 2.4, "property (ω) holds for $T+K$ for any finite rank operator $K \in B(H)$ commuting with T" cannot induce that "$T - \lambda I$ is the product of finitely many normal operators for any $\lambda \in \mathbb{C}$". For example, let $T \in B(\ell^2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots),$$

then T is isoloid and satisfies property (ω). Also, for any finite rank operator K

commuting with T, $T+K$ satisfies property (ω). But we can see that for any $|\lambda|<1$, $T-\lambda I$ is not the product of finitely many normal operators.

(2) We can prove that: if both T and T^* are isoloid and satisfy property (ω), then, for any $\lambda \in \mathbb{C}$ and for any finite rank operator $K \in B(H)$ commuting with T, $T+K-\lambda I$ is the product of finitely many normal operators if and only if $\sigma_{CI}(T)=\emptyset$.

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